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Irreversible compressible work and APE dissipation in turbulent stratified fluids

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Abstract. Although it plays a key role in the theory of stratified turbulence, the concept of available potential energy (APE) dissipation has remained until now a rather mysterious quantity, owing to the lack of rigorous result about its irreversible character or energy conversion type. Here, we show by using rigorous energetics considerations rooted in the analysis of the Navier-Stokes for a fully compressible fluid with a nonlinear equation of state that the APE dissipation is an irreversible energy conversion that dissipates kinetic energy into internal energy, exactly as viscous dissipation. These results are established by showing that APE dissipation contributes to the irreversible production of entropy, and by showing that it is a part of the work of expansion/contraction. Our results provide a new interpretation of the entropy budget, that leads to a new exact definition of turbulent effective diffusivity, which generalizes the Osborn-Cox model, as well as a rigorous decomposition of the work of expansion/contraction into reversible and irreversible components. In the context of turbulent mixing associated with parallel shear flow instability, our results suggests that there is no irreversible transfer of horizontal momentum into vertical momentum, as seems to be required when compressible effects are neglected, with potential consequences for the parameterisations of momentum dissipation in the coarse-grained Navier-Stokes equations.

1. Introduction

It has long been known that in the presence of stratification, part of the kinetic energy may ultimately be dissipated as a the result of work against vertical buoyancy forces. The physical mechanism that is traditionally evoked is that the conversion of kinetic energy into gravitational potential energy causes vertical displacements of the isentropic surfaces from a notional horizontal reference position, thus causing temperature fluctuations that are dissipated by molecular diffusion, e.g., [11, 1]. The rate at which kinetic energy is removed via work against buoyancy forces is usually denoted by ε_P and called the available potential energy dissipation rate. Until now, the concept of APE dissipation rate has been primarily defined in the context of a Boussinesq fluid with a linear equation of state, in which case ε_P is related to the dissipation of temperature variance $\chi_T = \|\nabla\theta'\|^2$, where θ' are potential temperature anomalies via $\varepsilon_P = \alpha g_0 \chi_T$, where α and g_0 are the (constant) thermal expansion coefficient and acceleration of gravity respectively. The concept of APE dissipation plays a central role in the theory of stratified turbulence, as it serves as the basis for the definition of the turbulent diapycnal diffusivity K_v in the Osborn-Cox model [8, 9] via the relationship:

$$K_v = \frac{\varepsilon_P}{N^2} = \frac{\Gamma \varepsilon_K}{N^2} \tag{1}$$

where $\Gamma = \varepsilon_P/\varepsilon_K$ is often called the mixed efficiency or dissipation ratio, e.g., [7], while N^2 is the squared buoyancy frequency.

Despite its central role in the theory of stratified turbulence, the concept of APE dissipation remains nevertheless somewhat mysterious. For instance, in the evolution equation for turbulent kinetic energy, the term $\varepsilon_P + \varepsilon_K$ appears as a form of total dissipation. As is well known, the viscous dissipation rate ε_K is an irreversible conversion of kinetic energy into internal energy; however, we are not aware of any proof establishing the irreversible character of ε_P, although this is expected owing to the link between ε_P and χ_T. Moreover, it has usually been assumed that ε_P is a conversion of available potential energy into background gravitational potential energy, e.g., [21]. Physically, this would make the total dissipation $\varepsilon_P + \varepsilon_K$ a rather strange quantity, as it seems awkward that two quantities that seem to so naturally combine would be each associated with different types of energy conversions.

The purpose of this paper is to further develop and extend some previous results of [13], with the aim of fully elucidating the nature of the concept of APE dissipation, which can only be done in the context of the fully compressible Navier-Stokes equations, in order to avoid any potential misunderstanding or misinterpretation that so naturally arise in approximate systems of equations, most notably in the incompressible system owing to the often unclear role played by internal energy. Section 2 recalls some results about the energetics of a turbulent mixing. Section 3 provides the main theoretical results. Section 4 shows how to decompose the pressure work into reversible and irreversible components. Section 5 offers a summary of the results and a discussion.

2. Energetics of a turbulent mixing event

2.1. Compressible Navier-Stokes equations

We seek to revisit the energetics of turbulent mixing, with the aim of uncovering the physical origin of APE dissipation, in the context of the fully compressible Navier-Stokes equations. As in [13], we assume the equation of state to be nonlinear and to depend on temperature and pressure. The relevant equations can be written under the form:

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla P = -\rho g_0 \mathbf{z} + \nabla \cdot \mathbf{S}, \tag{2}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \tag{3}$$

$$\rho \frac{D\eta}{Dt} = \frac{\nabla \cdot (\kappa \rho c_p \nabla T)}{T} + \frac{\rho \varepsilon_K}{T} = \frac{\rho \dot{Q}}{T}, \tag{4}$$

$$e = e(\eta, \upsilon), \tag{5}$$

$$T = \frac{\partial e}{\partial \eta}, \qquad P = -\frac{\partial e}{\partial \upsilon}, \tag{6}$$

where $\mathbf{v} = (u, v, w)$ is the three-dimensional Eulerian velocity field, $\upsilon = 1/\rho$ is the specific volume (ρ is the density), g_0 the (constant) acceleration of gravity, $\mathbf{z}$ the normal unit vector parallel to the gravity field pointing upwards, and κ the molecular diffusivity. The thermodynamics of the fluid is described by the specification of an equation of state for the specific internal energy $e = e(\eta, \upsilon)$, where η is the specific entropy, from which the in-situ temperature T and thermodynamic pressure P can be diagnosed, while c_p is the specific heat capacity at constant pressure. The quantity $\mathbf{S}$ denotes the deviatoric stress tensor, whose components are given by:

$$\mathbf{S}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \left(\lambda - \frac{2\mu}{3} \right) \delta_{ij} \frac{\partial u_\ell}{\partial x_\ell}, \tag{7}$$

in the classical tensorial notation, where Einstein's summation convention for repeated indices has been adopted and δ_{ij} is the Kronecker delta. The parameters μ and λ are the shear and bulk (or volume) viscosity respectively. The viscous dissipation rate ε_K is therefore given by:

$$\rho \varepsilon_K = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_\ell}{\partial x_\ell} \right)^2 + \lambda \left(\nabla \cdot \mathbf{v} \right)^2 \tag{8}$$

where again the summation convention for repeated indices has been used.

2.2. Standard energetics analysis of a turbulent mixing event

As in [13], we discuss the energetics of turbulent mixing by considering the specific case of a turbulent mixing event arising from shear flow instability, a popular topic of investigation, which so far has been mostly investigated in the context of a Boussinesq fluid with a linear equation of state, e.g., [21, 12, 2]. Physically, the phenomenology

considered is that of a shear flow that is initially in a laminar state with no APE, which then becomes unstable and associated with intense turbulent mixing, before returning to a laminar state with no APE. With regard to configuration, the fluid is assumed to lie between a horizontal solid boundary located at $z = 0$ and a horizontal free surface located at $z = D$, at which the constant external pressure P_0 is applied. All boundaries are insulated, so that no heat exchange occurs between the fluid and its surroundings at any time. The assumption of free surface allows for volume changes to take place upon irreversible mixing and therfore for the compressible work by the atmosphere to be performed on the fluid;

The standard energetic analysis of turbulent mixing traditionally relies on a description involving only three energy reservoirs, namely the total kinetic energy KE, the gravitational potential energy GPE, and the internal energy IE, whose mathematical expressions are:

$$\mathrm{KE} = \int_V \frac{\mathbf{v}^2}{2}\mathrm{d}m, \qquad \mathrm{GPE} = \int_V g_0 z \mathrm{d}m, \qquad \mathrm{IE} = \int_V e(\eta, \upsilon)\mathrm{d}m \qquad (9)$$

where $\mathrm{d}m = \rho \mathrm{d}V$ is the elementary mass of a fluid element. Standard manipulations allows one to write evolution equations for each energy reservoir as follows:

$$\frac{\mathrm{dKE}}{\mathrm{d}t} = -\underbrace{\int_V g_0 w\,\mathrm{d}m}_{W} + \underbrace{\int_V (P - P_0)\frac{D\upsilon}{Dt}\mathrm{d}m}_{B'} - \underbrace{\int_V \varepsilon_K\,\mathrm{d}m}_{D_{KE}}, \qquad (10)$$

$$\frac{\mathrm{dGPE}}{\mathrm{d}t} = \underbrace{\int_V g_0 w\,\mathrm{d}m}_{W}, \qquad (11)$$

$$\frac{\mathrm{dIE}}{\mathrm{d}t} = \int_V \dot{Q}\,\mathrm{d}m - \underbrace{\int_V (P - P_0)\frac{D\upsilon}{Dt}\,\mathrm{d}m}_{B'} - \underbrace{P_0 \int_V \frac{D\upsilon}{Dt}\,\mathrm{d}m}_{B_0}, \qquad (12)$$

e.g., see [13] an example of derivation. The main fundamental difference between the present energetic analysis for a fully compressible fluid and that for a Boussinesq fluid with a linear equation of state, e.g., [21], lies in the presence in the KE equation (10) of the thermodynamic work of expansion B' by the anomalous (or gauge) pressure $P' = P - P_0$.

Next, the above equations are time averaged over the duration of the turbulent mixing event, denoted by an overbar, which yields the following budget equations:

$$\Delta\mathrm{KE} = -\overline{W} + \overline{B'} - \overline{D}_{KE} = -\overline{D}_{APE} - \overline{D}_{KE}, \qquad (13)$$

$$\Delta\mathrm{GPE} = \overline{W}, \qquad (14)$$

$$\Delta\mathrm{IE} = \overline{D}_{KE} - \overline{B'} - \overline{B}_0, \qquad (15)$$

where in the IE budget equation (15), the volume integral of $\dot{Q}$ is only due to the Joule heating by viscous dissipation owing to the assumption of no heat exchange with the environment. The KE budget equation (13) reflects the well established result that over

a turbulent mixing event, kinetic energy appears to be dissipated ($\Delta KE < 0$) both by viscous dissipation $\overline{D}_{KE}$, as well as by work against gravity via the density flux $\overline{W}$, which are the only two terms appearing in the standard KE budget of a Boussinesq fluid with a linear equation of state, e.g., [21]. In a compressible fluid, however, there is also a contribution from the work of expansion/contraction $\overline{B}'$, which has been so far traditionally regarded as negligible for Boussinesq fluids. A key point of the present paper is to argue that such an assumption is at best misleading, as previously argued in [13], and that $\overline{B}'$ is actually a key energy conversion in turbulent stratified fluids. Leaving this point aside for the moment, we posit here that the term $\overline{D}_{APE} = \overline{W} - \overline{B}'$ should be regarded as the natural generalisation to a compressible fluid of the well-known concept of *APE dissipation*. The rest of the paper is devoted to establishing that this is indeed justified, by demonstrating that the expression for $\overline{D}_{APE}$ thus defined is similar at leading order with its classical Boussinesq counterpart.

2.3. Analysis in terms of available energetics

In turbulence theory, the traditional approach to obtaining an explicit expression for APE dissipation has usually relied on the consideration of an evolution equation for the APE density $E_a = N^2 \zeta^2 / 2$, which is valid only for small displacements ζ from a horizontally uniform reference state. This approach was generalised by [21] by using [5]'s exact APE framework, valid for arbitrarily large displacements from the reference state, to the case of a Boussinesq fluid with a linear equation of state, which [13] further generalised to the case of a fully compressible fluid with a nonlinear equation of state. By construction, [5]'s reference state is the state of minimum potential energy obtainable in an isentropic re-arrangement of mass, as illustrated in Fig. 1. As a result, the reference state can only evolve as the result of diabatic effects, so that in absence of heat exchange with the environment, the time evolution of the reference state will be a direct measure of the amount of irreversible mixing taking place in the fluid. Physically, the isentropic re-arrangement leading to [5]'s reference state can be a regarded as a rigorous and exact procedure for separating the mostly adiabatic *stirring* from the diabatic *irreversible mixing*. Mathematically, the isentropic re-arrangement assumes the existence of a mapping taking a parcel located at $(\mathbf{x}, t)$ in the actual state to its vertical position $z_r = z_r(\mathbf{x}, t)$ in the reference state. By construction, the isentropic re-arrangement of mass conserves the parcels' mass and entropy, and the reference state is in exact hydrostatic balance, which can be mathematically expressed by:

$$\eta(\mathbf{x}, t) = \eta_r(z_r, t), \tag{16}$$

$$\rho(\mathbf{x}, t)\mathrm{d}V = \rho_r(z_r, t)\mathrm{d}V_r, \tag{17}$$

$$\frac{\partial P_r}{\partial z_r} = -\rho_r(z_r, t)g_0. \tag{18}$$

In order to obtain an explicit expression for the APE dissipation, we need to derive evolution equations for the potential energy of the actual and reference states. For a
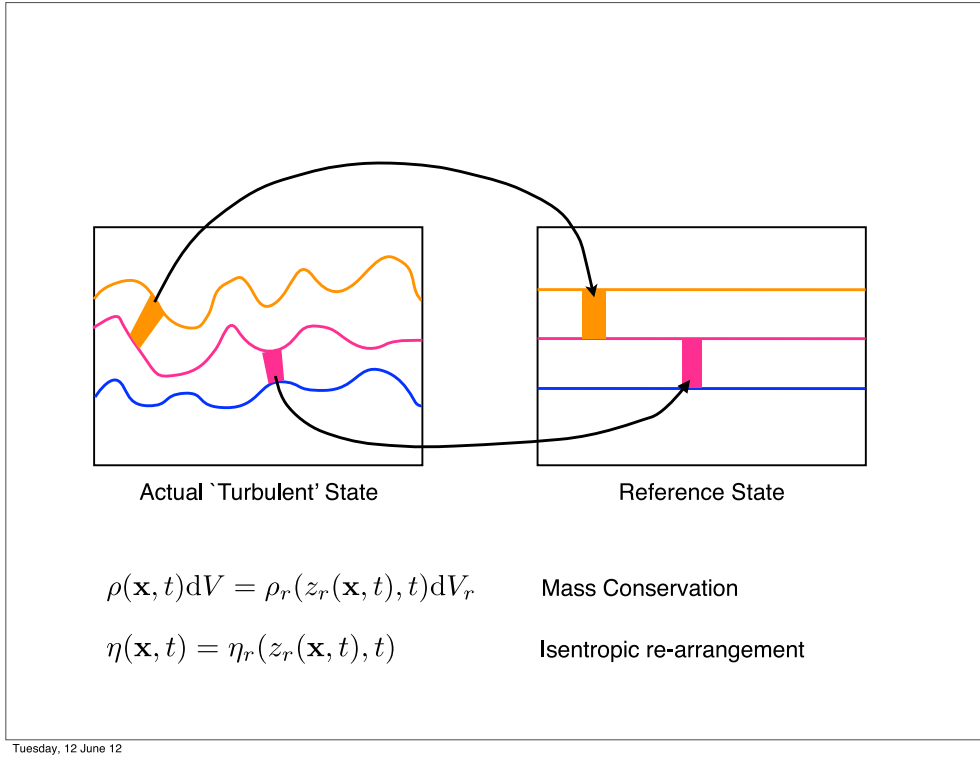
Figure 1 shows two boxes labeled "Actual 'Turbulent' State" and "Reference State". Below:

$$\rho(\mathbf{x}, t)\mathrm{d}V = \rho_r(z_r(\mathbf{x}, t), t)\mathrm{d}V_r \quad \text{Mass Conservation}$$

$$\eta(\mathbf{x}, t) = \eta_r(z_r(\mathbf{x}, t), t) \quad \text{Isentropic re-arrangement}$$

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Figure 1. The isentropic and mass-conserving re-arrangement underlying the construction of Lorenz's reference state of minimum potential energy.

compressible fluid, the total potential energy PE is the sum of the gravitational potential energy GPE, internal energy IE, and $P_0 V_{ol}$, where V_{ol} is the total volume of the fluid,

$$\text{PE} = \int_V \rho\left[e(\eta, \upsilon) + g_0 z\right]\mathrm{d}V + P_0 V_{ol}. \tag{19}$$

By definition, the potential energy of the background reference state and the available potential energy are given by the following expressions:

$$\text{PE}_r = \int_V \rho\left[e(\eta_r, \upsilon_r) + g_0 z_r\right]\mathrm{d}V + P_0 V_{ol,r}, \tag{20}$$

$$\text{APE} = \text{PE} - \text{PE}_r, \tag{21}$$

where $V_{ol,r}$ is the volume of the fluid in the reference state. As the evolution equation for PE is known from standard energetics, the key step to obtaining an evolution equation for APE is to first obtain one for the background PE. To that end, it is useful to first establish the following result:

$$\frac{\mathrm{dGPE}_r}{\mathrm{d}t} = \int_V \rho g w_r \,\mathrm{d}V = \int_V \rho P_r' \frac{D\upsilon_r}{Dt}\,\mathrm{d}V = B_r', \tag{22}$$

which states that the change in the background GPE can be expressed in terms of the thermodynamic work of expansion/contraction done by the pressure in the reference state, where $w_r = Dz_r/Dt$ is the vertical velocity of the parcels in the reference state,

while $P_r' = P_r - P_0$ is the gauge pressure. The proof follows from the result:

$$\int_V P_r' \frac{Dv_r}{Dt} \, \mathrm{d}m = \int_{V_r} \left[\frac{D[P_r'v_r]}{Dt} - v_r \frac{DP_r'}{Dt} \right] \rho_r \mathrm{d}V_r = \frac{d}{dt} \int_{V_r} P_r' \, \mathrm{d}V_r - \int_{V_r} \left[\frac{\partial P_r'}{\partial t} - \rho_r g_0 w_r \right] \mathrm{d}V_r$$

$$= \int_{V_r} \rho_r g_0 w_r \mathrm{d}V_r = \int_V \rho g_0 w_r \mathrm{d}V = \frac{\mathrm{dGPE}_r}{\mathrm{d}t}, \tag{23}$$

where in order to arrive at Eq. (23), we used mass conservation $\mathrm{d}m = \rho \mathrm{d}V = \rho_r \mathrm{d}V_r$ between the actual and reference states; moreover, the fact that $P_r' = 0$ at the free surface implies:

$$\frac{d}{dt} \int_{V_r} P_r' \mathrm{d}V_r = \int_{V_r} \frac{\partial P_r'}{\partial t} \, \mathrm{d}V_r. \tag{24}$$

By using Eq. (22), and the fact that the differential of reference internal energy is $\mathrm{d}e_r = T_r \mathrm{d}\eta_r - P_r \mathrm{d}v_r$, it is easy to show that evolution equation for PE_r is:

$$\frac{\mathrm{dPE}_r}{\mathrm{d}t} = \int_V \rho \left[T_r \frac{D\eta_r}{Dt} - P_r' \frac{Dv_r}{Dt} + g_0 w_r \right] \mathrm{d}V = \int_V \rho \left[T_r \frac{D\eta}{Dt} - P_r' \frac{Dv_r}{Dt} + g_0 w_r \right] \mathrm{d}V$$

$$= \int_V \frac{T_r}{T} \left[\nabla \cdot (\kappa \rho c_p \nabla T) + \rho \varepsilon_k \right] \mathrm{d}V. \tag{25}$$

By summing the evolution equations for GPE and IE written above, the evolution equation for the total potential energy can be written as:

$$\frac{\mathrm{dPE}}{\mathrm{d}t} = \int_V \rho \left[T \frac{D\eta}{Dt} - P' \frac{Dv}{Dt} + g_0 w \right] \mathrm{d}V$$

$$= \int_V \left[\nabla \cdot (\kappa \rho c_p \nabla T) + \rho \varepsilon_K \right] \mathrm{d}V - B' + W. \tag{26}$$

Given the fact that $\mathrm{APE} = \mathrm{PE} - \mathrm{PE}_r$, the evolution equation for APE is:

$$\frac{\mathrm{dAPE}}{\mathrm{d}t} = \int_V \left(\frac{T - T_r}{T} \right) \nabla \cdot (\kappa \rho c_p \nabla T) \, \mathrm{d}V + \int_V \left(\frac{T - T_r}{T} \right) \rho \varepsilon_K \, \mathrm{d}V + W - B'$$

$$= -\int_V \kappa \rho c_p \nabla T \cdot \nabla \left(\frac{T - T_r}{T} \right) \mathrm{d}V + \int_V \left(\frac{T - T_r}{T} \right) \varepsilon_K \, \mathrm{d}m + W - B'. \tag{27}$$

Finally, it is also useful to produce an evolution equation for $\mathrm{AGPE} = \mathrm{GPE} - \mathrm{GPE}_r$, which is given by:

$$\frac{\mathrm{dAGPE}}{\mathrm{d}t} = W - W_r = W - B_r'. \tag{28}$$

All the above results were previously derived in [13], but have been obtained more simply here. We can now return to the problem of elucidating the nature of APE dissipation. To that end, we take advantage of the fact that APE and AGPE are both zero in the laminar states preceding and following a turbulent mixing event. Thus, by using Eqs. (22) and (23), it follows from the AGPE budget that:

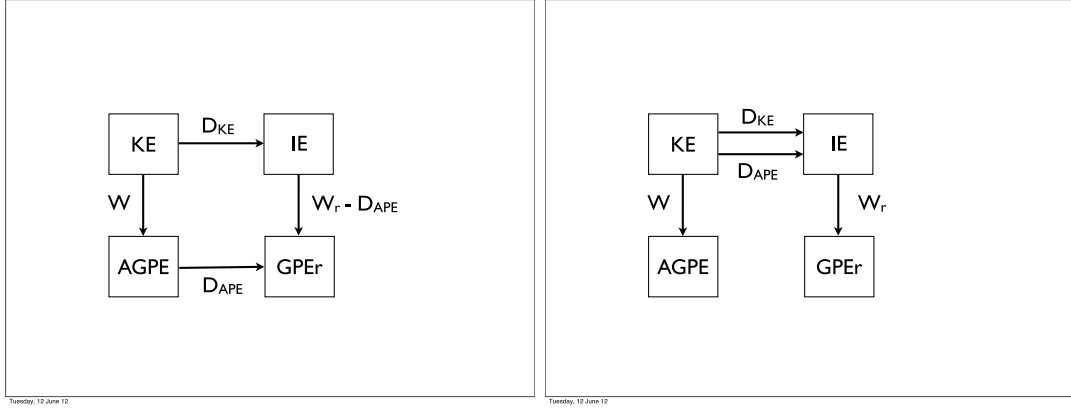$$\overline{W} = \overline{W}_r = \overline{B'}_r, \tag{29}$$

Figure 2. (Left panel) Traditional interpretation of the energetics of turbulent mixing according to Winters et al. (1995). (Right panel) New interpretation based on the rigorous analysis of the energetics of the fully compressible Navier-Stokes equations initially proposed by Tailleux (2009a).

which establishes that the net change in GPE (GPE$_r$) actually occurs at the expenses of internal energy, exactly as in the purely laminar case, not at all at the expenses of the kinetic energy of the initial shear flow as is often thought to be the case based on [21]'s analysis. Similarly, by using Eqs. (27) and (29), it follows from the APE budget that:

$$\overline{B'} = \overline{B'}_r - \int_V \kappa \rho c_p \nabla T \cdot \nabla \left(\frac{T - T_r}{T} \right) \, \mathrm{d}V + \int_V \left(\frac{T - T_r}{T} \right) \rho \varepsilon_k \, \mathrm{d}V. \qquad (30)$$

By combining the above results, it follows that the expression for the total APE dissipation averaged over the turbulent mixing event must be given by

$$\overline{D}_{APE} = \overline{W} - \overline{B'} = \overline{B}'_r - \overline{B'}$$

$$= \int_V \kappa \rho c_p \nabla T \cdot \nabla \left(\frac{T - T_r}{T} \right) \, \mathrm{d}V - \int_V \left(\frac{T - T_r}{T} \right) \rho \varepsilon_k \, \mathrm{d}V. \qquad (31)$$

This important result states that the APE dissipation can be expressed as $\overline{D}_{APE} = \overline{B}'_r - \overline{B}$ and hence as the part of the thermodynamic work of expansion/contraction not involved in the change of the background GPE$_r$; as a part of the work of expansion/contraction, the APE dissipation appears as a conversion between kinetic energy and internal energy, similarly as the viscous dissipation. The rest of the paper is devoted to further elucidating the properties of $\overline{D}_{APE}$.

3. Theoretical results on the nature of APE dissipation and irreversible mixing in compressible stratified fluids

3.1. Proof of the irreversible character of APE dissipation

We established earlier (Eq. (13)) that the total change in kinetic energy over a turbulent mixing event is given by:

$$\Delta KE = -(\overline{D}_{APE} + \overline{D}_{KE}), \qquad (32)$$

which states that both the viscous dissipation $\overline{D}_{KE}$ and APE dissipation $\overline{D}_{APE}$ contribute to dissipate the kinetic energy of the unstable parallel shear flow. Given that $\overline{D}_{APE}$ and $\overline{D}_{KE}$ appear to both convert dissipate kinetic energy into internal energy, it is important from a thermodynamic viewpoint to establish whether $\overline{D}_{APE}$ can also be regarded as an *irreversible* energy conversion. Indeed, while the irreversible character of viscous dissipation is well established, we are not aware of any such result for APE dissipation.

In order to prove the irreversible character of D_{APE}, we find it necessary to introduce the locally-defined APE dissipation rate ε_P as follows:

$$\rho\varepsilon_p = \kappa\rho c_p \nabla T \cdot \nabla\left(\frac{T-T_r}{T}\right) - \left(\frac{T-T_r}{T}\right)\rho\varepsilon_K, \tag{33}$$

which is such that $D_{APE} = \int_V \rho\varepsilon_K\,\mathrm{d}V$. From a thermodynamic viewpoint, the irreversible character of a particular process follows from such a process contributing to the irreversible production of entropy. To show that this is the case for ε_P, recall that in the present context, the specific entropy is governed by the following equation:

$$\rho\frac{D\eta}{Dt} = \frac{\dot{Q}}{T} = \nabla\cdot\left(\frac{\kappa\rho c_p\nabla T}{T}\right) - \rho\kappa c_p\nabla T\cdot\nabla\left(\frac{1}{T}\right) + \frac{\rho\varepsilon_K}{T} \tag{34}$$

where the irreversible entropy production is made up of the last two terms in the above expression. While ε_K explicitly appears in the latter, this is less obvious for ε_P. We find that it is possible to uncover the presence of ε_P in Eq. (34) by writing the term $\nabla(1/T)$ as follows:

$$\nabla\left(\frac{1}{T}\right) = \nabla\left(\frac{T_r}{T}\frac{1}{T_r}\right) = \frac{1}{T_r}\nabla\left(\frac{T_r}{T}\right) + \frac{T_r}{T}\nabla\left(\frac{1}{T_r}\right)$$

which allows one to rewrite Eq. (34) as follows:

$$\rho\frac{d\eta}{dt} = \nabla\cdot\left(\frac{\kappa\rho c_p\nabla T}{T}\right) + \kappa\rho c_p\frac{\nabla T\cdot\nabla T_r}{T\,T_r}\,\mathrm{d}V + \frac{\rho(\varepsilon_P + \varepsilon_K)}{T_r}. \tag{35}$$

Eq. (35) is an important new result, which provides new insight into the nature of irreversible entropy production. Specifically, it shows that the latter can be regarded as the sum of a term measuring the amount of 'work' ($= \varepsilon_P + \varepsilon_K$) being dissipated irreversibly in the system, plus a 'heat' term (the term proportional to $\nabla T\cdot\nabla T_r$) that arises from the irreversible character of heat transfer among fluid parcels with different transfer. In the following paragraph, we use the latter term as our starting point for introducing a new definition of the turbulent effective diffusivity and heat flux.

3.2. Generalisation of the Osborn-Cox model to fully compressible stratified fluids and definition of turbulent effective diffusivity

Given that the very concept of irreversible process takes its origin in the entropy budget, it is not a surprise that the latter often constitutes the starting point for the definition

of turbulent diapycnal mixing, e.g., [8, 4, 3]. Here, we show that if we locally define the *turbulent effective diffusivity* K_{eff} by:

$$K_{\text{eff}} = \kappa \frac{c_p T_r}{c_p^r T} \left(\frac{\partial T_r}{\partial z_r} \right)^{-1} \nabla T \cdot \nabla z_r = \kappa \frac{c_p T_r}{c_p^r T} \left(\frac{\partial T_r}{\partial z_r} \right)^{-2} \nabla T \cdot \nabla T_r, \tag{36}$$

then it is possible to rewrite the irreversible entropy production term in Eq. (35) due to the irreversible heat transfer entirely in terms of K_{eff} and the background temperature profile T_r as follows:

$$\kappa \rho c_p \frac{\nabla T \cdot \nabla T_r}{T \, T_r} = K_{\text{eff}} \rho c_p^r \frac{1}{T_r^2} \left(\frac{\partial T_r}{\partial z_r} \right)^2, \tag{37}$$

which makes it possible to write the total entropy production as follows:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = \int_V K_{\text{eff}} \rho c_p^r \frac{1}{T_r^2} \left(\frac{\partial T_r}{\partial z_r} \right)^2 \mathrm{d}V + \int_V \frac{\rho(\varepsilon_P + \varepsilon_K)}{T_r} \, \mathrm{d}V. \tag{38}$$

It appears also possible to rewrite the latter expression in terms of the following locally-defined effective turbulent heating rate:

$$\dot{Q}_{\text{eff}} = \frac{\partial}{\partial z_r} \left\{ K_{\text{eff}} \rho c_p^r \frac{\partial T_r}{\partial z_r} \right\} \tag{39}$$

as follows:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = \int_V \frac{\dot{Q}_{\text{eff}}}{T_r} \, \mathrm{d}V + \int_V \frac{\rho(\varepsilon_P + \varepsilon_K)}{T_r} \, \mathrm{d}V. \tag{40}$$

so that locally, the net heating can be express as $\dot{Q}_{net} = \dot{Q}_{\text{eff}} + \rho(\varepsilon_P + \varepsilon_K)$, noting here that the APE dissipation can therefore be regarded as a form of Joule heating, similarly as viscous dissipation. Note, however, that as far as we can judge:

$$\frac{\partial}{\partial z_r} \left(\frac{K_{\text{eff}} \rho c_{pr}}{T_r} \frac{\partial T_r}{\partial z_r} \right) \neq \nabla \cdot \left(\frac{\kappa \rho c_p \nabla T}{T} \right). \tag{41}$$

The above results are interesting, because they suggest that the effective turbulent heat flux should be down the gradient of T_r, rather than down the gradient of potential temperature θ, as is often assumed. We note from Eq. (36) that the turbulent effective diffusivity is only well defined as long as T_r remains a monotonic function of z_r, i.e., as long as $\partial T_r / \partial z_r > 0$, and therefore ceases to make sense if T_r were to become uniform. In contrast, it is also seen that K_{eff} remains well defined in the limit of uniform *in-situ* temperature, which in general will differ from a state of uniform T_r as long as the fluid remains stirred to ensure that isotherms remain different from isentropes.

3.3. Evolution of the background GPE due to turbulent mixing

An important result of turbulent mixing theory is that turbulent mixing causes a net increase in a centre of gravity of the fluid, which can be expressed in terms of the turbulent diapycnal diffusivity K_v and squared buoyancy frequency N^2 as follows:

$$\left. \frac{\mathrm{dGPE}}{\mathrm{d}t} \right|_{mixing} = \int_V \rho K_v N^2 \mathrm{d}V. \tag{42}$$

The possibility to characterise turbulent diapycnal mixing in terms of its energy signature in the GPE budget is important, for it is at the origin of the proposal by [6] that energetics considerations might be useful to constrain the mechanical sources of stirring by the wind and tides in the oceans, see [15, 16, 18] for a review of salient points. It is important to realize, however, that Eq. (42) has been established only for a Boussinesq fluid with a linear equation of state. We now seek the relevant generalisation of Eq. (42) to the case of a fully compressible fluid with a nonlinear equation of state by seeking an evolution equation for the background GPE. As previously shown in [13], one possible expression is as follows:

$$\frac{\mathrm{dGPE}_r}{\mathrm{d}t} = B_r' = \int_V P_r' \frac{D\upsilon_r}{Dt} \rho \, \mathrm{d}V = \int_V P_r \left[\frac{\partial \upsilon_r}{\partial \eta_r} \frac{D\eta_r}{Dt} + \frac{\partial \upsilon_r}{\partial P_r} \frac{DP_r}{Dt} \right] \mathrm{d}V$$

$$= \int_V \frac{P_r' \alpha_r T_r}{\rho_r c_{pr}} \left\{ \frac{\nabla \cdot (\kappa \rho c_p \nabla T) + \rho \varepsilon_K}{T} \right\} \mathrm{d}V, \tag{43}$$

which uses the result that $\partial \upsilon_r / \partial \eta_r = \alpha_r T_r / (\rho_r c_{pr})$ is the adiabatic lapse rate, and the fact that the term dependent upon pressure should vanish (see also [10] for a general demonstration of this point). It is useful to introduce the thermodynamic efficiency-like parameter Υ_r, defined by:

$$\Upsilon_r = \frac{\alpha_r (P_r - P_0)}{\rho_r c_{pr}} = \frac{\alpha_r P_r'}{\rho_r c_{pr}} \tag{44}$$

in which case, the above expression can be rewritten as follows:

$$\frac{\mathrm{dGPE}_r}{\mathrm{d}t} = - \int_V \kappa \rho c_p \nabla T \cdot \nabla \left(\frac{\Upsilon_r T_r}{T} \right) \mathrm{d}V + \int_V \frac{\Upsilon_r T_r}{T} \rho \varepsilon_K \mathrm{d}V$$

$$= \underbrace{- \int_V \frac{\kappa \rho c_p T_r}{T} \nabla T \cdot \nabla \Upsilon_r \, \mathrm{d}V}_{W_{r,mixing}} + \underbrace{\int_V \Upsilon_r \rho (\varepsilon_P + \varepsilon_K) \, \mathrm{d}V}_{W_{r,joule}}. \tag{45}$$

Eq. (45) states that the net increase in GPE over a turbulent mixing event is due to the irreversible mixing by turbulent molecular diffusion (the term $W_{r,mixing}$), as well as to the increase in the centre of gravity caused by the Joule heating due to the total dissipation $\varepsilon_P + \varepsilon_K$ (the term $W_{r,joule}$) and the associated fluid expansion. As the latter effect is usually neglected in the literature, it is the first term $W_{r,mixing}$ that needs to be expressed in terms of the effective diffusivity K_{eff} defined above.

$$\frac{\mathrm{dGPE}_r}{\mathrm{d}t} \bigg|_{mixing} = - \int_V \frac{\kappa \rho c_p T_r}{T} \nabla T \cdot \nabla z_r \frac{\partial \Upsilon_r}{\partial z_r} \mathrm{d}V = - \int_V K_{\mathrm{eff}} \rho c_{pr} \frac{\partial \Upsilon_r}{\partial z_r} \frac{\partial T_r}{\partial z_r} \mathrm{d}V$$

$$= \underbrace{\int_V \rho K_{\mathrm{eff}} \alpha_r g_0 \frac{\partial T_r}{\partial z_r} \mathrm{d}V}_{linear} - \underbrace{\int_V K_{\mathrm{eff}} \rho c_{pr} P_r' \frac{\partial}{\partial z_r} \left(\frac{\alpha_r}{\rho_r c_{pr}} \right) \frac{\partial T_r}{\partial z_r} \mathrm{d}V}_{nonlinear}. \tag{46}$$

Eq. (46) states that the change in background GPE due to turbulent mixing can be expressed as the sum of a 'linear' and 'nonlinear' terms, where the nonlinearity involved

stems from the nonlinearity of the equation of state. As a result, it is the linear term that is directly comparable with the Boussinesq expression Eq. (42). To clarify the link, note that one possible expression for the squared buoyancy frequency in a compressible fluid is: $N^2 = g_0 \rho \Gamma \partial \eta / \partial z = \alpha g_0 T / c_p \partial \eta / \partial z$, where $\Gamma = \alpha T / (\rho c_p)$ is the adiabatic lapse rate. By using the result that $d\eta = (c_p/T)dT - (\alpha/\rho)dP$, it follows that:

$$\alpha_r g_0 \frac{\partial T_r}{\partial z_r} = \alpha_r g_0 \left[\frac{T_r}{c_{pr}} \frac{\partial \eta_r}{\partial z_r} - \rho_r g_0 \Gamma_r \right] = N_r^2 - \frac{\alpha_r^2 g_0^2 T_r}{c_{pr}} = \left(1 - \frac{\alpha_r^2 g_0^2 T_r}{c_{pr} N_r^2} \right) N_r^2.$$

This shows, therefore, that it appears to be inaccurate to use N^2 in the Boussinesq expression Eq. (42), which appears to involve the pseudo squared buoyancy frequency $\alpha_r g_0 \partial T_r / \partial z_r$ instead. Note also that the examination of the nonlinear term indicates that a linear equation of state implies that the parameter $\alpha/(\rho c_p)$ be treated as constant, not just α. This will be the case if ρ and c_p are also treated as constant.

3.4. Link between background GPE changes and APE dissipation

As shown by [21], the case of a Boussinesq fluid with a linear equation of state has the interesting property that the net increase in background GPE to be directly linked to the rate of APE dissipation as follows:

$$\left. \frac{d\text{GPE}_r}{dt} \right|_{mixing} = D_{APE} + \left. \frac{d\text{GPE}_r}{dt} \right|_{laminar} \tag{47}$$

where the laminar contribution refers to the background rate of background GPE increase that would occur in absence of turbulence. It is such a property that led [21] to (erroneously) assume that turbulent mixing causes the irreversible conversion of AGPE into background GPE, as illustrated in the left panel of Fig. 2. This interpretation was challenged by [13], who showed that in general the net increase in background GPE is not as simply linked to D_{APE} in the general case of a compressible fluid with a nonlinear equation of state.

In [13], the differences between the change in background GPE and APE dissipation were only investigated by means of numerical experiments. Here, we seek to clarify these differences by seeking a clearer analytical expression for D_{APE}, which we had not been able to obtain in [13]. To make analytical progress, what is needed is an explicit understanding of the differences between the in-situ and background temperature gradients ∇T and ∇T_r, in order to understand the physical parameters controlling the term $\nabla(T/T_r)$ entering the definition of APE dissipation. To that end, we regard specific entropy as a function of temperature and pressure, and use the following well known relationships:

$$d\eta = \frac{c_p}{T}dT - \frac{\alpha}{\rho}dP \qquad \rightarrow \qquad dT = \frac{T}{c_p}d\eta + \Gamma dP$$

where $\Gamma = \alpha T / (\rho c_p)$ is the *adiabatic lapse rate*. This in turn implies:

$$\nabla T = \frac{T}{c_p}\nabla \eta + \Gamma \nabla P, \tag{48}$$

$$\nabla T_R = \left[\frac{T_R}{c_{pr}} + \Gamma_R \frac{dP_R}{d\eta}\right]\nabla\eta = \frac{T_r}{c_{pr}}\nabla\eta - \rho_r g_0 \Gamma_r \nabla z_r, \tag{49}$$

by recalling the fact that by definition, the reference temperature profile $T_r = T(\eta, P_r(\eta, t))$ can be regarded as a function of entropy at each particular time t, as well as by using the results that $dP_r/d\eta = dP_r/dz_r \times (d\eta_r/dz_r)^{-1}$ and hence that $(d\eta_r/dz_r)^{-1}\nabla\eta(d\eta/dz_r)^{-1}\nabla\eta_r(z_r) = \nabla z_r$. From Eqs. (48) and (49), it follows that $\nabla(T/T_r)$ can be written as:

$$\nabla\left(\frac{T_r}{T}\right) = \frac{\nabla T_r}{T} - \frac{T_r}{T^2}\nabla T = \frac{1}{T}\left[\frac{T_r}{c_{pr}} + \Gamma_r\frac{dP_r}{d\eta}\right]\nabla\eta - \frac{T_r}{T^2}\left[\frac{T}{c_p}\nabla\eta + \Gamma\nabla P\right]$$

$$= \frac{T_r}{T}\left(\frac{c_p - c_{pr}}{c_p c_{pr}}\right)\nabla\eta + \frac{\Gamma_r}{T}\frac{dP_r}{d\eta}\nabla\eta - \frac{\Gamma T_r}{T^2}\nabla P$$

$$= \frac{T_r}{T}\left(\frac{c_p - c_{pr}}{c_p c_{pr}}\right)\nabla\eta - \frac{\alpha_r T_r g_0}{c_{pr}T}\nabla z_r - \frac{\Gamma T_r}{T^2}\nabla P \tag{50}$$

As seen previously, the APE dissipation is made up of a diffusive part as well as a small viscous part. Here, we decompose the diffusive part as the sum of three terms:

$$D_{APE}^{\text{diffusive}} = -\int_V \kappa\rho c_p \nabla T \cdot \nabla\left(\frac{T_r}{T}\right)\,\mathrm{d}V = D_{APE}^{(1)} + D_{APE}^{(2)} + D_{APE}^{(3)}, \tag{51}$$

where all the components $D_{APE}^{(i)}$ are such that they individually vanish when the actual state coincides with the reference state. These are defined by:

$$D_{APE}^{(1)} = \int_V \kappa\rho c_p \frac{\alpha_r T_r g_0}{c_{pr}T}\nabla T \cdot \nabla z_r\,\mathrm{d}V + \int_V \frac{\kappa\alpha T_r}{T}\nabla T \cdot \nabla P_r(z, t)\,\mathrm{d}V$$

$$= \int_V \rho K_{\text{eff}}\alpha_r g_0 \frac{\partial T_r}{\partial z_r}\,\mathrm{d}V - \int_V \frac{\kappa\alpha\rho_r g_0 T_r}{T}\frac{\partial T}{\partial z}\,\mathrm{d}V,$$

$$D_{APE}^{(2)} = \int_V \kappa\rho c_p \frac{T_r}{T}\left(\frac{c_{pr} - c_p}{c_p c_{pr}}\right)\nabla T \cdot \nabla\eta\,\mathrm{d}V = \int_V \left(\frac{c_{pr} - c_p}{c_p}\right)\rho K_{\text{eff}}\frac{\partial T_r}{\partial z_r}\frac{\partial\eta_r}{\partial z_r}\,\mathrm{d}V,$$

$$D_{APE}^{(3)} = \int_V \frac{\kappa\alpha T_r}{T}\nabla T \cdot \nabla(P - P_r(z, t))\,\mathrm{d}V.$$

For a fluid of limited vertical extent typical, $|(c_{pr} - c_p)/c_{pr}| \ll 1$; also, because of closeness to hydrostatic equilibrium, $P \approx P_r(z, t)$, so that it seems clear that both $D_{APE}^{(2)}$ and $D_{APE}^{(3)}$ must be small in general, and hence that the APE dissipation is dominated by the first term, i.e., $D_{APE} \approx D_{APE}^{(1)}$. Interestingly, the latter is very close to the expression derived in the context of a fluid with a linear equation of state by [21], to which it reduces to in the limit $T = T_r$ and $\rho = \rho_r$. Unlike the expression Eq. (46) for the change in background GPE, the expression for D_{APE} is seen to be largely unaffected by the nonlinearities of the equation of state, which only affect the above expressions by making $T - T_r$ as well as $c_p - c_{pr}$ differ from zero. The conclusion that D_{APE} is largely insensitive to the nonlinearities of the equation of state was previously arrived at in [14] by means of numerical experimentation, which is confirmed here analytically.

Based on the structure of the $D_{APE}^{(i)}$, it is clear that when the flow is turbulent, D_{APE} is controlled at leading order by:

$$D_{APE} \approx \int_V \rho K_{\text{eff}} \alpha_r g_0 \frac{\partial T_r}{\partial z_r} \, \mathrm{d}V \geq 0, \tag{52}$$

which is identical to the linear term controlling the rate of change of the background GPE in Eq. (46). We can clarify the difference between the net change in background GPE and APE dissipation by writing down the difference $B_r' - D_{APE}$, viz.,

$$B_r' - D_{APE} = -\int_V K_{\text{eff}} \rho c_{pr} P_r' \frac{\partial}{\partial z_r} \left(\frac{\alpha_r}{\rho_r c_{pr}} \right) \frac{\partial T_r}{\partial z_r} \, \mathrm{d}V$$

$$- \int_V \left(\frac{c_{pr} - c_p}{c_p} \right) \rho K_{\text{eff}} \frac{\partial T_r}{\partial z_r} \frac{\partial \eta_r}{\partial z_r} \, \mathrm{d}V - \int_V \frac{\kappa \alpha T_r}{T} \nabla T \cdot \nabla P \, \mathrm{d}V. \tag{53}$$

For a linear equation of state $\alpha/(\rho c_p) = $ constant, we see that B_r' and D_{APE} nearly cancel out. For a nonlinear equation of state, however, we see that this is no longer the case. Interestingly, we recover the result of [14] arguing that D_{APE}, unlike B_r', is very little affected by the nonlinearities of the equation of state. We concluded in [14] that the dissipation ration $\varepsilon_P/\varepsilon_K$ (which is a measure of the mixing efficiency, e.g., [7]), was largely insensitive to the nonlinearities of the equation of state. Although the leading order term Eq. (52) is positive definite, the other contributions to D_{APE} do not appear to be sign definite, so that although we concluded in an earlier paragraph that D_{APE} is an irreversible conversion, it does not appear possible from the expressions $D_{APE}^{(i)}$ to ascertain that the exact expression for D_{APE} is also necessarily positive definite in all circumstances. Note that this does not violate the second law of thermodynamics, because the irreversible production of entropy remains positive definite even in the occasional instances where ε_P would be negative.

3.5. Link between turbulent effective diffusivity and APE dissipation

In the classical theory of stratified turbulence, the turbulent diffusivity is traditionally defined in the two equivalent ways:

$$K_v = \frac{\langle \|\nabla \theta'\|^2 \rangle}{(d\bar{\theta}/dz)^2} = \frac{\varepsilon_P}{N^2}, \tag{54}$$

that is either in terms of the dissipation of temperature variance $\chi_T = \|\nabla \theta'\|^2$ or in terms of the APE dissipation rate ε_P. For a Boussinesq fluid with a linear equation of state, these two approaches are strictly equivalent, as ε_P is equal to χ_T up to a multiplicative constant involve α and g_0. Here, note that the turbulent effective diffusivity $K_{\text{eff}} \to \kappa$ when the actual state collapses to the reference state. Since ε_P vanishes when the actual state coincides with the reference state, we may try to define an ε_P effective diffusivity from:

$$K_{\varepsilon_p} = \frac{\varepsilon_P}{\alpha_r g_0 \partial T_r/\partial z_r}. \tag{55}$$

By using the integrands for the different $D_{APE}^{(i)}$, we have:

$$K_{\varepsilon_p} = K_{\text{eff}} + \frac{K_{\text{eff}}}{\alpha_r g_0} \left(\frac{c_{pr} - c_p}{c_p} \right) \frac{\partial \eta_r}{\partial z_r} + \frac{\kappa \alpha T_r}{\rho g_0 \alpha_r T} \frac{\nabla T \cdot \nabla P}{\partial T_r / \partial z_r} \neq K_{\text{eff}} - \kappa. \quad (56)$$

As far as exact results are concerned, it appears from the above calculations that the χ_T and ϵ_P definition of turbulent diapycnal diffusivity are no longer simply related for a compressible fluid with a nonlinear equation of state, unless we overlooked a non-straightforward link between the two approaches. Based on our current understanding of the problem, we believe that the entropy production-based approach seems to be the most natural and and fundamental to define a turbulent effective diffusivity.

4. Reversible and irreversible pressure work, and the role of the divergent velocity field

Our result that the APE dissipation should be regarded as an irreversible energy conversion that is part of the thermodynamic work of expansion/contraction has important implications for our understanding of the role of the divergent velocity field, which appears to be important even in the context of Boussinesq fluids. To that end, we now discuss the issue of how to isolate the reversible and irreversible components of the pressure work, which is the second term in the right-hand side of the following evolution equation for the kinetic energy:

$$\rho \frac{D}{Dt} \frac{\mathbf{v}^2}{2} = -\rho \frac{D\Phi}{Dt} - \mathbf{v} \cdot \nabla P + \mathbf{v} \cdot \nabla \cdot \mathbf{S}. \quad (57)$$

As is well known, the pressure work is one out of three effects that can alter the kinetic energy, the two others being conversion with gravitational potential energy and work against viscous stresses. In order to understand the reversible and irreversible effects controlling the pressure work, let us first recall that the classical approach to decomposing the viscous work in such a way is by decomposing it as follows:

$$\mathbf{v} \cdot \nabla \cdot \mathbf{S} = \nabla \cdot \mathbf{F}_{ke} - \rho \varepsilon_K \quad (58)$$

where the components F_j of $\mathbf{F}_{ke}$ are given by $F_j = v_i S_{ij}$. From a thermodynamic viewpoint, the term $\nabla \cdot \mathbf{F}_{ke}$ is *reversible* for it is the one that allows the external mechanical forcing (such as boundary stress) to be communicated within the interior of the fluid, in contrast to the irreversible viscous dissipation term $\rho \varepsilon_K$, which is associated with the local destruction of kinetic energy with a corresponding increase in entropy production. We now turn to the pressure work term, which is often written as:

$$-\mathbf{v} \cdot \nabla P = -\nabla \cdot (P\mathbf{v}) + P \nabla \cdot \mathbf{v}_d = -\nabla \cdot (P\mathbf{v}) + \rho P \frac{Dv}{Dt} \quad (59)$$

where $\mathbf{v}_d$ is the divergent part of the velocity field. From a thermodynamic viewpoint, the term $\nabla \cdot (P\mathbf{v})$ is reversible, for it is responsible for transmitting the information about the externally applied pressure within the interior of the fluid, while the reversible or irreversible character of the work of expansion/contraction PDv/Dt cannot be immediately determined and hence needs further discussion. Note that the work of

expansion/contraction is neglected in the traditional incompressible approximation, which is equivalent to assuming that at leading order, only the reversible part of the pressure work is important and that only viscous effects contribute to the irreversible dissipation of horizontal momentum. It is important to realize, however, that from a thermodynamic viewpoint, the divergent velocity field is the only source of irreversibility in the pressure work term, so that its contribution cannot be discarded solely on the basis of its assumed smallness, given that it can potentially cause additional dissipation of horizonal momentum – an important point in the context of turbulent parameterisations of momentum dissipation.

In order to elucidate the reversible and irreversible aspects of the work of expansion/contraction, let us regard the specific volume v as a function of specific entropy η and pressure P, so that we can write:

$$\rho P \frac{Dv}{Dt} = \rho P \left[\frac{\partial v}{\partial \eta} \frac{D\eta}{Dt} + \frac{\partial v}{\partial P} \frac{DP}{Dt} \right]$$

where $\partial v / \partial \eta = \alpha T/(\rho c_p) = \Gamma$ is the adiabatic lapse rate and $\partial v/\partial P = -1/(\rho^2 c_s^2)$, where c_s is the speed of sound. We first focus on the pressure term, which can be manipulated as follows:

$$\rho P v_P \frac{DP}{Dt} = \rho \frac{D}{Dt} \int_{P_r}^{P} \tilde{P} v_P(\eta, \tilde{P}) \, d\tilde{P} + \rho \left\{ P_r v_{P\eta}(\eta, P_r) \frac{DP_r}{Dt} - \int_{P_r}^{P} \tilde{P} v_{\eta P}(\eta, \tilde{P}) \, d\tilde{P} \frac{D\eta}{Dt} \right\}$$

$$= -\rho \frac{D\text{aee}}{Dt} + \rho P_r v_{P\eta}(\eta, P_r) \frac{DP_r}{Dt} - \rho \left\{ [P\Gamma - P_r \Gamma_r] - \int_{P_r}^{P} \Gamma(\eta, \tilde{P}) d\tilde{P} \right\} \frac{D\eta}{Dt}$$

where the thermodynamic quantity aee is defined by:

$$\text{aae}(\eta, P; P_r) = -\int_{P_r}^{P} P' \frac{\partial v}{\partial P}(\eta, \tilde{P}) d\tilde{P} = \int_{P_r}^{P} \frac{\tilde{P}}{\rho^2 c_s^2} d\tilde{P}.$$

By noting that the background work of expansion $P_r Dv_r/Dt$ and temperature $T - T_r$ can be written as follows:

$$B_r = \rho P_r \frac{Dv_r}{Dt} = \rho P_r \left[\Gamma_r \frac{D\eta}{Dt} + \frac{\partial v}{\partial \eta \partial P}(\eta, P_r) \frac{DP_r}{Dt} \right],$$

$$\int_{P_r}^{P} \Gamma(\eta, P') \, dP' = T - T_r,$$

it follows that the work of expansion/contraction can eventually be written in the following form:

$$\rho P \frac{Dv}{Dt} = -\rho \frac{D\text{aee}}{Dt} + \rho P_r \frac{Dv_r}{Dt} + (T - T_r)\rho \frac{D\eta}{Dt}$$

$$= -\rho \frac{D\text{aee}}{Dt} + \rho P_r' \frac{Dv_r}{Dt} + \rho_0 P_0 \frac{Dv_r}{Dt} + \nabla \cdot \left[\left(\frac{T - T_r}{T} \right) \kappa \rho c_p \nabla T \right] - \rho \varepsilon_P. \quad (60)$$

Eq. (60) is a key result of this paper, which makes it clear that the APE dissipation rate ε_p can be regarded to be part of the overall expression for the thermodynamic work of expansion/contraction. Integrating Eq. (60) thus yields the following expression:

$$B = \int_V \rho P \frac{Dv}{Dt} \, dV = -\frac{d}{dt}\text{AEE} + \frac{d}{dt}\text{GPE}_r + P_0 \frac{d}{dt}V_{ol,r} - D_{APE}. \tag{61}$$

Eq. (61) makes it clearer that B is made up of mostly reversible components, except for the irreversible APE dissipation term. Note that integrating Eq. (61) over a turbulent mixing event allows one to recover our earlier result that:

$$\overline{B}' = \Delta GPE_r - \overline{D}_{APE} = \overline{B}'_r - \overline{D}_{APE}, \tag{62}$$

which we obtained previously from the consideration of the APE and AGPE budget, and which we discussed in relation to Eq. (53). Eqs. (60) and (61) are important new results in the context of the thermodynamic understanding of turbulent fluids, given that as far as we are aware, exact fundamental results on the work of expansion/contraction have been lacking. The present results greatly improve on the results by [15], who attempted the first discussion of how to split B into reversible and irreversible components.

5. Summary of results and conclusions

5.1. Summary of results

In this paper, we re-examined and further extended several recent results by [13], with the primary aim of elucidating the nature of "APE dissipation", which although a key quantity in the theory of stratified turbulence, has nevertheless remained quite obscure until now. Our approach relies on the use of [5]'s theory of available potential energy, which enables an exact separation of the adiabatic process of stirring from that of irreversible mixing, as first demonstrated by [21]. As in [13], the approach was extended here to a fully compressible fluid with an arbitrary nonlinear equation of state in pressure and temperature, for which we showed that the exact definition of ε_P should be:

$$\varepsilon_P = \kappa c_p \nabla T \cdot \nabla \left(\frac{T - T_r}{T}\right) - \left(\frac{T - T_r}{T}\right)\varepsilon_K, \tag{63}$$

where $T_r = T(\eta, P_r(\eta, t))$ is the reference temperature profile, a function of entropy (and time when diabatic effects modify the reference profile). The definition Eq. (63) is local in space and time; by expressing T and T_r in terms of pressure and entropy, an alternative expression for the diffusive part of ε_P is:

$$\varepsilon_P^{\text{diff}} = \kappa c_p \nabla T \cdot \left[\frac{T_r}{T}\left(\frac{c_{pr} - c_p}{c_p}\right) + \frac{\alpha_r T_r g_0}{c_{pr} T}\nabla z_r + \frac{\Gamma T_r}{T}\nabla P\right], \tag{64}$$

where $\Gamma = \alpha T/(\rho c_p)$ is the adiabatic lapse rate. Both the diffusive and viscous parts of ε_P vanish when the actual state coincides with the laminar reference state, so that to some degree, ε_P can be regarded as a "turbulent" quantity.

The introduction of [5]'s reference state of minimum potential energy allows one to show that averaged over a turbulent mixing event, the density flux $\overline{W}$ represents

the net change in background GPE, which can be expressed in terms of the work of expansion/contraction in the reference state, so that $\overline{W} = \overline{W}_r = \overline{B}'_r$. As a result, it is possible to show that averaged over a turbulent mixing event, we have:

$$\overline{D}_{APE} = \overline{B}_r - \overline{B} \tag{65}$$

which states that the APE dissipation is the part of the work of expansion/contraction that is not associated with modifying the background GPE. We showed that it is possible to rewrite the irreversible entropy production as follows:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = \int_V \frac{\kappa \rho c_p \nabla T \cdot \nabla T_r}{T T_r} \mathrm{d}V + \int_V \frac{\rho(\varepsilon_P + \varepsilon_K)}{T_r} \, \mathrm{d}V, \tag{66}$$

which demonstrates that ε_P should be interpreted as an irreversible conversion of kinetic energy into internal energy, similarly as the viscous dissipation ε_K, and hence that molecular diffusion contributes to the irreversible destruction of kinetic energy. Alternatively, the result means that the integrated expansion work $\overline{B}$ can be written as $\overline{B} = \overline{B}_{reversible} + \overline{B}_{irreversible}$, where $\overline{B}_{reversible} = \overline{B}_r$ and $\overline{B}_{irreversible} = -\overline{D}_{APE}$, where the reversible character of B_r comes from the fact that $B_r = d\mathrm{IE}_r/dt$ is an exact differential, and that although it is controlled by molecular diffusion, it has no signature in the irreversible entropy production.

Eq. (66) expresses the irreversible entropy production as the sum of 'heat' and 'work' terms, where the work term is the one proportional to the sum $\varepsilon_P + \varepsilon_K$, which measures the amount of kinetic energy dissipated by irreversible processes. The heat term can be used to locally define a turbulent effective diffusivity as follows:

$$K_{\mathrm{eff}} = \kappa \frac{c_p T_r}{c_{pr} T} \left(\frac{\partial T_r}{\partial z_r}\right)^{-2} \nabla T \cdot \nabla T_r = \kappa \frac{c_p T_r}{c_{pr} T} \left(\frac{\partial T_r}{\partial z_r}\right)^{-1} \nabla T \cdot \nabla z_r, \tag{67}$$

which is the generalisation of the Osborn-Cox model for the diapycnal diffusivity to a compressible stratified fluid. This makes it possible to write the 'heat' component of irreversible entropy production as follows:

$$\frac{d\Sigma}{dt}\bigg|_{heat} = \int_V \rho c_{pr} K_{\mathrm{eff}} T_r^2 \left(\frac{\partial T_r}{\partial z_r}\right)^2 \mathrm{d}V. \tag{68}$$

This definition is associated with a turbulent heat flux

$$F_{turb} = -\rho K_{\mathrm{eff}} c_{pr} \frac{\partial T_r}{\partial z_r}, \qquad \dot{Q}_{turb} = -\frac{\partial F_{turb}}{\partial z_r} \tag{69}$$

which allows to write the heat part of the irreversible entropy production as:

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t}\bigg|_{heat} = \int_V \frac{\dot{Q}_{turb}}{T_r} \, \mathrm{d}V = \int_V \frac{1}{T_r} \frac{\partial F_{turb}}{\partial z_r} \, \mathrm{d}V. \tag{70}$$

The definition of the turbulent effective diffusivity allows one to write the change in background GPE as the sum of a Boussinesq, non-Boussinesq and Joule heating terms that are respectively defined by:

$$\frac{\mathrm{dGPE}_r}{\mathrm{d}t}\bigg|_{bouss} = \int_V \rho K_{\mathrm{eff}} \alpha_r g_0 \frac{\partial T_r}{\partial z_r} \, \mathrm{d}V, \tag{71}$$

$$\frac{\mathrm{dGPE}_r}{\mathrm{d}t}\bigg|_{non-bouss} = \int_V \rho K_{\mathrm{eff}} P_r' \frac{\partial}{\partial z_r}\left(\frac{\alpha_r}{\rho_r c_{pr}}\right)\frac{\partial T_r}{\partial z_r}\,\mathrm{d}V, \tag{72}$$

$$\frac{d\mathrm{GPE}_r}{\mathrm{d}t}\bigg|_{joule} = \int_V \Upsilon_r \rho\left(\varepsilon_P + \varepsilon_K\right)\,\mathrm{d}V. \tag{73}$$

Likewise, it is also possible to rewrite the local APE dissipation rate in terms of the effective diffusivity as follows:

$$\varepsilon_P = K_{\mathrm{eff}}\alpha_r g_0 \frac{\partial T_r}{\partial z_r} + \left(\frac{c_{pr}-c_p}{c_p}\right)K_{\mathrm{eff}}\frac{\partial T_r}{\partial z_r}\frac{\partial \eta_r}{\partial z_r} + \frac{\kappa\alpha T_r}{\rho T}\nabla T \cdot \nabla P. \tag{74}$$

These results confirm that at leading order, the leading order terms determining the net change in background GPE and APE dissipation are equal for a fluid with a linear equation of state, as previously shown by [21]. The results also confirm that the formula for D_{APE} and the net change in background GPE are otherwise fundamentally different, as first shown by [13], and hence that it cannot be correct to regard turbulent mixing as converti ng the AGPE into background GPE, in contrast to what what suggested by [21] and illustrated in the left panel of Fig. 2. Instead, what appears to be the correct interpretation is illustrated in the right panel of Fig. 2.

5.2. Discussion and conclusions

In this paper, we fully elucidated the physical origin of the concept of APE dissipation, which although a key quantity in the theory of stratified turbulence, has remained quite mysterious so far. Specifically, we showed that the APE dissipation can be regarded as the irreversible part of the work of expansion/contraction, the reversible part of the latter being involved in transferring the internal energy to the background gravitational potential energy, and is therefore ultimately caused by the divergent velocity field. Although this may seem surprising at first, since until now the concept of APE dissipation has been defined and predicted accurately by the incompressible Boussinesq approximation, there is actually no contradiction because as shown by [17], the Boussinesq model actually admits a representation of compressible effects in the form of apparent changes in gravitational potential energy due to density changes.

This new interpretation of APE dissipation is important, because it implies that in the context of parallel shear flow instability, there is actually no irreversible transfer of horizontal momentum into vertical momentum, as seems to be necessary if the thermodynamic work of expansion/contraction is neglected altogether, which seems odd from the viewpoint of momentum conservation. In the new interpretation, the vertical momentum associated with the net vertical motion of the centre of gravity due to turbulent mixing is found to be entirely caused by the vertical thermodynamic forces caused by the departure of the vertical stratification from thermodynamic equilibrium, so that the net change in background GPE occurs solely at the expenses of internal energy, as is well known to be the case in the purely laminar case. Simultaneously, turbulent molecular diffusion enhances the magnitude of the divergent velocity field,

which appears to be the part of the velocity field causing the irreversible dissipation of the parallel shear flow kinetic energy into internal energy. We speculate that this new interpretation should have implication for how to parameterise the turbulent dissipation of momentum in the coarse grained Navier-Stokes equations, but further work is needed to understand how to do this precisely.

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