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Accepted Version

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Identification of fractional-order transfer functions using a step excitation

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***Abstract*—This brief proposes a new method for the identification of fractional order transfer functions based on the time response resulting from a single step excitation. The proposed method is applied to the identification of a three-dimensional RC network, which can be tailored in terms of topology and composition to emulate real time systems governed by fractional order dynamics. The results are in excellent agreement with the actual network response, yet the identification procedure only requires a small number of coefficients to be determined, demonstrating that the fractional order modelling approach leads to very parsimonious model formulations.**

***Index Terms*—Fractional order systems, system identification, analog circuit simulation, RC circuits.**

I. INTRODUCTION

THE need for adopting fractional order calculus in engineering stems from its ability to describe phenomena that cannot be fully described by a local theory. Examples include propagating acoustic or electromagnetic waves in the presence of boundaries, where according to Huygens' principle a propagating wave may be seen as a non-local process in space and time inducing a memory effect (a delayed reaction) on a propagating wave [1], as well as when a smooth transition between a local (Newtonian) and a non-local (quantum) theory of motion is required. By incorporating non-local processes and phenomena in a mathematical framework, it becomes possible to extend the application of systems theory to a much wider range of problems across the physical sciences. The current contribution proposes an intuitive systems identification framework for complex systems with emergent behaviours emulated using a network of resistive and capacitive components, when these are subjected to a step excitation. The new algorithm aims to complement existing literature on the subject. Applications include filtering [2]-[10], analysis of dielectric responses [11]-[13], and control [14]-[18].

Manuscript received October 03, 2014. This work was supported by FAPESP (grants 2011/17610-0, 2011/13777-8), CNPq (research fellowships) and CAPES (DINTER program).

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A. Notation

The Laplace transform of a signal $y(t),\ t \geq 0$, will be denoted by $Y(s) = \mathcal{L}[y(t)]$. The notation $\mathcal{L}^{-1}$ will indicate the inverse transform. Definitions will be stated by using the $\triangleq$ symbol. The limit values of a signal $y(t)$ will be denoted as $y(0^+) \triangleq \lim_{t\to 0^+} y(t)\,,\ \ y(\infty) \triangleq \lim_{t\to\infty} y(t)$.

II. PROPOSED IDENTIFICATION METHOD

It is assumed that the system under consideration is stable and can be described by a transfer function of the form

$$G(s) = \frac{b_0 + b_1 s^{\beta_1} + \cdots + b_m s^{\beta_m}}{1 + a_1 s^{\alpha_1} + \cdots + a_n s^{\alpha_n}} \tag{1}$$

where $b_0, b_1, \ldots, b_m, a_1, \ldots, a_n$ are coefficients to be identified and $\beta_1, \ldots, \beta_m, \alpha_1, \ldots, \alpha_n$ are positive real-valued exponents. Initially, these exponents will be assumed to be known.

The proposed identification method is based on the step response of the system, which is used to generate a set of auxiliary signals, as depicted in Fig. 1.

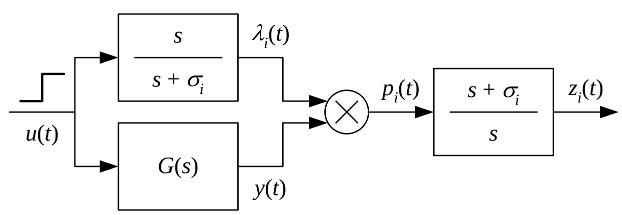

Fig. 1. Generation of the signals employed in the proposed identification procedure, using a given filtering parameter $\sigma_i > 0$. The rectangular blocks correspond to transfer functions, whereas the circle with a $\times$ sign indicates a pointwise multiplication of the signals in the time domain.

In Fig. 1, $y(t)$ represents the response of the system to a unit step $u(t),\ t \geq 0$, such that

$$U(s) = 1/s,\ \ Y(s) = G(s)U(s) = G(s)/s \tag{2}$$

In addition, $\lambda_i(t)$ is the result of passing the step input $u(t)$ through a filter with transfer function $s/(s+\sigma_i)$, i.e.

$$\lambda_i(t) \triangleq \mathcal{L}^{-1}\left[\frac{s}{s+\sigma_i}U(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{s+\sigma_i}\right] = e^{-\sigma_i t},\ t \geq 0 \tag{3}$$

where $\sigma_i > 0$ is a given filtering parameter. Signal $p_i(t)$ is obtained as $p_i(t) \triangleq y(t)\lambda_i(t) = y(t)e^{-\sigma_i t},\ t \geq 0$, which corresponds to the following Laplace transform [19]:

$$P_i(s) = \mathcal{L}[y(t)e^{-\sigma_i t}] = Y(s+\sigma_i) \tag{4}$$

From (2) and (4), it follows that

$$P_i(s) = \frac{G(s+\sigma_i)}{s+\sigma_i} \tag{5}$$

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Finally, $z_i(t),\ t \geq 0$, is the result of passing $p_i(t)$ through a filter with transfer function $(s+\sigma_i)/s$, i.e.

$$Z_i(s) \triangleq \frac{s+\sigma_i}{s} P_i(s) = \frac{G(s+\sigma_i)}{s} \tag{6}$$

In view of the Final Value Theorem, which is also valid for fractional calculus [20], $z_i(\infty)$ can be related to $Z_i(s)$ as $z_i(\infty) = \lim_{s\to 0} sZ_i(s)$. By using this relation together with (1), (6), it follows that

$$z_i(\infty) = \lim_{s\to 0} G(s+\sigma_i) = \frac{b_0 + b_1\sigma_i^{\beta_1} + \cdots + b_m\sigma_i^{\beta_m}}{1 + a_1\sigma_i^{\alpha_1} + \cdots + a_n\sigma_i^{\alpha_n}} \tag{7}$$

which can be rewritten as

$$\begin{aligned} &z_i(\infty)a_1\sigma_i^{\alpha_1} + \cdots + z_i(\infty)a_n\sigma_i^{\alpha_n} \\ &- b_0 - b_1\sigma_i^{\beta_1} - \cdots - b_m\sigma_i^{\beta_m} = -z_i(\infty) \end{aligned} \tag{8}$$

By evaluating $z_i(\infty)$ with $n+m+1$ different values of σ_i ($i = 1, 2, \ldots, n+m+1$), Eq. (8) can be used to derive a system of linear equations of the form $A\theta = c$, with

$$A = \begin{bmatrix} z_1(\infty)\sigma_1^{\alpha_1} & \cdots & z_1(\infty)\sigma_1^{\alpha_n} & -1 & -\sigma_1^{\beta_1} & \cdots & -\sigma_1^{\beta_m} \\ z_2(\infty)\sigma_2^{\alpha_1} & \cdots & z_2(\infty)\sigma_2^{\alpha_n} & -1 & -\sigma_2^{\beta_1} & \cdots & -\sigma_2^{\beta_m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{n+m+1}(\infty)\sigma_{n+m+1}^{\alpha_1} & \cdots & z_{n+m+1}(\infty)\sigma_{n+m+1}^{\alpha_n} & -1 & -\sigma_{n+m+1}^{\beta_1} & \cdots & -\sigma_{n+m+1}^{\beta_m} \end{bmatrix} \tag{9}$$

$$\theta = [a_1 \cdots a_n\, b_0\, b_1 \cdots b_m]^T,\ c = \begin{bmatrix} -z_1(\infty) \\ -z_2(\infty) \\ \vdots \\ -z_{n+m+1}(\infty) \end{bmatrix}. \tag{10}$$

Therefore, if $det(A) \neq 0$, the model coefficients can be obtained by calculating $\theta = A^{-1}c$.

To conclude the presentation of the proposed method, there remain the issues of choosing appropriate values for the filtering parameters $\sigma_1, \sigma_2, \ldots, \sigma_{n+m+1}$ employed in the identification procedure and determining the transfer function exponents $\beta_1, \ldots, \beta_m, \alpha_1, \ldots, \alpha_n$.

For simplicity, the filtering parameters can be chosen as $\sigma_2 = \rho\,\sigma_1,\ \sigma_3 = \rho\,\sigma_2,\ \ldots, \sigma_{n+m+1} = \rho\,\sigma_{n+m}$, where ρ is a constant in the range $0 < \rho < 1$. It is worth noting that the largest parameter σ_1 corresponds to the exponential function in (3) with fastest decay. In this work, the value of σ_1 is adjusted to match the delay times of the exponential $e^{-\sigma_1 t}$ and the step response $y(t)$. In this context, the delay time t_d is defined as the time required for a function to reach the midpoint between its initial and final values, i.e. $y(t_d) = [y(\infty) + y(0^+)]/2$. Therefore, σ_1 can be calculated as $\sigma_1 = (\ln 2)/t_d$.

Finally, let $\gamma \triangleq [\beta_1 \cdots \beta_m\ \alpha_1 \cdots \alpha_n]^T$ denote the vector of exponents to be determined. For given values of γ and ρ, the identification procedure can be carried out in order to obtain the vector of transfer function coefficients as $\theta = A^{-1}c$. The following cost function $J(\gamma, \rho)$ can then be calculated as the root-mean-square error between the step response $y(t)$ and the output $\hat{y}(t)$ of the resulting model:

$$J(\gamma, \rho) \triangleq \sqrt{\frac{1}{N}\sum\nolimits_{k=1}^{N} [y(kT) - \hat{y}(kT)]^2} \tag{11}$$

where T is the sampling period employed in the acquisition of the step response and N is the number of acquired samples. Therefore, suitable values of γ and ρ can be obtained by using numerical optimization techniques to minimize $J(\gamma, \rho)$.

In the present work, the FOTF Matlab Toolbox [21] was employed to evaluate the step response $\hat{y}(t)$ of the fractional-order model, and the flexible polyhedron algorithm [22] implemented in the Matlab Optimization Toolbox was adopted to obtain γ and ρ through the minimization of $J(\gamma, \rho)$.

Remark 1: To simplify the optimization procedure involved in the determination of the transfer function exponents, the following particularization can be adopted, as in [23]:

$$\begin{aligned} &\beta_1 = 1,\ \ldots,\ \beta_{m-1} = (m-1),\ \beta_m = \beta \\ &\alpha_1 = 1,\ \ldots,\ \alpha_{n-1} = (n-1),\ \alpha_n = \alpha \end{aligned} \tag{12}$$

where β and α will be the only real-valued exponents to be determined. Suitable values for n and m can be chosen by using parsimony criteria on the basis of the resulting approximation errors, as in integer-order identification [24].

Remark 2: A limitation of the proposed method is the need to choose a suitable value of ρ. It may be argued that this parameter affects the distribution of the approximation error over different frequencies. Indeed, smaller values of ρ result in the use of exponential functions with faster decay, which may lead to greater emphasis on the reduction of error at higher frequencies. This relation could be exploited to guide the selection of ρ, as an alternative to tuning this parameter through numerical optimization. Such a possibility will be pursued in future studies. Additional research will also be required for the analytical or numerical derivation of uncertainty bounds on the identified coefficients and exponents.

Remark 3: Sufficient theoretical conditions for the invertibility of the matrix A defined in (9) still need to be investigated. However, no singularity or ill-conditioning problems were experienced in the present work.

III. Case study: Three-dimensional RC network

Fig. 2 presents an example of the RC networks under consideration, which can be used as circuit models for the dielectric response of composite materials containing insulating and conducting particles [12]. The topology comprises N_Z layers, each consisting of a matrix of nodes with N_X rows and N_Y columns. A resistor R_S is included to model the internal resistance of the source. The numbers of resistors (not including the source) and capacitors are denoted by N_R and N_C, respectively, so that $N_R + N_C = N_Z[N_Y(N_X + 1) + N_X(N_Y - 1)] + (N_Z - 1)N_X N_Y$ [23]. The fraction of capacitors is defined as $f_C = N_C/(N_R + N_C)$.

As shown in [23], a dynamic model in descriptor form can be derived from Kirchoff's laws to relate the input voltage $u(t)$ with the current $y(t)$ entering the network. A randomized incidence matrix can be employed to place the R, C components at aleatory positions. The resulting model can

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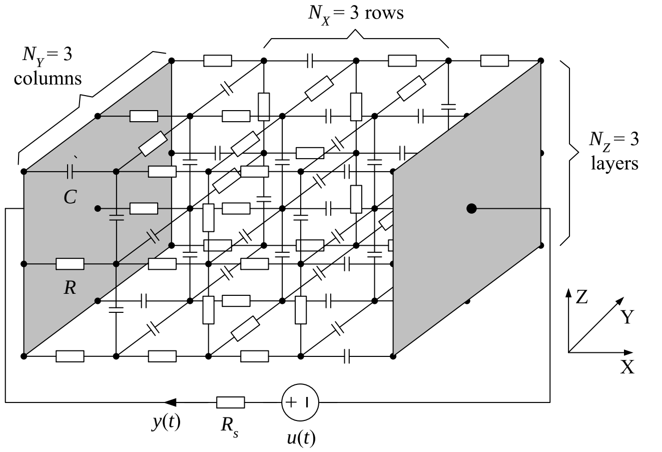

Fig. 2. Diagram of a three-dimensional RC network with random allocation of the R, C elements. The grey plates indicate electrodes employed for connection to a voltage source with output resistance R_S.

be converted to an integer-order transfer function $G(s)$, which facilitates the time-domain simulation of the network, as well as the calculation of the admittance in the frequency domain.

It is worth noting that the model order can be very high, even after taking into account algebraic constraints associated to closed paths of capacitors, and eliminating non-controllable or non-observable modes. For example, the network employed in [23], with 100 resistors and 100 capacitors, resulted in a transfer function with order 73. However, it was found that a more compact representation could be obtained by using a fractional order transfer function of the form:

$$G_{frac}(s) = \frac{b_0 + b_1 s + \cdots + b_{n-1}s^{n-1} + b_n s^{\beta}}{1 + a_1 s + \cdots + a_{n-1}s^{n-1} + a_n s^{\alpha}} \quad (13)$$

where single fractional exponents α, β are employed at the denominator and numerator, as stated in Eq. (12).

The identification procedure adopted in [23] was aimed at matching the exact admittance $G(j\omega)$ of the network and the approximated admittance $G_{frac}(j\omega)$ in fractional-order form, over a given range of frequencies ω. For this purpose, the coefficients $b_0, b_1, \ldots, b_n, a_1, \ldots, a_n$ and the fractional exponents α, β were adjusted to minimize the following cost:

$$J = \sum_{k=1}^{N} w_k |G(j\omega_k) - G_{frac}(j\omega_k)|^2 \quad (14)$$

where $\omega_1, \omega_2, \ldots, \omega_N$ are frequencies of interest chosen by the designer and $w_1, w_2, \ldots, w_N > 0$ are frequency-dependent weights. The flexible polyhedron algorithm [22] was employed for the minimization of J with respect to the coefficients and fractional exponents. As a result, a transfer function $G_{frac}(s)$ with $n = 2$, involving only seven parameters $(b_0, b_1, b_2, a_1, a_2, \beta, \alpha)$, was sufficient to provide a good approximation of the admittance over a broad frequency range.

In what follows, the results of the proposed identification method will be compared with those reported in [23], which were obtained by minimizing the cost (14) over $N = 500$ frequencies logarithmically spaced between $\omega_1 = 10^{-2}$ rad/s and $\omega_{500} = 10^3$ rad/s. The weights were set to $w_k = |G(j\omega_k)|^{-2}$ in order to normalize the cost [23]. The same network will be employed, with parameters $R_S = 0.1\,\Omega$, $R = 1\,\Omega$, $C = 0.5\,F$, $N_Z = 3$, $N_X = N_Y = 5$ and $f_C = 0.5$.

IV. Results

A. Preliminary analysis of the RC network response

Fig. 3a and 3b present the admittance and unit step response of the network, including the source resistance R_S. As can be seen in Fig. 3a, the admittance converges to -10 dB at low frequencies, which amounts to a resistance of $3.16\,\Omega$. This value corresponds to the association of R_S with the equivalent resistance of the RC network, since the capacitors behave as open circuits at low frequencies. As a result, the steady-state value of the input current following the 1 V step excitation is equal to $1\ \text{V}/3.16\,\Omega$ = 0.32 A, which is consistent with the limit value of the curve shown in the inset of Fig. 3b.

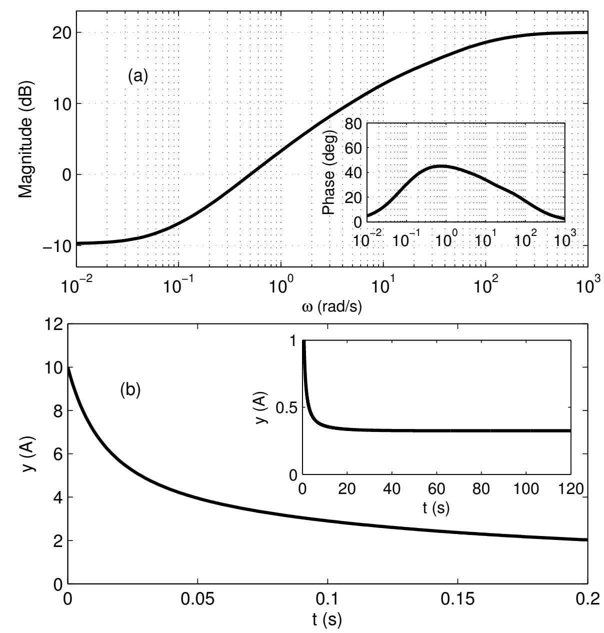

Fig. 3. (a) Admittance and (b) unit step response of the RC network.

The convergence of the admittance to +20 dB at high frequencies can be ascribed to a path of capacitors between the network terminals. Indeed, since the capacitors behave as short circuits at high frequencies, the overall admittance becomes $20\log_{10}(1/R_S) = 20\log_{10}(1/0.1) = 20$ dB. This is also the reason why the input current reaches an initial value of 10 A in Fig. 3b. The capacitors impose that the voltage across the network remains equal to zero immediately after the application of the 1 V step excitation and thus the initial current is equal to $1\ \text{V}/R_S$ = 1 V / 0.1 Ω = 10 A.

B. Identification results

The proposed identification method was initially employed with a fractional-order transfer function of the following form:

$$G_{frac}^{n=1}(s) = \frac{b_0 + b_1 s^{\alpha}}{1 + a_1 s^{\alpha}} \quad (15)$$

where $\alpha > 0$ is a real-valued exponent. This is a particular case of (13) with $n = 1$, in which the same fractional exponent is used in the numerator and denominator. This configuration was adopted in view of the asymptotic behavior of the network admittance at high frequencies, as discussed above. Therefore, the cost (11) becomes a function of only two parameters, namely α and ρ. In view of the possible convergence to poor

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local minima, the optimization was carried out from different initial values of (α, ρ) in a grid formed by varying α from 0.1 to 1.0 and ρ from 0.1 to 0.9, both with step size of 0.1.

The proposed method was also employed to identify a fractional-order transfer function of the following form:

$$G_{frac}^{n=2}(s) = \frac{b_0 + b_1 s + b_2 s^{\alpha}}{1 + a_1 s + a_2 s^{\alpha}} \quad (16)$$

which is a particular case of (13) with $n = 2$ and the same fractional exponent α in the numerator and denominator. The initial (α, ρ) pairs employed in the optimization were taken from a grid formed by varying α from 0.1 to 2.0 and ρ from 0.1 to 0.9, both with step size of 0.1. For comparison, the following integer-order transfer functions were also identified:

$$G_{int}^{n=1}(s) = \frac{b_0 + b_1 s}{1 + a_1 s}, \quad G_{int}^{n=2}(s) = \frac{b_0 + b_1 s + b_2 s^2}{1 + a_1 s + a_2 s^2} \quad (17)$$

In this case, the cost (11) becomes a function of a single parameter ρ. This parameter was optimized by using initial values ranging from 0.1 to 0.9, with step size of 0.1.

The identification was based on the step response of the RC network, with time range of $0 - 80$ s and sampling period $T = 0.001$ s. By using a computer with an i5 processor (1.70 GHz), the optimization times for each initial (α, ρ) point were approximately 3 and 11 minutes for the integer and fractional-order model, respectively. No substantial differences between $n = 1$ and $n = 2$ were observed. The workload is mainly associated to the generation of the model output $\hat{y}(kT)$ employed in the cost function (11). The computational demand of such a procedure is much larger in the fractional order case.

Table I presents the resulting integer and fractional-order transfer functions, along with the results reported in [23].

TABLE I: SUMMARY OF RESULTS.

	$n=1$	$n=2$
(a)	$\frac{1.449s + 0.358}{0.229s + 1}$	$\frac{0.875s^2 + 2.73s + 0.339}{0.116s^2 + 1.614s + 1}$
(b)	$\frac{1.348s^{0.682} + 0.287}{0.145s^{0.682} + 1}$	$\frac{5.311s + 0.792s^{0.439} + 0.333}{0.481s + 2.994s^{0.439} + 1}$
(c)	$\frac{1.395s^{0.698} + 0.281}{0.181s^{0.651} + 1}$	$\frac{0.067s + 3.015s^{0.852} + 0.346}{0.088s + 1.288s^{0.472} + 1}$

(a) Proposed method (integer order), (b) Proposed method (fractional order), (c) Results reported in [23] (Fractional order).

Fig. 4 compares the step responses of the integer and fractional order models obtained with the proposed method. As can be seen, the fractional order models provide a much better match of the network response. For $n = 2$, the model response is almost indistinguishable from that of the network.

Fig. 5 compares the step response of the fractional order model obtained herein with that reported in [23], which are labeled "step response identification" and "frequency response identification", respectively. The comparison is restricted to $n = 2$, because the transfer function obtained in [23] for $n = 1$ is not proper (i.e. the dominating term in the numerator has a larger exponent compared to the denominator), as seen in Table I. Therefore, the corresponding step response is not well-defined. In fact, the parametrization adopted in [23] did not impose that the fractional exponents should be equal in

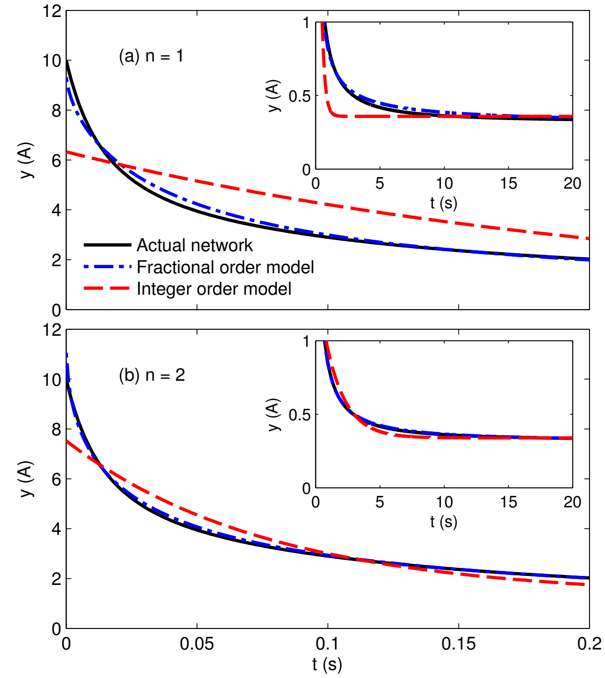

Fig. 4. Step response results: Fractional and integer order models.

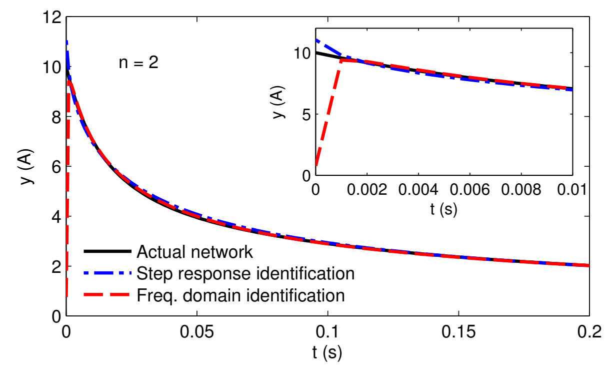

Fig. 5. Step response results (fractional order model with $n = 2$): Comparison with the frequency-domain identification method reported in [23].

the numerator and denominator. Since the identification was aimed at matching the network within a limited frequency range, there was no need to impose an *a priori* relation between these exponents. Regarding the $n = 2$ case in Fig. 5, it can be seen that both models provide a good match of the network response. However, the model reported in [23] yields a larger discrepancy in the initial part of the transient response, as shown in the inset. This finding may be ascribed to a mismatch between the admittance of the network and the identified model at frequencies higher than those employed in the cost function (14). Since the method proposed herein is directly aimed at matching the step response, the resulting model provides a much better match of the initial response.

A dramatic improvement using the fractional over the integral order formulation is shown in Fig. 6, which corroborates the time-domain findings. Finally, Fig. 7 compares the frequency responses of the fractional order models obtained herein with those reported in [23]. For $n = 2$ the results show excellent agreement to the network frequency response and are consistent to those in [23]. The proposed method, tailored at matching the step response in the time domain, also provides an excellent match in the frequency domain.

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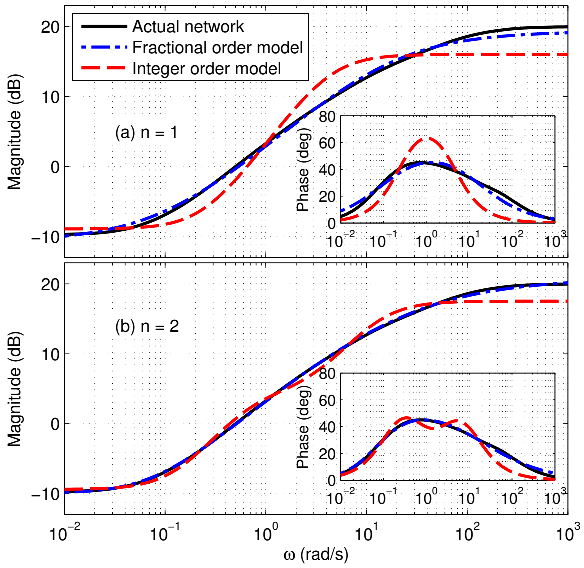

Fig. 6. Frequency domain results: Fractional and integer order models.

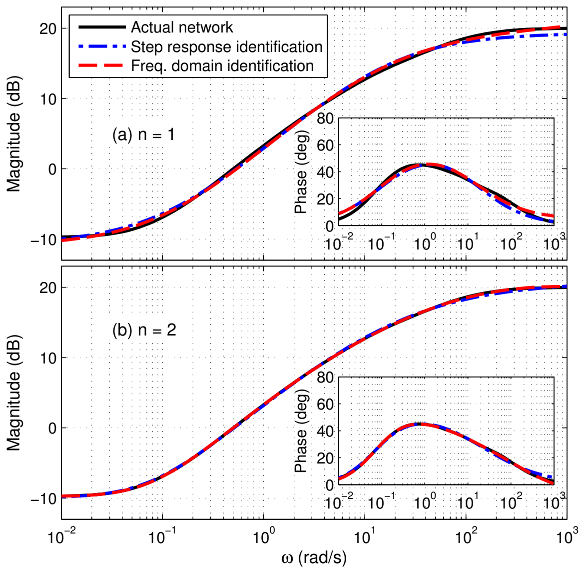

Fig. 7. Frequency domain results (fractional order models): Comparison with the frequency-domain identification method reported in [23].

V. Conclusion

This brief discussed the identification of fractional order transfer functions using a step excitation. The main advantage of this formulation is that fewer parameters need to be numerically optimized. Instead of optimizing the coefficients and fractional exponents of the transfer function, the proposed method only requires the optimization of the fractional exponents, as well as the parameter ρ employed in the definition of the auxiliary filtering parameters. Algorithmic implementation is straightforward using existing computational toolboxes.

Fractional order systems were generated in software using a nodal analysis of 3D RC circuits. These networks can be tailored in terms of topology and composition to emulate real time systems governed by fractional order dynamics. A single fractional exponent α was sufficient to obtain an excellent matching of the network dynamics in the time and frequency domains. Therefore, only two parameters (α, ρ) had to be numerically optimized. The proposed step response identification procedure has a very parsimonious formulation, matches very well the network step response, and is consistent with previous frequency domain formulations [23].

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