University of Reading

Cross-correlations and cross-bicorrelations in Sterling exchange rates

Article

Accepted Version

Brooks, C. ORCID: https://orcid.org/0000-0002-2668-1153 and Hinich, M. J. (1999) Cross-correlations and cross-bicorrelations in Sterling exchange rates. Journal of Empirical Finance, 6 (4). pp. 385-404. ISSN 0927-5398 doi: 10.1016/S0927-5398(99)00007-9 Available at https://centaur.reading.ac.uk/35967/

It is advisable to refer to the publisher's version if you intend to cite from the work. See Guidance on citing.
Published version at: http://dx.doi.org/10.1016/S0927-5398(99)00007-9
To link to this article DOI: http://dx.doi.org/10.1016/S0927-5398(99)00007-9

Publisher: Elsevier

All outputs in CentAUR are protected by Intellectual Property Rights law, including copyright law. Copyright and IPR is retained by the creators or other copyright holders. Terms and conditions for use of this material are defined in the End User Agreement.

www.reading.ac.uk/centaur

CentAUR

Central Archive at the University of Reading

Reading's research outputs online

NOTICE: this is the author's version of a work that was accepted for publication in research in the *Journal of Empirical Finance*. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in the *Journal of Empirical Finance*, 6.4 (1999), DOI: 10.1016/s0927-5398(99)00007-9

CROSS-CORRELATIONS AND CROSS-BICORRELATIONS IN STERLING EXCHANGE RATES

Chris Brooks[1], ISMA Centre, The University of Reading

and

Melvin J. Hinich, University of Texas at Austin

Abstract

This paper proposes two new tests for linear and nonlinear lead/lag relationships between time series based on the concepts of cross-correlations and cross-bicorrelations respectively. The tests are then applied to a set of Sterling-denominated exchange rates. Our analysis indicates that there existed periods during the post-Bretton Woods era where the temporal relationship between different exchange rates was strong, although these periods have become less frequent over the past twenty years. In particular, our results demonstrate the episodic nature of the nonlinearity, and have implications for the speed of flow of information between financial series. The method generalises recently proposed tests for nonlinearity to the multivariate context.

J.E.L. Classifications: C32, F31

Keywords: cross-correlations, cross-bicorrelations, exchange rates, nonlinearity

[1] Corresponding author, ISMA Centre, P.O. Box 242, The University of Reading, Whiteknights, Reading RG6 6BA, England, Tel. (+44) 1734 31 67 68; Fax: (+44) 1734 31 47 41; e-mail: C.Brooks@reading.ac.uk

1. Introduction

Researchers in economics and finance have been interested in testing for nonlinear dependence in time series for almost a decade now. Following relatively early work by Brock (1986), Hsieh (1989), and Scheinkman and LeBaron (1989a & 1989b), the number of applications has increased dramatically. There appear to be at least two reasons for the popularity of this line of research. First, if evidence of nonlinearity is found in the residuals from a linear model applied to a financial time series, this must cast doubt on the adequacy of the linear model as an adequate representation of the data. Second, if the nonlinearity is present in the conditional first moment, it may be possible to devise a trading strategy based on nonlinear models which is able to yield higher returns than a buy-and-hold rule.

The most popular portmanteau tests for nonlinearity employed have been the BDS test of Brock, Dechert and Scheinkman (1987), now published as Brock, Dechert, Scheinkman and LeBaron (1996), and the bispectrum test of Hinich (1982). The vast majority of researchers to use these tests have found strong evidence for nonlinearity (see Brock *et al.* (1991), and Brooks (1996) for surveys and applications), although the usefulness of nonlinear time series models for yielding superior predictions of asset returns is still undecided; see, Nachane and Ray (1993), Weigend and Gerschenfeld (1993), LeBaron (1993) etc. Although Baek and Brock (1992), Hiemstra and Jones (1994) and Gallant, Rossi and Tauchen (1994) provide contradictory results, the majority of studies to date examining the issue of nonlinearity have been entirely univariate in nature, considering each series in isolation. This is highly restrictive, since relationships between variables over time are clearly of importance.

There also exists a parallel literature which seeks to determine whether observed nonlinearities in financial time series are due to the existence of stochastic nonlinear relationships or fully deterministic (chaotic) dynamics. Although there is almost no evidence in favour of the latter; see Cecen and Erkal (1996a & 1996b), Ramsey, Sayers and Rothman, (1990) or Brooks (1998), it appears that most of the nonlinearity can be explained by reference to the GARCH family of models; for example see Baillie and Bollerslev (1989) and Hsieh, (1989a & 1989b).

This paper attempts to draw the two somewhat disparate areas of research into nonlinearity and multivariate time series analysis together proposing a new test for nonlinearity which allows for cross-bicorrelations between pairs of series. Tests of simple cross-correlations are also considered. These tests can be viewed as natural multivariate extensions of the Hinich (1996) portmanteau bicorrelation and whiteness statistics which search for nonlinear co-features between time series. The method is more general than the tests for common features that are proposed by Engle and Kozicki (1993), since no knowledge of the kind of dynamics purported to be present in the data is required to detect the dependence[2]. The present paper hopefully provides an additional tool to the nonlinear Granger causality tests employed in the t literature (by Baek and Brock (1992) and Hiemstra and Jones (1994). The test proposed in this paper is able to pick up any form of nonlinear dependence of the third order statistic between two series and might also help researchers to determine the functional form of the nonlinear relationship between the two series by determining in which directions the bicorrelations flow and which of the lags are significant .

The remainder of this paper is organised as follows. Section 2 outlines the testing methodology used; section 3 describes the data employed, while section 4 offers some analysis and concluding remarks.

2. Testing Methodology

Let the data be a sample of length N, from two jointly covariance stationary time series $\{x(t_k)\}$ and $\{y(t_k)\}$ which have been standardised to have a sample mean of zero and a sample variance of one by subtracting the sample mean and dividing by the sample standard deviation in each case. Since we are working with small sub-samples of the whole series, stationarity is not a stringent assumption. The null hypothesis for the test is that the two series are independent pure white noise processes, against an alternative that some cross-covariances, $C_{xy}(r) = \mathrm{E}[x(t_k)y(t_k+r)]$ or cross-bicovariances $C_{xxy}(r,s) = \mathrm{E}[x(t_k)x(t_k+r)y(t_k+s)]$ are non-zero. As a consequence of the invariance of $\mathrm{E}[x(t_1)x(t_2)y(t_3)]$ to permutations of (t_1,t_2), stationarity implies that the expected value is a function of two lags and that $C_{xxy}(-r,s) = C_{xxy}(r,s)$. If the maximum lag used is $L < N$, then the principal domain for the bicovariances is the rectangle $\{1 \le r \le L, -L \le s \le L\}$.

Under the null hypothesis that $\{x(t_k)\}$ and $\{y(t_k)\}$ are pure white noise, then $C_{xy}(r)$ and $C_{xxy}(r,s) = 0\ \forall\ r,s$ except when $r = s = 0$. This is also true for the less restrictive case when the two processes are merely uncorrelated, but the theorem given below to show that the test statistic is asymptotically normal requires independence between the two series. If there is second or third order lagged dependence between the two series, then , $C_{xy}(r)$ or $C_{xxy}(r,s) \neq 0$ for at least one r value or one pair of r and s values respectively. The following statistics give the r sample xy cross-correlation and the r,s sample xxy cross-bicorrelation respectively:

$$C_{xy}(r) = (N-r)^{-1} \sum_{t=1}^{N-r} x(t_k)y(t_k+r),\ r \neq 0 \tag{1}$$

and

$$C_{xxy}(r,s) = (N-m)^{-1} \sum_{t=1}^{N-m} x(t_k)x(t_k+r)y(t_k+s) \text{ where } m = \max(r,s). \tag{2}$$

[2] Although generality can be viewed as a virtue of a test, one might also reasonably argue that it reduces the test's power.

The cross-bicorrelation can be viewed as a correlation between the current value of one series and the value of previous cross-correlations between the two series. Note that the summation in the second order case (1) does not include contemporaneous terms, and is conducted on the residuals of an autoregressive fit to filter out the univariate autocorrelation structure so that contemporaneous correlations will not cause rejections. For the third order test, we estimate the test on the residuals of a bivariate vector autoregressive model containing a contemporaneous term in one of the equations. The motivation for this pre-whitening step is to remove any traces of linear correlation or cross-correlation so that any remaining dependence between the series must be of a nonlinear form. It can then be shown that

$$E[C_{xy}(r)] = 0 \tag{3}$$

$$E[C_{xxy}(r,s)] = 0 \tag{4}$$

$$E[C_{xy}^{2}(r)] = (N-r)^{-1} \tag{5}$$

$$E[C_{xxy}^{2}(r,s)] = (N-m)^{-1} \tag{6}$$

under the null hypothesis. Let $L=N^c$ where $0 < c < 0.5$ [3]. The test statistics for non-zero cross-correlations and cross-bicorrelations are given by

$$H_{xy}(N) = \sum_{r=1}^{L} (N-r)\ C_{xy}^{\ 2}(r) \tag{7}$$

and

$$H_{xxy}(N) = \sum_{s=-L}^{L}{}' \sum_{r=1}^{L} (N-m)\ C_{xxy}^{\ 2}(r,s), \quad ('-s \neq -1,1,0) \tag{8}$$

respectively. These tests are joint or composite tests for cross-correlations and cross-bicorrelations (in a similar vein to the Ljung-Box Q* test for autocorrelation), where the number of correlations tested for is L and the number of cross-bicorrelations tested for is $L(2L-1)$. We use theorem 1 from Hinich (1996), namely

[3] In this application, we use c=0.25, although the results and the null distribution of the test are not very sensitive to changes in this parameter.

Theorem 1: H_{xy} and H_{xxy} are asymptotically chi-squared with L and $L(2L-1)$ degrees of freedom respectively as $N \rightarrow \infty$,

which is proved in the appendix to Hinich (1996) for the univariate bicorrelation test statistic. An extension of this theorem to the multivariate test proposed in this paper, is presented in abbreviated form in the appendix to this article. The full version is available from the authors upon request.

3. The Data and Preliminaries

The analysis presented here is based on 5192 daily mid-price spot exchange rates of the Austrian schilling, the Danish krone, the French franc, the German mark, the Italian lira, the Japanese yen, and the U.S. dollar data, denominated against the UK pound. The sample period taken covers the whole of the post-Bretton Woods era, specifically from 2 January 1974 until 1 July 1994 inclusive. We analyse the differences of the log of the exchange rates, which can be interpreted as continuously compounded daily returns. The cross-correlations and cross-bicorrelations are examined via pair-wise comparisons between all combinations of two of the exchange rates from the set of seven (21 pairs). The three currencies with the largest world turnover[4] denominated against the pound are the U.S. dollar / pound (8.5% of average daily world turnover), the German mark / pound (4.9%), and the Japanese yen / pound (<1%). These three exchange rates are considered together with a number of less frequently traded European currencies[5] to consider whether these smaller-volume currencies returns follow those of the other European currencies, or whether they take their lead from the larger (mostly non-European) currencies.

---

[4]All figures quoted in this section refer to the year 1992, and are taken from *International Capital Movements and Foreign Exchange Markets: A Report to the Ministers and Governors by the Group of Deputies* Rome, 23 April, 1993.
[5]Excluding the German Mark / Pound, all other intra-EMS currency pairs make up only 7% of world daily average turnover.

The data are split into a set of 148 non-overlapping windows of length 35 observations (i.e. about 7 trading weeks). Samples of this size suggest a use of $L = 35^{0.25}$, which is rounded to 2. The reason for using many short windows is that potential arbitrage opportunities induced by non-contemporaneous cross-correlations or cross-bicorrelations are not likely to last long. Hence the use of long data series would probably yield very little[6], and hence nonlinearities which persist only for short periods of time would remain hidden. This is a major advantage of the testing approach used here relative to many of its competitors which require large volumes of data to have sufficient power, and which have poor small sample properties.

The results of a small Monte Carlo study to determine size of the test for samples of the length used here are given in table 1. Two series, each of length 35 are generated using a Gaussian, uniform or Student's *t* distribution with 5 or 10 degrees of freedom. The two series drawn from the same distribution are then tested for cross-correlations or cross-bicorrelations. This procedure is repeated 6000 times.

[table 1 here]

The results of the simulation clearly demonstrate that the tests are conservative at small samples for the uniform distribution, and the empirical sizes of the tests are close to their nominal values for the Gaussian data. The last two columns of table 1 also show the empirical size of the test when data are drawn from a *t*-distribution with 5 and 10 degrees of freedom respectively; these distributions are more likely to be representative of financial asset return series since they are fat-tailed. The simulation shows that the test is only modestly over-sized for the *t* with 5 degrees of freedom, and is appropriately sized for the slightly less fat-tailed distribution. Thus the test statistic is well behaved

[6]Indeed, an application of an identical procedure to that used here for the whole data series used as one single window gave no significant cross-correlations or cross-bicorrelations for any of the currencies, even at the 10% level.

with respect to the asymptotic theory, even for rather small samples. One should also be able to obtain similar results for *x* and *y* being drawn from different distributions (e.g. one set of Gaussian draws and one set of uniform), so long as the two were independent processes with finite first six moments .

4. Results

The *p*-values for the cross-correlations that are significant at the 1% level are shown in table 2 together with the dates of the windows in which this occurred.

[table 2 here]

These cross-correlation statistics are calculated on the residuals of an AR(3) fit to each series to filter out any linear autoregressive dependence[7]. Many significant test statistics are caused by contemporaneous cross-correlations, but there are also many that are not contemporaneous. The former is hardly surprising, and could be interpreted as arising from Sterling-related news which affected two bilateral exchange rates against sterling in a similar fashion. The lead / lag cross-correlations are, however, of considerably greater interest, and indicate that for some currencies, there may have been a degree of predictability at certain times over the past twenty years. For example, there was a correlation of 0.66 between the Austrian schilling / pound lagged two periods, and the and the German mark / pound, indicating that if the German mark rises one day during that period, we would have expected the Austrian Schilling to rise two trading days later. Many such relationships exist between the currencies, although there are many more cross-correlations between the intra-European currency pairs than between pairs containing the Japanese Yen or U.S. Dollar.

[7] The test is asymptotically invariant to linear filtering, and so may be applied to the residuals of a linear model, or to the raw data. It is important that linear dependence in the data is removed, for its presence could lead to spurious rejections of the null hypothesis. In theory, it would also be possible to apply the tests to the residuals from a nonlinear model - for example, the MA(1)-GARCH(1,1) model is often used to summarise the first two moments of financial returns. However, such a step is unnecessary with the correlation and bicorrelation tests since the presence or otherwise of ARCH-effects will not cause a rejection of the null hypotheses. This arises

The number and percentage of significant cross-bicorrelation windows for each pair of exchange rates are given in column 2 of table 3.

[table 3 here]

The results for cross-bicorrelations outlined in the ensuing analysis are estimated on the residuals of a bi-variate vector autoregression of order 3 in each equation (a BVAR(3,3)). The proportion of significant cross-bicorrelation windows is much larger than the nominal 1% threshold used, indicating that significant nonlinear lead/lag relationships existed between currencies. Correlations between the values of the *xxy* and *yyx* statistics are given in column 3 of table 3. On the whole, they show a very high degree of correlation, indicating that the nonlinear relationships may be bi-directional. The correlation between the values of the cross-correlation (*xy*) and the cross bicorrelation (*xxy*) statistics are much lower, however, indicating that linear and nonlinear relationships between the series need not occur at the same time. A more detailed analysis of the significant cross-bicorrelation is given in table 4.

[table 4 here]

Only bicorrelations with *xxy* or *yyx* values that are greater than 0.5 in absolute value are shown in table 4 due to space constraints, so that we concentrate on only the very largest bicorrelations. It is evident that there are many more significant cross-bicorrelations than cross-correlations, although the former are much more difficult to interpret. The majority of the significant cross-bicorrelations occur for the smaller-volume European exchange rates, particularly the Austrian Schilling / Pound and the Italian Lira / Pound. The *p*-values associated with the test statistics are typically much smaller than would be generated by a fat-tailed distribution if the data were iid (such as those given in the Monte Carlo study outlined above). It is also evident that there are more significant cross-bicorrelations during the earlier part of the series. The significant windows appear to occur in clusters; the most

---

from the fact that the tests are effectively tests for cross-relationships in the conditional mean rather than in the conditional variance.

recent prolonged period of dependence was during late 1992, around the time of Sterling's departure from the European Exchange Rate Mechanism (ERM).

A recent paper by Karolyi and Stulz (1996), has shown that cross-correlations between the shares of U.S. and Japanese companies trading in the U.S. are not significantly affected by macroeconomic announcements, or interest rate shocks. They show that co-movements between the series are high when the individual markets are volatile, or when "the markets move a lot" (p984). The cross-correlation framework proposed here provides a natural testing ground for this conjecture. If the markets do indeed move closely together, this will imply that the cross-correlation and cross-bicorrelation statistics (the latter being calculated after pre-whitening using a VAR), should have small values when the individual variances of the series are high. In other words, we would expect *Corr(xy,VarX), Corr(xy,VarY), Corr(xxy,VarX), Corr(xxy,VarY), Corr(yyx,VarX), Corr(yyx,VarY)* to be negative and fairly large. The results of table 5 show, however, that this hypothesis is not borne out, with no strong relationship (either positive or negative) between the test statistics and the variances, except in the case of the Danish Krone / U.S. Dollar, where the simple cross-correlation statistics are negatively correlated with the individual variances. These results contrast with those of Karolyi and Stulz (1996), where comovements and variances did tend to be positively related. However, Karolyi and Stulz considered only linear cross-correlations, and they examined stock returns rather than exchange rates.

[insert table 5 here]

Our findings have important implications for the ability of investors to internationally diversify portfolios, since strong contemporaneous co-movements between series coupled with high individual variances imply that fewer apparently country-specific risks are internationally diversifiable, so that the riskiness of the portfolio overall increases. This issue is becoming increasingly important following increases in capital mobility and the openness of trade. Also, countries which are part of the

European Exchange Rate Mechanism (ERM) co-ordinating fiscal and monetary policies more closely in order to meet the "convergence criteria" for forming a single currency mans that the correlations between currencies within Europe are likely to become stronger over the next few years.

5. Conclusions

In this paper, we have examined a new approach to testing for nonlinear interactions between series, and we have illustrated the method on a set of exchange rates. The method provides a complement to Granger causality analysis, and is general enough to detect many types of nonlinear dependence between series in their conditional means. We find a much larger number of significant cross-correlations and cross-bicorrelations than one would expect if the data were generated by independent white noise processes. Moreover, this type of structure cannot be generated by one of the GARCH family of models, so long as the GARCH model is a Martingale difference sequence. A Martingale difference has zero bicorrelations except for $E[x(t)x(t+r)y(t+r)]$, which is not included in the sum for the bicorrelation statistics. Therefore GARCH models should give rise to third order statistics that are not significantly different from zero.

The episodic nature of the observed linear and nonlinear co-dependence should be noted. We find that, in common with Ramsey and Zhang's (1997) analysis of the univariate case, multivariate activity in financial markets are relatively short-lived and surrounded by longer periods of apparent randomness. It is, perhaps, also not surprising that the cross-correlations and cross-bicorrelations all feature a small-volume European exchange rate on at least one side, and that there is little dependence between, for example, the Japanese Yen and the U.S. Dollar. The currencies which are less frequently traded and which are likely to be less closely scrutinised by dealers, are also likely to be slower to respond to new information. So, for example, the return on one of these currencies today may still be reflecting information that was fully incorporated into the "bigger" currencies yesterday. Thus the return of the “smaller” exchange rate today will be correlated with the return of the larger exchange rate yesterday. This will manifest itself as a non-zero cross-correlation or cross-bicorrelation, for the relationship between the two need not necessarily be linear. This argument was first suggested by Fisher (1966), to explain serial correlation in stock market indices and portfolios containing the stocks of small firms (see Perry, 1985 or Chelley-Steeley and Steeley, 1995, for more recent applications of this logic). This argument has been played down in much of the recent literature, which argues that the effects of this phenomenon will be small for data sampled at daily or lower frequencies. Boudoukh *et al*. (1994), however, argue that this nonsynchronous trading effect has been understated in the literature, and that most of the apparent predictability observed by, for example, Cohen *et al.* (1986), can be explained by this effect.

The dependencies observed in this paper must, by definition, be present for more than a few days to be detected. Hence we conjecture that the observed level of cross-correlation and cross-bicorrelation between currencies cannot be entirely attributed to nonsynchronous trading, and their existence must be considered evidence inconsistent with the weak form of the efficient markets hypothesis. Although further research is required to determine whether profitable trading strategies could be developed from this analysis, and building an appropriate multivariate nonlinear model of the switching type is not a simple task, our results are encouraging, and suggest that further investigation is worth while. The cross-bicorrelation test is, however, suggestive of an appropriate functional form for a nonlinear model since the cross-bicorrelation is essentially a test of $E[x(t)x(t+r)y(t+s)]$. If we restrict ourselves to consider the case where r and s are negative, then one might be able to predict $x(t)$ on the basis of the lags of $x(t+r)y(t+s)$; a brief description of one method of implementing such models is given in Brooks and Hinich (1998).

Acknowledgements

The authors are grateful to three anonymous referees and an editor of this journal for comments on an earlier version of this paper. The usual disclaimer applies.

References

Baek, E. and W. Brock, 1992, A Nonparameteric Test for Independence of a Multivariate Time Series, Statistica Sinica 2, 137-156

Baillie, R.T. and Bollerslev, T. (1989) The Message in Daily Exchange Rates: A Conditional-Variance Tale *Journal of Business and Economic Statistics* 7(3), 297-305

Boudoukh, J., M.P. Richardson, and R.F. Whitelaw ,1994, A Tale of Three Schools: Insights on Short-Horizon Stock Returns, The review of Financial Studies 7, 539-573

Brillinger, D. ,1975, Time Series, Data Analysis and Theory, Holt, Rinehard and Winston, New York

Brock, W.A. ,1986, Distinguishing Random and Deterministic Systems: Abridged Version, Journal of Economic Theory 40, 168-195

Brock, W.A., W.D. Dechert, J.A. Scheinkman ,1987, A Test for Independence Based on the Correlation Dimension, Mimeo. Department of Economics, University of Wisconsin at Madison, and University of Houston

Brock, W.A., W.D. Dechert, J.A. Scheinkman and B. LeBaron, 1996, A Test for Independence Based on the Correlation Dimension, Econometric Reviews 15, 197-235

Brock, W.A., D.A. Hsieh and B. LeBaron ,1991, Nonlinear Dynamics, Chaos, and Instability: Statistical Theory and Economic Evidence, M.I.T. Press, Reading, Mass.

Brooks, C., 1998, Forecasting Stock Return Volatility: Does Volume Help?, Journal of Forecasting 17, 59-80

Brooks, C. ,1996, Testing for Nonlinearities in Daily Sterling Exchange Rates, Applied Financial Economics 6, 307-317 .

Brooks, C. and M.J. Hinich, 1998, Forecasting High Frequency Exchange Rates Using Cross Bicorrelations forthcoming as *Proceedings of Computational Finance Conference* London Business School

Cecen, A.A. and C. Erkal, 1996a, Distinguishing between Stochastic and Deterministic Behaviour in Foreign Exchange Rate Returns: Further Evidence, Economics Letters 51, 323-329.

Cecen, A.A. and C. Erkal, 1996b, Distinguishing between Stochastic and Deterministic Behaviour in High Frequency Foreign Exchange Rate Returns: Can Nonlinear Dynamics Help Forecasting, International Journal of Forecasting 12, 465-473

Chelley-Steeley, P.L. and J.M. Steeley ,1995, Conditional Volatility and Firm Size: An Empirical Analysis of U.K. Equity Portfolios, Applied Financial Economics 5, 433-440

Cohen, K., et al. ,1986, The Microstructure of Securities Markets, Prentice-Hall, Englewood Cliffs, N.J.

Engle, R.F. and S. Kozicki, 1993, Testing for Common Features Journal of Business and Economic Statistics 11, 369-380

Fisher, L. ,1966, Some New Stock Market Indices, Journal of Business 39, 191-225

Gallant, AR., P.E. Rossi, and G. Tauchen, 1993, Nonlinear Dynamic Structures, Econometrica, 61, 871-907

Hiemstra, C. and J.D. Jones, 1994, Testing for Linear and Nonlinear Granger Causality in the Stock Price-Volume Relation, Journal of Finance 49, 1639-1664

Hinich, M.J. ,1996, Testing for Dependence in the Input to a Linear Time Series Model, Journal of Nonparametric Statistics 6, 205-221

Hinich, M.J. ,1982, Testing for Gaussianity and Linearity of a Stationary Time Series, Journal of Time Series Analysis 3, 169-176

Hsieh, D.A. ,1989a, Testing For Nonlinear Dependence in Daily Foreign Exchange Rates, Journal of Business 62, 339-368

Hsieh, D.A., 1989b Modelling Heteroscedasticity in Daily Foreign Exchange Rates, Journal of Business and Economic Statistics 7, 307-317

Karolyi, G.A., and R.M. Stulz ,1996, Why Do Markets Move Together? An Investigation of U.S.-Japan Stock Return Comovements, Journal of Finance 51, 951-986

LeBaron, B., 1993, Nonlinear Diagnostics and Simple Trading Rules for High Frequency Foreign Exchange Rates in Weigend, A.S. and N.A. Gershenfeld, eds., Time Series Prediction: Forecasting the Future and Understanding the Past SFI Studies in the Sciences of Complexity, Proceedings Volume 15, Addison Wesley, Reading, Mass., 457-474

Leonov, V.P. and A.N. Shiryaev, 1959, On a Method of Calculation of Semi-Invariants *Theory of Probability and its Applications* 4, 319-329

Nachane, D.M. and D. Ray ,1993, Modelling Exchange Rate Dynamics: New Perspectives From the Frequency Domain, Journal of Forecasting 12, 379-394

Perry, P.R. ,1985, Portfolio Serial Correlation and Nonsynchronous Trading, Journal of Financial and Quantitative Analysis 20, 517-523.

Ramsey, J.B., C. Sayers, and P. Rothman, 1990, The Statistical Properties of Dimension Calculations Using Small Data Sets: Economic Applications, International Economic Review 31, 991-1020

Ramsey, J.B. and Z. Zhang, 1997, The Analysis of Foreign Exchange Data Using Waveform Dictionaries, Journal of Empirical Finance 4, 341-372

Scheinkman, J.A. and B. LeBaron ,1989a, Nonlinear Dynamics and GNP Data in Barnett, W.A., Geweke, J. and Shell, K. ,eds., Economic Complexity: Chaos, Sunspots, Bubbles and Nonlinearity - International Symposium in Economic Theory and Econometrics Chapter 9, 213-227, Cambridge University Press.

Scheinkman, J.A. and B. LeBaron ,1989b, Nonlinear Dynamics and Stock Returns, Journal of Business 62, 311-337.

Stoll, H.R. and R.E. Whaley, 1990, The Dynamics of Stock Index and Stock Index Futures Returns Journal of Financial and Quantitative Analysis 25, 441-468.

Weigend, A.S. and N.A. Gershenfeld ,1993, Time Series Prediction: Forecasting the Future and Understanding the Past SFI Studies in the Sciences of Complexity, Volume 15, Addison Wesley, Reading, Mass.

Table 1: Size of the Cross-Correlation and Cross-Bicorrelation Test Statistics for Small Samples

Test Under Study	Nominal Size of Test	Actual Size of Test for Gaussian Data	Actual Size of Test for Uniform Data	Actual Size of test for Student's *t* with 5 Degrees of Freedom	Actual Size of test for Student's *t* with 10 Degrees of Freedom
xxy	5%	3.8%	2.3%	4.9%	4.5%
	1%	1.4%	0.4%	2.2%	1.4%
	0.1%	0.4%	0.1%	0.8%	0.3%
yyx	5%	4.1%	2.7%	5.4%	4.4%
	1%	1.6%	0.6%	2.2%	1.4%
	0.1%	0.6%	0.1%	0.9%	0.4%
xy	5%	3.3%	3.7%	4.1%	3.9%
	1%	0.4%	0.8%	0.6%	0.7%
	0.1%	0.1%	0.1%	0.2%	0.2%

Table 2: Dates and p-Values for Test Statistics for Cross-Correlations, and Values of Most Significant Cross-Correlations

Series (x / y)	Dates (start - end)	p-value for xy statistic	Most significant correlation (at lag[a])
Austrian Schilling / Danish Krone	No significant cross-correlations		
Austrian Schilling / French Franc	10/9/74-28/10/74	0.0043	0.49(-1)
	20/6/85-7/8/85	0.0098	0.52(-1)
	27/3/92-19/5/92	0.0082	0.53(1)
Austrian Schilling / German Mark	2/8/77-20/9/77	0.0000	0.66(-2)
	15/1/81-4/3/81	0.0038	0.48(2)
	20/6/84-7/8/85	0.0070	0.52(1)
Austrian Schilling / Italian Lira	2/8/77-20/9/77	0.0003	0.39(-2)
Austrian Schilling / Japanese Yen	15/12/78-6/2/79	0.0047	0.46(-2)
Austrian Schilling / U.S. Dollar	17/3/80-7/5/80	0.0078	0.35(2)
Danish Krone / French Franc	30/4/85-19/6/85	0.0070	0.33(1)
	13/3/89-3/5/89	0.0032	0.43(2)
Danish Krone / German Mark	28/8/75-15/10/75	0.0054	0.49(1)
	15/1/81-4/3/81	0.0093	0.49(-2)
	30/4/85-19/6/85	0.0065	0.31(-1)
	13/3/89-3/5/89	0.0002	0.42(2)
Danish Krone / Italian Lira	13/9/82-29/10/82	0.0018	0.62(2)
	13/3/89-3/5/89	0.0011	0.44(2)
Danish Krone / Japanese Yen	15/12/78-6/2/79	0.0011	0.48(-2)
Danish Krone / U.S. Dollar	No significant cross-correlations		
French Franc / German Mark	10/9/74-28/10/74	0.0009	-0.56(2)
	15/1/81-4/3/81	0.0009	0.58(2)
	26/10/87-11/12/87	0.0049	-0.55(1)
	13/3/89-3/5/89	0.0065	0.34(-2)
French Franc / Italian Lira	27/10/78-14/12/78	0.0037	0.39(2)
French Franc / Japanese Yen	20/8/84-8/10/84	0.0075	0.29(2)
French Franc / U.S. Dollar	20/8/84-8/10/84	0.0031	0.31(2)
German Mark /Italian Lira	10/9/74/28/10/74	0.0020	0.44(-1)
	2/8/77-20/9/77	0.0031	0.46(2)
	15/1/81-4/3/81	0.0091	0.50(-2)
	13/3/89-3/5/89	0.0043	0.33(2)
German Mark / Japanese Yen	No significant cross-correlations		
German Mark / U.S. Dollar	No significant cross-correlations		
Italian Lira / Japanese Yen	15/12/78-6/2/79	0.0054	0.45(0)
Italian Lira / U.S. Dollar	20/8/84-8/10/84	0.0036	-0.63(1)
Japanese Yen / U.S. Dollar	28/8/92-15/10/92	0.0076	0.49(1)

[a] x leads for positive lags, y leads for negative lags

Table 3: Number and Percentage of Significant (at the 1% level) Cross-Bicorrelation Windows and Correlations Between *xxy* and *yyx*, and between *xxy* and the simple cross-correlation for all windows

Series (x / y)	No. (%) sig. cross-bicorrelation windows	Corr(xxy,yyx)	Corr(xxy,xy)
Austrian Schilling / Danish Krone	40 (27.0%)	0.646	0.205
Austrian Schilling / French Franc	40(27.0%)	0.680	0.285
Austrian Schilling / German Mark	44(29.7%)	0.643	0.238
Austrian Schilling / Italian Lira	37(25.0%)	0.490	0.160
Austrian Schilling / Japanese Yen	36(24.3%)	0.438	0.132
Austrian Schilling / U.S. Dollar	29(19.6%)	0.452	0.320
Danish Krone / French Franc	42(28.4%)	0.695	0.275
Danish Krone / German Mark	44(29.7%)	0.695	0.271
Danish Krone / Italian Lira	46(31.1%)	0.620	0.118
Danish Krone / Japanese Yen	29(19.6%)	0.446	0.202
Danish Krone / U.S. Dollar	28(18.9%)	0.365	0.281
French Franc / German Mark	43(29.1%)	0.728	0.322
French Franc / Italian Lira	47(31.8%)	0.652	0.069
French Franc / Japanese Yen	31(20.9%)	0.406	0.334
French Franc / U.S. Dollar	29(19.6%)	0.395	0.201
German Mark /Italian Lira	42(28.4%)	0.478	0.207
German Mark / Japanese Yen	32(21.6%)	0.415	0.256
German Mark / U.S. Dollar	26(17.6%)	0.394	0.197
Italian Lira / Japanese Yen	34(23.0%)	0.413	0.187
Italian Lira / U.S. Dollar	36(24.3%)	0.413	0.072
Japanese Yen / U.S. Dollar	30(20.3%)	0.521	0.239

Table 4: Dates and *p*-values for Cross-Bicorrelation Tests Statistics together with Values of Most Significant Bicorrelations

Series (*x* / *y*)	Dates (start - end)	*p*-value for *xxy* statistic	*p*-value for *yyx* statistic	Most significant *xxy* bicorrelations (at lags)	Most significant *yyx* bicorrelations (at lags)
Austrian Schilling / Danish Krone	No significant cross-bicorrelations				
Austrian Schilling / French Franc	1/4/75-19/5/75	0.0121	0.0003	0.41 (2,0)	0.78 (2,1)
	27/1/76-15/3/76	0.0003	0.0302	0.56 (2,1)	0.47 (1,1)
	25/6/76-12/8/76	0.0200	0.0029	0.61 (2,2)	0.48 (1,2)
	26/2/86-17/4/86	0.0301	0.0671	0.54 (1,1)	0.26 (1,2)
Austrian Schilling / German Mark	29/10/74-16/12/74	0.0499	0.0005	0.51 (2,1)	0.28 (1,1)
	24/9/81-11/11/81	0.0076	0.0001	0.35 (1,2)	0.51 (1,0)
	9/10/84-26/11/84	0.7268	0.0019	-	0.53 (1,2)
	8/8/85-27/9/85	0.0001	0.4955	0.75 (1,2)	-
	15/11/85-7/1/86	0.3863	0.0001	-	0.58 (2,1)
Austrian Schilling / Italian Lira	23/4/90-12/6/90	0.0047	0.7926	0.62 (2,2)	-
	1/1/91-18/2/91	0.7781	0.0033	-	0.55 (1,2)
	17/8/93-10/5/93	0.4963	0.0001	-	0.80 (1,2)
Austrian Schilling / Japanese Yen	No significant cross-bicorrelations				
Austrian Schilling / U.S. Dollar	10/9/74-28/10/74	0.0239	0.0005	0.64 (2,1)	0.69 (1,0)
	15/4/82-6/3/82	0.6949	0.0098	-	0.59 (1,1)
Danish Krone / French Franc	28/8/75-16/10/75	0.0000	0.0303	0.44 (2,2)	0.56 (2,2)
	27/1/76-15/3/76	0.0000	0.0066	0.58 (2,1)	0.59 (1,1)
Danish Krone / German Mark	21/9/77-8/4/77	0.0001	0.9981	0.62 (1,1)	-
	13/9/82-29/10/82	0.0045	0.9714	0.64 (2,2)	-
	1/11/82-17/12/82	0.0000	0.2576	0.87 (2,0)	-
	10/6/86-28/7/86	0.0068	0.4884	0.62 (1,0)	-
Danish Krone / Italian Lira	6/5/76-24/6/76	0.6116	0.0000	-	0.75 (1,1)
	13/8/76-1/10/76	0.0693	0.0000	0.48 (1,2)	0.63 (1,0)
	20/6/85-7/8/85	0.0001	0.7395	0.69 (1,1)	-
	28/8/92-15/10/92	0.0008	0.9582	0.56 (2,1)	-
Danish Krone / Japanese Yen	24/5/83-12/7/83	0.8143	0.0032	-	0.65 (1,2)
Danish Krone / U.S. Dollar	21/9/77-11/8/77	0.0000	0.4406	0.54 (1,2)	-
	15/4/82-3/6/82	0.8218	0.0044	-	0.52 (1,1)
	8/3/85-29/4/85	0.0003	0.1219	0.60 (2,1)	-
French Franc / German Mark	27/1/76-15/3/76	0.0230	0.0000	0.52 (1,1)	0.35 (2,1)
	2/8/77-20/9/77	0.0000	0.0021	0.61 (1,2)	0.56 (1,2)
	15/1/81-4/3/81	0.0006	0.8692	0.77 (2,2)	-
	15/11/85-7/1/86	0.5314	0.0008	-	0.56 (1,2)
	26/2/86-17/4/86	0.0018	0.8432	0.51 (1,1)	-
French Franc / Italian Lira	6/5/76-24/6/76	0.0038	0.0061	0.59 (2,1)	0.53 (1,1)
	13/8/76-1/10/76	0.3210	0.0000	-	0.59 (1,0)
	4/6/82-22/7/82	0.0092	0.4661	0.77 (1,1)	-
	1/11/82-17/12/82	0.0049	0.5174	0.56 (1,1)	-
	20/12/82-9/2/83	0.0000	0.8099	0.59 (2,2)	-
	13/3/89-3/5/89	0.0054	0.0971	0.53 (1,1)	0.32 (1,1)
	28/8/92-15/10/92	0.0006	0.9862	0.70 (2,1)	-
	17/8/93-5/10/93	0.5460	0.0064	-	0.55 (1,1)
	7/3/94-26/4/94	0.0202	0.0084	0.52 (2,1)	0.45 (2,1)
French Franc / Japanese Yen	13/8/76-1/10/76	0.4363	0.0001	-	0.60 (1,2)
French Franc / U.S. Dollar	27/1/76-15/3/76	0.0479	0.0000	0.54 (1,1)	0.50 (2,1)
	22/4/77-13/6/77	0.0000	0.88660	0.53 (2,2)	-
	10/2/83-30/3/83	0.0421	.0008	0.32 (1,1)	0.68 (1,1)
German Mark /Italian Lira	10/9/74-28/10/74	0.0068	0.9694	0.71 (1,1)	-
	6/5/76-24/6/76	0.3530	0.0057	-	0.59 (1,1)
	13/8/76-1/10/76	0.0864	0.0006	0.37 (1,2)	0.58 (1,0)
	24/9/81-11/11/81	0.0142	0.0013	0.54 (1,2)	0.30 (1,0)
	20/12/82-9/2/83	0.0000	0.5858	0.54 (2,2)	-

Series (x / y)	Dates (start - end)	p-value for *xxy* statistic	p-value for *yyx* statistic	Most significant *xxy* bicorrelations (at lags)	Most significant *yyx* bicorrelations (at lags)
	26/12/86-13/2/87	0.00660	0.6078	0.60 (1,1)	-
	28/8/92-15/10/92	.0002	0.9992	0.73 (2,1)	-
	19/8/93-10/5/93	0.7725	0.0024	-	0.57 (1,1)
German Mark / Japanese Yen	No significant cross-bicorrelations				
German Mark / U.S. Dollar	10/9/74-28/10/74	0.0023	0.0377	0.56 (1,1)	0.44 (1,0)
Italian Lira / Japanese Yen	27/1/76-15/3/76	0.8231	0.0051	-	0.53 (1,2)
	13/8/76-1/10/76	0.0000	0.0000	0.71 (1,0)	0.55 (1,2)
	8/9/78-26/10/78	0.0072	0.7623	0.69 (1,1)	-
	9/10/84-26/11/84	0.0000	0.2005	0.65 (1,1)	-
	18/1/85-7/3/85	0.0001	0.3889	0.57 (2,0)	-
Italian Lira / U.S. Dollar	4/12/75-26/1/76	0.0000	0.5887	0.72 (2,0)	-
	6/5/76-24/6/76	0.0041	0.0249	0.52 (1,0)	0.44 (2,1)
	13/8/76-1/10/76	0.9434	0.0021	-	0.69 (1,2)
	22/11/76-11/1/77	0.0000	0.9870	0.66 (1,1)	-
	20/2/78-10/4/78	0.0006	0.8740	0.58 (2,2)	-
	18/1/85-7/3/85	0.0057	0.0299	0.55 (2,0)	-
	8/8/85-26/9/85	0.9151	0.0075	-	0.66 (1,2)
Japanese Yen / U.S. Dollar	No significant cross-bicorrelations				

Table 5: Correlation of the Correlation and Bicorrelation Test Statistics with the Individual Variances of the Series.

Series (x / y)	Corr (xy,$VarX$)	Corr (xy,$VarY$)	Corr (xxy,$VarX$)	Corr (xxy,$VarY$)	Corr (yyx,$VarX$)	Corr (yyx,$VarY$)
Austrian Schilling / Danish Krone	0.06	0.04	0.01	-0.03	0.05	-0.09
Austrian Schilling / French Franc	0.05	0.04	0.05	0.04	0.12	0.16
Austrian Schilling / German Mark	0.01	-0.21	0.14	0.14	0.07	0.12
Austrian Schilling / Italian Lira	-0.08	-0.12	0.18	0.22	0.16	0.16
Austrian Schilling / Japanese Yen	-0.01	0.00	0.10	-0.09	-0.01	-0.02
Austrian Schilling / U.S. Dollar	0.08	-0.03	0.14	0.05	0.01	-0.03
Danish Krone / French Franc	0.00	0.02	-0.14	0.02	0.00	0.11
Danish Krone / German Mark	-0.03	-0.01	0.05	0.04	-0.05	0.03
Danish Krone / Italian Lira	-0.08	-0.04	0.15	0.28	0.11	0.18
Danish Krone / Japanese Yen	-0.09	-0.06	0.13	0.03	0.03	-0.03
Danish Krone / U.S. Dollar	-0.37	-0.37	0.03	0.19	-0.01	-0.01
French Franc / German Mark	-0.05	-0.05	-0.03	0.11	0.00	0.12
French Franc / Italian Lira	-0.05	-0.01	0.13	0.01	0.20	0.15
French Franc / Japanese Yen	-0.08	0.00	0.02	-0.02	0.02	0.04
French Franc / U.S. Dollar	-0.04	-0.05	0.03	-0.16	0.11	0.00
German Mark /Italian Lira	-0.05	0.00	0.23	0.21	0.16	0.19
German Mark / Japanese Yen	0.01	0.10	0.16	-0.05	0.06	0.04
German Mark / U.S. Dollar	-0.02	0.03	0.05	-0.03	0.06	0.11
Italian Lira / Japanese Yen	-0.08	-0.09	0.10	-0.02	0.02	0.08
Italian Lira / U.S. Dollar	-0.03	-0.14	0.11	-0.08	-0.06	-0.01
Japanese Yen / U.S. Dollar	0.18	0.14	0.04	-0.04	0.01	0.07

Proof of theorem 1 in the paper: The null hypothesis is that $\{x(t_k)\}$ and $\{y(t_k)\}$ are mutually independent i.i.d. zero mean series. Set $\sigma_x = \sigma_y = 1$. Redefine the three time points in the triple product $x(t_k)$ $x(t_k+r)$ $y(t_k+s)$ for a given (r,s) as follows: $t_{k_1} = t_k, t_{k_2} = t_k + r, t_{k_3} = t_k + s \; (k = 1, l)$. Then a) $\mathrm{E}[x(t_{k_1})x(t_{k_2})y(t_{k_3})] = 0$, and b) $\mathrm{E}[x(t_{k_1})x(t_{k_2})y(t_{k_3})]^2 = 1$.

The n^{th} order cumulant of a product of variates can be related to the joint cumulants of the variates, but the relationship is more complicated than the one between the moments and cumulants stated above. There is no simple approach to deal with the combinatorial relationships between the n^{th} order joint cumulants of the triple product $P(t_{k_1}t_{k_2}t_{k_3}) = x(t_{k_1})x(t_{k_2})y(t_{k_3})$ for various values of t_k, r, and s, and the cumulants of $u(t)$ even though the $x(t_k)$'s and $y(t_k)$'s are independent. The relationships rest on a definition of *indecomposable partitions* of two dimensional tables of subscripts of the t's (see Leonov and Shiryaev, 1959, and Sec. 2.3 of Brillinger, 1975). We display the table of the t's next to the table of their subscripts which Brillinger uses in his exposition.

Consider the following $l \times 3$ tables of $t_{k_1} = t_k, t_{k_2} = t_k + r_k, t_{k_3} = t_k + s_k \;\; (k = 1, l)$:

Times			Using Delay Notation		
t_{11}	t_{12}	t_{13}	t_1	$t_1 + r_1$	$t_1 + s_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t_{l1}	t_{l2}	t_{l3}	t_l	$t_l + r_l$	$t_l + s_l$

Let $\upsilon = \upsilon_1 \cup \cdots \cup \upsilon_M$ denote a partition of the k_{ji} in this table into M sets where j=1,...,l and i=1,2,3. There are many partitions of the l X 3 times from the single set of all the elements to $l \times 3$ sets of one element.

The m^{th} set in the partition is denoted $\upsilon_m = (k_{j_{1(m)}i_{1(m)}}, \ldots, k_{j_{\vartheta(m)}i_{\vartheta(m)}})$ where $\vartheta(m)$ is the number of elements in the set. The cumulant of $[x(k_{j_{1(m)}i_{1(m)}}), \ldots, x(k_{j_{\vartheta(m)}i_{\vartheta(m)}})]$ is $\kappa[x(k_{j_{1(m)}i_{1(m)}}) \cdots x(k_{j_{\vartheta(m)}i_{\vartheta(m)}})]$. The symbol $\kappa[\upsilon(m)]$ will be used for this joint cumulant.

If no two ji are equal for a set $\upsilon(m)$, then $\upsilon(m)$ is called a *chain*. A partition is called *indecomposable* if there is a set with at least one chain going through each row of the table (all the rows are chained together). A partition is *decomposable* if one set or a union of some set in υ equals a subset of the rows of the table. Consider, for example, the following 2 X 3 table:

$$\begin{matrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{matrix}$$

The decomposable partitions are: $(k_{11}, k_{12}, k_{13}) \cup (k_{21}, k_{22}, k_{23})$, which is the union of the two rows and all its sub-partitions. Three indecomposable partitions of this 2 × 3 table are $(k_{11}, k_{21}) \cup (k_{12}, k_{22}) \cup (k_{13}, k_{23})$, $(k_{11}, k_{22}) \cup (k_{12}, k_{21}) \cup (k_{13}, k_{23})$, and $(k_{11}, k_{21}, k_{12}, k_{22}) \cup (k_{13}, k_{23})$. Each pair of these partitions are chains.

Let $\upsilon_r = \upsilon_1 \cup \cdots \cup \upsilon_{M_r}$ denote the r^{th} indecomposable partition of table (A3) into M_r sets. The joint cumulant of $(x(t_{k_{11}})x(t_{k_{12}})y(t_{k_{13}}),\ldots,(x(t_{k_{\vartheta 1}})x(t_{k_{\vartheta 2}})y(t_{k_{\vartheta 3}}))$ is the sum over r of the products of the M_r cumulants $\kappa[\upsilon(m)]$ of the υ_m in each indecomposable υ_r.

It is easy to check that $\kappa[x(t_{k_1})x(t_{k_1}+r)y(t_{k_1}+s),\ldots,x(t_{k_l})x(t_{k_l}+r)y(t_{k_l}+s)]=0$ unless $t_{k_1} = \cdots t_{k_l} = t$. It then follows by an enumeration of each of the cumulants of the sets in the indecomposable partitions $\upsilon = \upsilon_1 \cup \cdots \cup \upsilon_p$ that most of the products of cumulants are zero for a given partition. A summary of the second order joint cumulants of the triples is as follows: $\kappa[x(t_1)x(t_1+r_1)y(t_1+s_1)x(t_2)x(t_2+r_2)y(t_2+s_2)]=0$ unless $t_1=t_2=t$, $r_1=r_2=r$, and $s_1=s_2=s$. If so, then $\kappa[p^2(t,t+r,t+s)]=1$.

The covariance of $C_{xxy}(r_1,s_1)$ and $C_{xxy}(r_2,s_2)$ is $[(N-s_1)(N-s_2)]^{-1/2}$ times a double sum of covariances of the P's. There are N-s non-zero terms (all equal to one) in the double sum of covariances. Then from the theorem. Var[$C_{xxy}(r,s)$] = $(N-s)/(N-s_2)$ = 1 and Cov[$C_{xxy}(r_1,s_2)$, $C_{xxy}(r_2,s_2)$] = 0.

To obtain the third order joint cumulants, consider the following 3 X 3 table of $0 < r_k < s_k$:

$$\begin{matrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{matrix} \qquad \begin{matrix} t_1 & t_1+r_1 & t_1+s_1 \\ t_2 & t_2+r_2 & t_2+s_2 \\ t_3 & t_3+r_3 & t_3+s_3 \end{matrix}$$

Using the delay notation for indices, first consider the following indecomposable partition: $\nu_1 = (t_1,t_2,t_3) \cup (t_1+r_1,t_2+r_2,t_3+r_3) \cup (t_1+s_1,t_2+s_2,t_3+s_3)$. If 1) $t_1 = t_2 = t_3$, 2) $r_1 = r_2 = r_3$,

3) $s_1 = s_2 = s_3$, then the third order cumulants of the three columns equal γ and thus the product of the cumulants is γ^3. If any one of these equalities in 1), 2), or 3) do not hold then the product is zero.

Now consider the indecomposable partition $\nu_2 = (t_1, t_2, t_3) \cup (t_1 + r_1, t_2 + r_1, t_3 + r_3, t_1 + s_1) \cup (t_2 + s_2, t_3 + s_3)$. The only non-zero product of cumulants holds for $t_1 = t_2 = t_3 = t$, $r_1 = r_2 = r_3 = r$, and $s_1 = s_2 = s_3 = s$, which yields $\kappa[p^3(t, t + r_1, t + s_1)] = \gamma^3$.

Suppose that $s_3 \neq r_1$. Consider the indecomposable partition (a sub-partition of ν_2) $\nu_3 = (t_1, t_2, t_3) \cup (t_1 + r_1, t_2 + r_1) \cup (t_3 + r_3, t_1 + s_1) \cup (t_2 + s_2, t_3 + s_3)$. Then if $t_1 = t_2 = t_3 = t$ and $r_1 = r_2$, $s_2 = s_3$, $r_3 = s_1$, the product of the cumulants is near zero since $x(t+s_1)$ and $y(t+s_1)$ are independent. The pattern should be clear. All the other indecomposable partitions have at least one zero cumulant. From the theorem, the third order joint cumulant of the C_{xxy} are zero.

We also require an understanding of the higher order joint cumulants to prove the asymptotic properties of our test statistic. The general form can be deduced from the fourth order case by enumerating the sets in the indecomposable partitions of the 4 × 3 table:

t_{11}	t_{12}	t_{13}	t_1	$t_1 + r_1$	$t_1 + s_1$
t_{21}	t_{22}	t_{23}	t_2	$t_2 + r_2$	$t_2 + s_2$
t_{31}	t_{32}	t_{33}	t_3	$t_3 + r_3$	$t_3 + s_3$
t_{41}	t_{42}	t_{43}	t_4	$t_4 + r_4$	$t_4 + s_4$

The major term in the error of the approximation is a function of the non-zero products of the following two types of indecomposable partitions of this table:

$$\nu_4 = (t_1, t_3) \cup (t_2, t_4) \cup (t_1 + r_1, t_2 + r_2) \cup (t_3 + r_3, t_4 + s_4) \cup (t_1 + s_1, t_2 + s_2) \cup (t_3 + s_3, t_4 + s_4)$$
$$\nu_5 = (t_1, t_2, t_3, t_4) \cup (t_1 + r_1, t_2 + r_2) \cup (t_3 + r_3, t_4 + s_4) \cup (t_1 + s_1, t_2 + s_2) \cup (t_3 + s_3, t_4 + s_4)$$

If 4) $t_4 = t_3 = t_2 = t_1 = t$, 5) $r_1 = r_3 \neq r_2 = r_4$, and 6) $s_1 = s_3 \neq s_2 = s_4$ then the cumulants of all the pairs in ν_4 are one and the product then one, and the cumulant of the first column in ν_5 is κ, which is the product of the cumulants of ν_5. The cumulant products of the other indecomposable partitions of the table are all zero given constraints 4), 5), and 6). Thus $\kappa[p^2(t, t+r, t+s) p^2(t, t+r_2, t+s_2)] = (1+\kappa)$.

For each (r_1, s_1) and (r_2, s_2), these non-zero cumulant products equalities hold for at most N t's. Thus the fourth order joint cumulant $\kappa[c^2_{xxy}(r_1, s_2) c^2_{xxy}(r_2, s_2)]$ is

$[(N-s_1)(N-s_2)(N-s_3)(N-s_4)]^{-1/2}\mathrm{O}(N)(1+\kappa)=\mathrm{O}(N^{-1})(1+\kappa)$. There are of order $\mathrm{O}(L^4)$ such pairs of indices where $r_1 \neq r_2$ and $s_1 \neq s_2$ which have these joint cumulants. If $r_1=r_2$ or $s_1=s_2$, then there are a lot more fourth order non-zero cumulants. An enumeration of indecomposable partitions with non-zero cumulant products yields the following two results:

(A.1) $\kappa[p^4(t,t+r,t+s)]=\kappa^3+9\kappa^2+27\kappa+24$

and

$$\kappa[p^2(t,t+r,t+s_1)p^2(t,t+r,t+s_2)]=\kappa^2+6\kappa+8$$

The same pattern holds of partitions of the general l X 3 table of subscripts into pairs with identical indices. The major non-zero cumulants are $\kappa[p^2(t,t+r_1,t+s_1)\cdots p^2(t,t+r_{l/2},t+s_{l/2})]=\mathrm{O}(N^{-1})$ when l is even, and $\kappa[p^2(t,t+r_1,t+s_1)\cdots p^2(t,t+r_{(l-1)/2},t+s_{(l-1)/2})p(r_l,s_l)]=\mathrm{O}(N^{-1})$ for a restricted set of $(r_k - s_k)$ when l is odd. Thus the l^{th} joint cumulant of the C_{xxy}'s is of order $\kappa\mathrm{O}(N^{1-l/2})$.

These results will now be applied to prove that the test statistic H_N is asymptotically normal. It has already been shown that $\mathrm{Var}[C_{xxy}(r_m,s_m)] = \mathrm{E}[C_{xxy}^2(r_m,s_m)] = 1$ and thus $\mathrm{E}(H_N)=0$ under the null hypothesis. From the relationship between the covariances and the fourth order cumulants,

$Var[C_{xxy}{}^2(r_1,s_1)C_{xxy}{}^2(r_2,s_2)]=\kappa[c_{xxy}{}^2(r_1,s_1)c_{xxy}{}^2(r_2,s_2)]+2,$

$Var[C_{xxy}{}^2(r_1,s_1)C_{xxy}{}^2(r_2,s_2)]=\kappa[c_{xxy}{}^2(r_1,s_1)c_{xxy}{}^2(r_2,s_2)]+2\kappa^2[c_{xxy}(r_1,s_1)c_{xxy}(r_2,s_2)]$. Suppose that $r_4=r_3=r_2=r_1=r$, and $s_4=s_3=s_2=s_1=s$. Then from (A.1), $\kappa[c_{xxy}{}^4(r,s)]=\mathrm{O}(\mu(\kappa)/N)$ where $\mu(\kappa)=\kappa^3+9\kappa^2+27\kappa+24$. Thus $Var[C_{xxy}{}^2(r,s)]=2+\mathrm{O}(\mu(\kappa)/N)$.

If $r_1=r_3 \neq r_2=r_4$ and $s_1=s_3 \neq s_2=s_4$ (constraints 5) and 6)), then $\kappa[c_{xxy}{}^2(r_1,s_1)c_{xxy}{}^2(r_2,s_2)]=\mathrm{O}((1+\kappa)/N)$. If $r_1=r_2$, then the joint cumulant is $\mathrm{O}(\nu(\kappa)/N)$, where $\nu(\kappa)=\kappa^2+6\kappa+8$. Thus $Cov[C_{xxy}{}^2(r_1,s_1)C_{xxy}{}^2(r_2,s_2)]=\mathrm{O}(N^{-1})$. Since the number of $C_{xxy}{}^2(r_m,s_m)-1$ terms in the sum is $L^2/2$ ($L=N^c$), $\mathrm{Var}(H_N)=1+\mathrm{O}(L^2/N) \to 1$ as $N \to \infty$ since $0 < c < 1/2$. There are approximately L^3 such (r_1,s_1), (r_2,s_2), (r_3,s_3), and (r_4,s_4) in the double sum which satisfies constraints 5) and 6) and thus the error in the variance of H_N due to these covariances is of the order $L^{-2}L^3N^{-1}=N^{c-1}$.

To complete the proof, we will now demonstrate that the cumulants of H_N of order $l \geq 3$ go to zero as $N \rightarrow \infty$. The l^{th} cumulant of H_N depends on the $2l$ order joint cumulant of the $C_{xxy}{}^2(r_k, s_k)$ for k=1,l. From above, these cumulants are of order $L^2 L^{-l} N^{1-l/2}$ which goes to zero as $N \rightarrow \infty$ for $l \geq 3$.