Space-efficient Indexing of Chess Endgame Tables

Article

Published Version

Nalimov, E. V., Haworth, G. M. and Heinz, E. A. (2000) Space-efficient Indexing of Chess Endgame Tables. ICGA Journal, 23 (3). pp. 148-162. ISSN 1389-6911 Available at https://centaur.reading.ac.uk/4562/

SPACE-EFFICIENT INDEXING OF CHESS ENDGAME TABLES[1]

E.V. Nalimov[2], G.McC. Haworth[3] and E.A. Heinz[4]

USA and England

ABSTRACT

Chess endgame tables should provide efficiently the value and depth of any required position during play. The indexing of an endgame's positions is crucial to meeting this objective. This paper updates Heinz' previous review of approaches to indexing and describes the latest approach by the first and third authors.

Heinz' and Nalimov's endgame tables (EGTs) encompass the *en passant* rule and have the most compact index schemes to date. Nalimov's EGTs, to the Distance-to-Mate (DTM) metric, require only 30.6×10^9 elements in total for all the 3-to-5-man endgames and are individually more compact than previous tables. His new index scheme has proved itself while generating the tables and in the 1999 World Computer Chess Championship where many of the top programs used the new suite of EGTs.

1. INTRODUCTION

The method used to index an endgame positions' values and depths largely determines both the space required and the speed of access during play over the board. It may aim to optimise the one or the other. A variety of approaches have been adopted as the challenges of larger and more complex endgames have been faced.

In this paper, Section 2 is an update of Heinz' review of indexing methods and Section 3 describes in detail Nalimov's new and more compact index scheme. Section 4 describes results achieved and Section 5 summarises and looks ahead to potential developments.

2. A REVIEW OF SOME INDEX SCHEMES

A previous paper (Heinz, 1999) surveyed, highlighted and analysed interesting work in the EGT field by Ströhlein (1970), Van den Herik and Herschberg (1985, 1986), Stiller (1989, 1991, 1994, 1995), Thompson (1986, 1991, 1996; ICCA J. Editors, 1992, 1993) and Edwards (1995). It presented a quantitative comparison of the index methods of Thompson (1986, 1996), Edwards (1995) and Heinz for all 3-to-4-man endgames.

Table 1, *q.v.* also (Heinz, 2000), extends that comparison to 5-man endgames using Thompson's indexes as the baseline. It infers the index range where the authors did not create the EGT, e.g., 4-1 and two-Pawn endgames. *X* = *Q*, *R*, *B* or *N* in Table 1 which makes it clear that different constraints were used by the EGT authors to reduce the size of the set of positions which they indexed.

Table 2, which includes the work of Wirth (1999), elicits these constraints and defines which of them have, in effect if not literally, been used by the EGT authors. The list below indicates that Edwards constrains the possible positions the least and Nalimov constrains them the most. For this reason, Edwards' index ranges are the largest and Nalimov's are the smallest. Heinz' EGTs made savings on the indexes of Thompson and Edwards which increase with the number of men, e.g., 3.13% for KxK, 7.67% for KxKy and 13.44% for KxyKz relative to Thompson's indexes.

The next subsections explain the rationale for three of the constraints.

[1] This is an edited version of the presentation by Ernst Heinz, delivered on June 18, 1999 at the *Advances in Computer Games 9* Conference in Paderborn, Germany, *q.v.* the Proceedings of the ACG 9 Conference.

[2] Microsoft Corporation, One Microsoft Way, Redmond, WA 98052-6399, USA: eugenen@microsoft.com.

[3] ICL, Sutton's Park Avenue, Reading, RG6 1AZ, UK: guy.haworth@icl.com.

[4] M.I.T. Laboratory for Computer Science (NE 43-228). 545 Technology Square, Cambridge, MA 02139, USA: heinz@mit.edu.

End-game	Edwards # Elements	Edwards +Δ%	Thompson # Elements	Thompson +Δ%	Heinz # Elements
KPK	32 * 64 * 64	33.33	24 * 64 * 64	-11.82	3612 * 24
KXK	10 * 64 * 64	38.53	462 * 64	-3.13	462 * 62
KPKP	32 * 64 * 64 * 64	77.78	24 * 48 * 64 * 64	-13.65	3612 * 24 * 47
KPPK	32 * 64 * 64 * 64	77.78	24 * 48 * 64 * 64	-55.90	3612 * 576
KPKX	32 * 64 * 64 * 64	33.33	24 * 64 * 64 * 64	-15.95	3612 * 24 * 61
KPXK	32 * 64 * 64 * 64	33.33	24 * 64 * 64 * 64	-15.95	3612 * 24 * 61
KXXK	10 * 64 * 64 * 64	38.53	462 * 64 * 64	-53.83	462 * 1891
KXYK	10 * 64 * 64 * 64	38.53	462 * 64 * 64	-7.67	462 * 62 * 61
KXKY	10 * 64 * 64 * 64	38.53	462 * 64 * 64	-7.67	462 * 62 * 61
KPPKP	32 * 64 * 64 * 64 * 64	137.04	24 * 48 * 48 * 64 * 64	-58.63	3612 * 24 * 1081
KPPPK	32 * 64 * 64 * 64 * 64	137.04	24 * 48 * 48 * 64 * 64	-86.15	3612 * 8684
KPPKX	32 * 64 * 64 * 64 * 64	77.78	24 * 48 * 64 * 64 * 64	-58.66	3612 * 576 * 60
KPPXK	32 * 64 * 64 * 64 * 64	77.78	24 * 48 * 64 * 64 * 64	-58.66	3612 * 576 * 60
KPXKP	32 * 64 * 64 * 64 * 64	77.78	24 * 48 * 64 * 64 * 64	-19.05	3612 * 24 * 47 * 60
KPXXK	32 * 64 * 64 * 64 * 64	33.33	24 * 64 * 64 * 64 * 64	-60.60	3612 * 24 * 1830
KXXKP	32 * 64 * 64 * 64 * 64	33.33	24 * 64 * 64 * 64 * 64	-60.60	3612 * 24 * 1830
KPXKY	32 * 64 * 64 * 64 * 64	33.33	24 * 64 * 64 * 64 * 64	-21.20	3612 * 24 * 61 * 60
KPXYK	32 * 64 * 64 * 64 * 64	33.33	24 * 64 * 64 * 64 * 64	-21.20	3612 * 24 * 61 * 60
KXYKP	32 * 64 * 64 * 64 * 64	33.33	24 * 64 * 64 * 64 * 64	-21.20	3612 * 24 * 61 * 60
KXXXK	10 * 64 * 64 * 64 * 64	38.53	462 * 64 * 64 * 64	-85.57	462 * 37820
KXXKY	10 * 64 * 64 * 64 * 64	38.53	462 * 64 * 64 * 64	-56.72	462 * 62 * 1830
KXXYK	10 * 64 * 64 * 64 * 64	38.53	462 * 64 * 64 * 64	-56.72	462 * 62 * 1830
KXYKZ	10 * 64 * 64 * 64 * 64	38.53	462 * 64 * 64 * 64	-13.44	462 * 62 * 61 * 60
KXYZK	10 * 64 * 64 * 64 * 64	38.53	462 * 64 * 64 * 64	-13.44	462 * 62 * 61 * 60

Table 1: Comparison of index range computations.

#	Identity	Constraint	KT	SE	EH	CW	EN
		Positions encoded					
1	C_W	wtm positions indexed	—	yes	yes	yes	yes
2	C_B	btm positions indexed	yes	yes	yes	yes	yes
		Placement of the Kings					
		Pawnless endgames					
3	C_8	stmK in a1-d1-d4	used	used	used	used	used
4	C_{KK1}	stmK and sntmK on separate squares	used	—	used	used	used
5	C_{TE}	if stmK on a1-d4, stmK in a1-h1-h8	used	—	used	used	used
6	C_{KKnP}	exactly 462 wK-bK positions used	used	—	used	used	used
		Endgames with Pawns					
7	C_{ad}	stmK in a-d	used	used	used	used	used
8	C_{KK2}	stmK and sntmK on separate squares	—	—	used	used	used
9	C_{KKP}	exactly 1806 wK-bK positions used	—	—	used	used	used
		Encoding Pawn positions					
10	C_P	Pawns constrained to ranks 2-7	used	—	used	used	used
11	C_{EP}	Pawns capturable *en passant* included	—	—	used	used	used
		Like men, i.e. of the same type and colour					
12	C_{LM}	Saving of *k!* for *k* like men	—	—	used	used	used
		Constraints on squares with more than one man					
13	$C_{S1\text{-}MM}$	No square with two men	—	—	—	—	—
14	$C_{S2\text{-}KPC}$	No square with K and another piece	—	—	used	used	used
15	$C_{S3\text{-}KPW}$	No square with K and a Pawn	—	—	—	—	used
16	$C_{S4\text{-}L1}$	No square with two like pieces	—	—	used	used	used
17	$C_{S5\text{-}L2}$	No square with two like Pawns	—	—	—	used	used
18	$C_{S6\text{-}SNTM1}$	No square with stm man and sntm piece	—	—	used	used	used
19	$C_{S7\text{-}SNTM2}$	No square with man and sntm Pawn	—	—	—	—	—
		Unblockable checks by the stm					
20	C_{UC}	No unblockable checks allowed	—	—	—	—	used
		Trimming the index-range					
21	C_F	First positions in a range not *broken*	—	—	—	—	—
22	C_L	Last positions in a range not *broken*	—	—	—	—	used

Table 2: Constraints available to limit the position-sets indexed.[5]

[5] Thompson (KT), Edwards (SE), Heinz (EH), Wirth (CW) and Nalimov (EN).

2.1 Constraining a King

A King is typically constrained to files a-d for endgames with Pawns and to the octant a1-d1-d4 for endgames without Pawns. The choice of the side-to-move King, *stmK,* as the man to constrain has two advantages:

- the stm King is always present so the constraint can always be exercised,
- there is only one stm King so the effect of the constraint is unambiguous.

In contrast, had a Rook been the constrained man, the software generating and accessing the EGTs would have to decide between the positions below with a Rook on b1 and on d3 respectively.

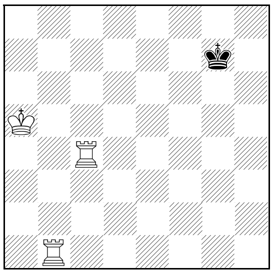

Figure 1: Version 1.

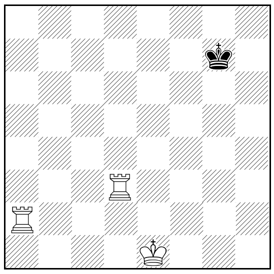

Figure 2: Version 2.

2.2 Like Men of the Same Type

Where one side has k men of one type, the index range may be reduced by a factor of $k! = k \times (k - 1) \times ... \times 1$. The $k!$ arrangements of k like, labelled men on q given squares are equivalent if the like men are unlabelled. There are $d = C_{q,k} = q!/[k!\,(q - k)!]$ placements of k like men on q squares where $0! \equiv 1$ by definition.

Let 0 ... (q - 1) be the numbers of the available squares and 0 ... (d - 1) the numbers of the *k-tuple* placements of the k like men. The method of transforming one k-tuple into the next determines the numbering:

$\{0, 1\} \rightarrow \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ ... advancing the highest-numbered man,
$\{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 3\}, \{1, 3\} \leftarrow \{2, 3\}$ ... retreating the lowest-numbered man.

For the first ordering, the placement $\{s_1, s_2, ... , s_k\}$ of the men on squares $\{s_i \mid i < j \Rightarrow s_i < s_j\}$ is given index r by the algorithm:

$r = 0$;
while $k > 0$ **do**
  while $s_1 \neq 0$ **do** $r \leftarrow r + (q - 1)!/[(k - 1)!(q - k)!]$; $q \leftarrow q - 1$;
    {'discard' square 0} **for** $i = 1$ **to** k **do** $s_i \leftarrow s_i - 1$ **end_do**;
  end_do;
  {'discard' square 0 and the man on square 0}
  $k \leftarrow k - 1$; $q \leftarrow q - 1$; **for** $i = 1$ **to** k **do** $s_i = s_{i+1} - 1$ **end_do**;
end_do

In the second ordering, the placement $\{s_1, s_2, ... , s_k\}$ of the men on squares $\{s_i \mid i < j \Rightarrow s_i < s_j\}$ is preceded by placements $\{\{t_1, t_2, ... , t_k\} \mid t_i < s_j\ (i \leq j)\ \&\ t_i = s_i, (i = j + 1, ..., k); j = 1, ... k\}$. It is therefore given index r by the succinct formula:

n_j = the number of j-tuples of ordered integers taken from $[0, s_j - 1] = s_j \times (s_j - 1) \times ... \times ... (s_j - j + 1)/j!$; $r = \Sigma_j\, n_j$.

Thompson, Stiller (1991, 1994, 1995) and Edwards did not take advantage of this economy. Heinz (1999, 2000), Wirth (1999) and Nalimov (1999) do and constrain like pieces[6], but not necessarily like Pawns, from sharing squares. The appendix features some studies with the theme of esoteric force, that is, unlikely numbers of like men.

[6] A *piece* is a *man* which is not a Pawn.

2.3 First and Last Index not *Broken*

If the highest indices in an *addressable subrange* of the index are marked *broken*[7] during the EGT initialisation process, they may simply be removed. If the lowest indices in an addressable subrange are marked broken, they may also be removed but the baseline of the remaining index subrange must be correspondingly reduced. Some illegal positions need not require access to the EGT if the access code incorporates illegality tests.

3. NALIMOV'S INDEX SCHEME

The first author has made publicly available an EGT generator and a complete set of 3-to-5-man and some 6-man EGTs to the Distance to Mate metric (cf. Hyatt, 2000). The main objectives of their construction are that:

- the colours White and Black are treated symmetrically
  separate indexes and files for wtm and btm positions; data on both 1-0 and 0-1 wins,
- the EGTs should be practical and efficient to use during play over the board
  the index for each endgame is the most compact yet produced,
  time-optimal 8KB EGT blocks of compressed data are decompressed in store,
  positions for a set configuration of the stm men are clustered together.

This latest index scheme uses the following approach, many of whose principles and optimisations were first articulated by Heinz (1999, 2000):

- the men are notionally placed on the board in the following order:
  stmK, sntmK, stm men (Q-R-B-N-P), sntm men (Q-R-B-N-P),
- the stmK-sntmK positions are used explicitly: 462 (no Ps) and 1806 (Ps)
  the *index range* therefore consists of 462 or 1806 *index subranges,*
- 'available' squares are numbered 0 ... q - 1 in order a1-...- h1 - a2 - ... - h8,
- the number of squares available to men of a type is calculated knowing:
  the positions of the Kings and the presence of previously-placed men.
  Each *index subrange* for an stmK-sntmK placement is therefore an *n-space*
- k like men of one colour are placed as a set with economy factor $k!$
- stm men cannot be placed giving an *unblockable check*[8] to the sntmK,
- positions allowing an *en passant* capture are indexed in a separate zone.

The net effect is that:

- the squares occupied by the two Kings are not available to any other man,
- the sntm's pieces occupy only previously-unoccupied squares,
- different types of stm pieces share squares in some indexed 'positions',
- 'positions' with Pawns on pieces' squares are indexed.

Nalimov's work can be seen as a significant evolution of Edwards' work which addressed the same objectives but which used less of the available constraints while indexing the positions. The next subsections focus on:

- avoiding unblockable checks, reducing the size of each index subrange
- calculating the dimensions of the n-space index subrange
- creating the complete EGT index
- calculating the index of a given position
- indexing positions with the features of *en passant* and/or castling rights
- improving the performance of EGT access.

3.1 Avoiding Unblockable Checks

Let us suppose White is to move: Black cannot be in check. Figure 3 shows that White's men cannot be placed on certain squares as they would give a check which could not be blocked by placing a further man on the board. Thus, Black's King and White's forces constrain the number of arrangements of White's men.

---

[7] A *broken* index entry denotes an illegal, unwanted or no position.
[8] An *unblockable check* cannot be blocked by placing a man on the board.

The index range for wtm positions will therefore in general be different from that for btm positions. Given the lexicographical way in which endgames are listed, the wtm index range is almost always[9] less than the btm index range. Where White and Black have the same men, only the btm half of the EGT is computed: the access method flips colours if presented with a wtm position.

White checks from other squares, as in Figure 4, may or may not be blocked by the placement of further men. Positions featuring such checks are indexed but if the sntmK is in check, their indexes are marked as *broken* during the initialisation phase.

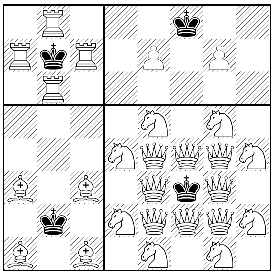

Figure 3: wtm, unblockable checks.

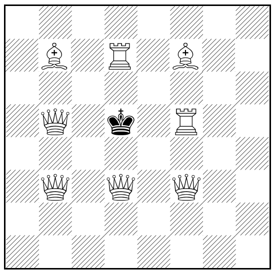

Figure 4: wtm, blockable checks.

wK	bK	wQ	wR	wB	wN	wP
any	a1	59	60	61	60-61	47-48
any	b1	57	59	60	59-60	47-48
any	c1	57	59	60	58-59	47-48
any	b2	54	58	58	58-59	46-47
a1	c2	54	58	58	56	47
a2	c2	54	58	58	56	46
a3	c2	54	58	58	57	46
any	a3	57	59	60	58-59	45-46
any	c3	54	58	58	54-55	44-45

Table 3: The squares ‘available’ to each white man with wtm.

52	53	54	55	56	57	58	59
44	45	46	47	48	49	50	51
36	37	38	39	40	41	42	43
28	29	30	31	32	33	34	35
20	21	22	23	24	25	26	27
12	13	14	15	16	17	18	19
6	7	8	9	10	11		
0	1	2	3	4	5		♚

Table 4: wQ squares for bKh1, wtm.

47	48	49	50	51	52	53	54
39	40	41	42	43	44	45	46
31	32	33	34	35	36	37	38
23	24	25	26	27	28	29	30
15	16	17	18	19	20	21	22
10				11	12	13	14
5		♚		6	7	8	9
0				1	2	3	4

Table 5: wQ squares for bKc2, wtm.

With White to move, each of the black King’s 64 positions determines the number of squares available for each white man, Q, R, B, N or P, as in Table 3. To improve efficiency, Nalimov computes for each man a 64 × 64 table giving the reference numbers, for each position of the sntmK, of the squares available to that man. These numbers are modified, given the position of the stmK.

Thus, Tables 4 and 5 give the numbers of the squares available to the wQ in wtm positions with the bK on h1 and c2 respectively. When the square of the wK is known, the numbers of the higher-numbered squares decrement by one. The chief reason for the compactness of the indexes described here is the reduction in the number of squares available to men of type *i* by the avoidance of unblockable checks.

[9] The 3-5 man exceptions are KBKN, KBKP, KRKN, KRKP, KBBKQ, KRBKQ, KBPKQ and KRBKP.

3.2 The *N*-Space Index Subranges

The wtm and btm index ranges are 462 or 1806 subranges, each an *n*-space associated with a specific wK-bK placement. Let the *qi* squares available to the *ki* non-King men of type *i* (*i* = 1, ... *t*) be numbered 0 ... *qi* - 1. Then:

- *qi* is determined as above by the stm, the King positions, the type *i*, and the prior men placed
- there are $di = C_{qi,\,ki} = qi! / [ki!(qi - ki)!]$ placements, 0 ... *di*-1, of the type *i* men
- the index subrange is the *n*-space [*d1*, *d2*, ... , *dt*], dimension *t*, size $\Pi_i\, di$
- the subranges' first entries $\{ind_{KK}\}$ index the wK-bK-position subsets.

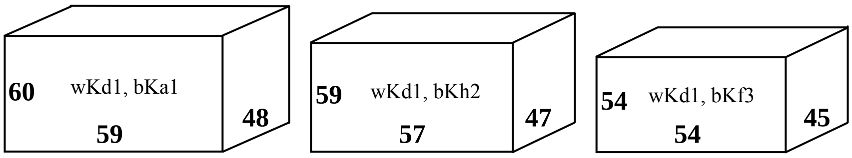

Figure 5: The wtm KQRPK index subranges for three bK positions.

Figure 5 illustrates the index subranges for wtm KQNPK with the wK on d1 and the bK on a1, h2 and f3. The wQ ranges in turn over 59, 57 and 54 squares, the wN ranges over 60, 59 and 54 squares, and the wP ranges over 48, 47 and 45 squares.

A more complex wtm example in the endgame KRRNKP illustrates a calculation involving two like men and also the wK occupying a square denied to the wN. With the wK on a1 and the bK on c2, the white Rooks have 58 squares available and, placed as a set, have 58×57/2 = 1653 placements. The wK occupies a square from which a wN would give an unblockable check. Therefore, the number of squares available to the wN, ignoring as Nalimov does the prior placement of the Rooks, is 57. There are 47 squares at most available to the bP and on some of these, the bP will be sharing a square with a white man. This sub-index *n*-space therefore has dimensions and size 1653 × 57 × 47 = 4,428,387.

3.3 EGT Index Size

Table 6 illustrates, with the wtm index of endgame KQRK, the impact of minimising the number of squares *qi* available to men of type *i*. The economy of this index approach is clear when compared with other possibilities.

The lookup tables which effect and expedite the indexing occupy some 200KBytes per 3-2 endgame and up to 350KBytes for 4-1 endgames.

Constraints	Notes	Computation	Size
————	The naive index-scheme	64 * 64 * 64 * 64	16,777,216
$C_{s1\text{-}mm}$	no square shared	64 * 63 * 62 * 61	15,249,024
C_8	Edwards' index-range	10 * 64 * 64 * 64	2,621,440
C_8 & $C_{s1\text{-}mm}$	wK in octant; no square shared	10 * 63 * 62 * 61	2,382,660
C_{KKnP}	Thompson's index-range	462 * 64 * 64	1,892,352
C_{KKnP} & $C_{s1\text{-}mm}$	Heinz' and Wirth's index-range	462 * 62 * 61	1,747,284
C_{KKnP} & $C_{uc(Q)}$	3 squares denied to the wQ	462 * 59 * 61	1,662,738
C_{KKnP} & $C_{uc(QR)}$	... and 2 sq. denied to the wR	462 * 59 * 59	1,608,222
C_{KKnP} & C_{uc} & C_L	Nalimov's index-range	(57*58 + ...) - 366	1,500,276
C_{KKnP} & C_{uc} & $C_{s1\text{-}mm}$ & C_L	Nalimov, but no square shared	(57*57 + ...) - 360	1,474,713

Table 6: Index ranges for wtm KQRK positions under various constraints.

The calculations for different types of man allow men to occupy the same square, e.g., in KQRK, KQPK or KQKP. However, the net reduction in the index ranges is significant and certainly much greater than the workspace required for the lookup tables.

3.4 The Index of a Position

As in Subsection 2.2, let the men of type *i* be placed on squares $\{s_{i,1}, \dots, s_{i,ki}\}$ as numbered for their type given prior placements. Then:

- the type *i* men are deemed to be in placement $ri \in [0, di - 1]$, $i = 1 \dots t$,
- the position has co-ordinates $[r1, \dots, rt]$ in the *n*-space $[d1, d2, \dots, dt]$,
- the position's *n-space index*, $subind = \sum_i r_i \times \Pi_{j>i}\, d_j$ where $j \le t + 1$ and $d_{t+1} \equiv 1$,
- assuming KK-placement $\kappa\kappa$, the position's index in the EGT is $ind_{\kappa\kappa} + subind$.

3.5 Indexing the *En Passant* Positions

RETRO (Forthoffer, Rasmussen and Dekker, 1989) uniquely generated EGTs recognising both *en passant* capture and castling. Recently, Heinz, Moreland, Nalimov (Heinz, 2000) and Wirth (1999) have indexed the positions featuring a possible *en passant* capture. Nalimov does so in a separate zone of the stm index after the main index. Let us assume that it is btm. A white Pawn will be on *x4*, *x* in *a-h*, and a black Pawn will be on an adjacent file, giving 14 potential placements of these two Pawns instead of 2,256. Further, as White has just moved a Pawn from *x2* to *x4*, squares *x2* and *x3* are not available to be occupied by other men.

Kings are still placed in their 1806 positions and stm pieces are still constrained by the avoidance of unblockable checks.

The concept of a separate index zone for positions with a specific feature, in this case *potential e.p. capture*, generalises to the provision of separate index zones for positions with specific subsets of the five features:

stm can make an *en passant* capture,
White and/or Black can castle on the a-side and/or the h-side.

The full representation of *en passant* and castling rights, not included in Nalimov's EGTs, involves 2^5 zones of positions rather than the usual one zone. However, as each feature constrains at least one man and reduces the index range by a factor of at least 60, 31 of the zones are relatively small. It may be helpful to place constrained men first but no fundamentally new principles of indexing are required.

3.6 EGT Access Performance

Because White, for example, submits a number of btm positions to the EGT, the placement of stm (black) men before their sntm equivalents also tends to cluster White's accesses to the file. Also, because chess engines probe the EGT at several nodes in their search tree, Nalimov wrote an efficient lookup function which manages an LRU, least-recently used, cache of EGT values. Experiments with CRAFTY show that the new index scheme facilitates much better caching behaviour than others, particularly with parallel search on symmetric multiprocessors.

Nalimov's EGT files are compressed into 8KB blocks, the technique exploiting common sequences and Huffman coding. The block size optimises runtime performance rather than space. It is usually more efficient to decompress the blocks at runtime in store than to work with uncompressed files.

All Endgames	Nalimov	Heinz	Thompson	Edwards
# Elements, wtm	14,702,353,093	16,807,619,304	25,936,842,240	37,046,484,992
Extra Elements	———	2,105,266,211	11,234,489,147	22,344,131,899
+Δ%	———	14.32	76.41	151.98
# Elements, btm	15,909,833,876	16,807,619,304	25,936,842,240	37,046,484,992
Extra Elements	———	897,785,428	10,027,008,364	21,136,651,116
+Δ%	———	5.64	63.02	132.85
# Elements, all	30,612,186,969	33,615,238,608	51,873,684,480	74,092,969,984
Extra Elements	———	3,003,051,639	21,261,497,511	43,480,783,015
+Δ%	———	9.81	69.45	142.04

Table 7: Summary of 3-to-5-man index range sizes.

	1-0		0-1	
Endgame	**wtm**	**btm**	**wtm**	**btm**
KBBBK	16	19	—	—
KBBK	19	19	—	—
KBBKB	22	22	1	2
KBBKN	78	78	0	1
KBBKP	74	73	82	83
KBBKQ	21	20	81	81
KBBKR	23	22	30	31
KBBNK	33	33	—	—
KBBPK	30	31	—	—
KBK	—	—	—	—
KBKB	1	0	0	1
KBKN	1	0	0	1
KBKP	1	0	19	29
KBNK	33	33	—	—
KBNKB	39	39	1	2
KBNKN	107	106	0	1
KBNKP	104	104	54	55
KBNKQ	36	35	53	53
KBNKR	36	35	39	41
KBNNK	34	34	—	—
KBNPK	33	33	—	—
KBPK	31	31	—	—
KBPKB	51	50	2	3
KBPKN	100	96	7	8
KBPKP	67	67	50	51
KBPKQ	35	34	50	50
KBPKR	45	44	38	39
KBPPK	25	32	—	—
KK	—	—	—	—
KNK	—	—	—	—
KNKN	1	0	0	1
KNKP	7	6	28	29
KNNK	1	0	—	—
KNNKB	4	3	0	1
KNNKN	7	6	0	1
KNNKP	115	114	73	74
KNNKQ	1	0	72	72
KNNKR	3	2	40	41
KNNNK	21	21	—	—
KNNPK	28	28	—	—
KNPK	27	28	—	—
KNPKB	43	42	8	9
KNPKN	97	97	3	7
KNPKP	57	57	57	58
KNPKQ	41	33	62	55
KNPKR	44	43	66	67
KNPPK	32	32	—	—
KPK	28	28	—	—
KPKP	33	33	33	33

	1-0		0-1	
Endgame	**wtm**	**btm**	**wtm**	**btm**
KPPK	32	32	—	—
KPPKB	43	43	3	4
KPPKN	50	50	16	17
KPPKP	127	127	42	43
KPPKQ	124	100	41	41
KPPKR	54	53	41	40
KPPPK	33	33	—	—
KQBBK	6	19	—	—
KQBK	8	10	—	—
KQBKB	17	17	1	2
KQBKN	21	21	0	1
KQBKP	32	33	17	24
KQBKQ	33	33	23	24
KQBKR	40	40	25	30
KQBNK	7	33	—	—
KQBPK	9	31	—	—
KQK	10	10	—	—
KQKB	17	17	—	—
KQKN	21	21	—	—
KQKP	28	28	10	29
KQKQ	13	12	12	13
KQKR	35	35	18	19
KQNK	9	10	—	—
KQNKB	17	17	0	1
KQNKN	21	21	0	1
KQNKP	30	41	22	29
KQNKQ	41	41	23	24
KQNKR	38	38	38	41
KQNNK	8	9	—	—
KQNPK	9	27	—	—
KQPK	10	28	—	—
KQPKB	28	29	1	2
KQPKN	30	30	7	8
KQPKP	105	122	14	34
KQPKQ	124	123	28	29
KQPKR	37	43	27	33
KQPPK	9	32	—	—
KQQBK	4	8	—	—
KQQK	4	10	—	—
KQQKB	15	17	—	—
KQQKN	19	21	—	—
KQQKP	22	30	2	13
KQQKQ	30	30	12	13
KQQKR	35	35	2	19
KQQNK	4	9	—	—
KQQPK	4	10	—	—
KQQQK	3	4	—	—
KQQRK	4	6	—	—
KQRBK	5	16	—	—

	1-0		0-1	
Endgame	**wtm**	**btm**	**wtm**	**btm**
KQRK	6	16	—	—
KQRKB	29	29	—	—
KQRKN	40	40	0	1
KQRKP	40	67	35	43
KQRKQ	67	67	37	38
KQRKR	34	35	2	20
KQRNK	5	16	—	—
KQRPK	7	16	—	—
KQRRK	4	7	—	—
KRBBK	12	19	—	—
KRBK	16	16	—	—
KRBKB	30	30	1	2
KRBKN	40	40	0	1
KRBKP	28	36	65	70
KRBKQ	21	20	70	70
KRBKR	65	64	26	30
KRBNK	29	33	—	—
KRBPK	16	31	—	—
KRK	16	16	—	—
KRKB	29	29	—	—
KRKN	40	40	0	1
KRKP	26	32	42	43
KRKR	19	19	19	19
KRNK	16	16	—	—
KRNKB	31	31	0	1
KRNKN	37	40	0	1
KRNKP	29	29	63	68
KRNKQ	20	19	69	69
KRNKR	37	36	39	41
KRNNK	15	16	—	—
KRNPK	17	27	—	—
KRPK	16	28	—	—
KRPKB	73	73	1	2
KRPKN	54	54	7	8
KRPKP	56	68	100	103
KRPKQ	68	59	103	104
KRPKR	74	74	28	33
KRPPK	15	32	—	—
KRRBK	10	16	—	—
KRRK	7	16	—	—
KRRKB	29	29	—	—
KRRKN	40	40	0	1
KRRKP	33	40	40	50
KRRKQ	29	28	49	49
KRRKR	31	31	2	20
KRRNK	10	16	—	—
KRRPK	14	16	—	—
KRRRK	5	7	—	—

Table 8: Maximal DTM figures for 1-0 and 0-1 wins, wtm and btm.

4. RESULTS

The first author has computed all 3-to-5-man DTM EGTs (Hyatt, 2000; Tamplin, 2000). His robust code also generated KQQKQQ on request for the Kasparov-World game (Nalimov, Wirth, and Haworth, 1999) and has now produced further 6-man EGTs including the deepest to date, KRNKNN.

The space-efficient index scheme incorporates the *en passant* rule and requires only 30.6×10^9 elements in total for the 3-to-5-man endgames. It is better for each endgame than previous schemes. By comparison, Heinz' scheme would have required 33.6×10^9 (+9.81%), Thompson's 51.9×10^9 elements (+69.45%) and Edwards' 74.1×10^9 elements (+142.04%), see Table 7.

The question of data integrity always arises with results which are not self-evidently correct. Nalimov runs a separate self-consistency phase on each EGT after it is generated. Both his EGTs and those of Wirth (1999) yield exactly the same number of mutual zugzwangs of each type (=/1-0, 0-1/= and 0-1/1-0) for all 2-to-5-man endgames (Haworth, 2000) and no errors have yet been discovered.

DARKTHOUGHT (Heinz, 1997), using Heinz' index-scheme and EGTs, competed in WMCC 1997 (Hamlen and Feist, 1997) and WCCC 1999 (Beal, 1999). Nalimov's new index scheme has proved its practicality over the board, particularly in WCCC 1999 where it was used by ten competitors including the leading SHREDDER, FRITZ, JUNIOR and NIMZO.

Table 8 gives the depths of DTM-maximal 1-0 and 0-1 wins, wtm and btm. The tables in the Appendix compare Nalimov's index sizes with others' and the statistics on residual broken positions in Nalimov's EGTs. This is the most complete tabulation of 2-5-man endgame data published so far.

5. SUMMARY

The index design is the key to computing compact and efficiently used chess endgame tables. The first author has exploited the available constraints on the positions to be indexed in the best way to date.

The result is that a robust and efficient EGT generation code, a complete suite of 145 3-to-5-man EGTs, and some 30 6-man EGTs are now publicly available.

Further progress in the compression of index ranges is possible. There can be less occurrences of men sharing squares if Pawns are notionally placed first (Karrer, 2000) and the presence of prior stm men is recognised.

6. ACKNOWLEDGEMENTS

Our thanks to an anonymous referee who contributed the succinct formula of Subsection 2.2 for indexing k like men. Also, thanks to Helmut Conrady, Peter Karrer and Lars Rasmussen whose combined contribution confirmed the maxDTM figures of Table 8 with two corrections, and substantiated them with the sets of maxDTM positions.

7. REFERENCES

Beal, D.F. (1999). The 9th World Computer-Chess Championship: the Search-Engine Features of the Programs. *ICCA Journal*, Vol. 22, No. 3, pp. 160-163. ISSN 0920-234X.

Beasley J. & Whitworth, T. (1996). *Endgame Magic*, esp. p. 158. Batsford. ISBN 0-7134-7971-X.

Bondar, I. (1998). Four Queen study. *Shakhmatnaya kompozitsia* No. 22. See also *EG* No. 137, p. 188 (2000).

Edwards, S.J. (1995). Comments on Barth's Article "Combining Knowledge and Search to Yield Infallible Endgame Programs." *ICCA Journal*, Vol. 18, No. 4, p. 219-225.

Elkies, N. (2000). SSSS-Q. *EG*, No. 137, p. 158. ISSN-0012-7671.

Forthoffer, D., Rasmussen, L. and Dekker, S.T. (1989). A Correction to Some KRKB-Database Results. *ICCA Journal*, Vol. 12, No. 1, pp. 25-27.

Hamlen, J. and Feist, M. (1997). Report on the 15th World Microcomputer Chess Championship. *ICCA Journal*, Vol. 20. No. 4, pp. 254-255.

Haworth, G.McC. (2000). Private communication to Karrer, Nalimov, Rasmussen and Wirth.

Heinz, E.A. (1997). How DARKTHOUGHT plays Chess. *ICCA Journal*, Vol. 20, No. 3, pp. 166-176.

Heinz, E.A. (1999). Endgame Databases and Efficient Index Schemes for Chess. *ICCA Journal*, Vol. 22, No. 1, pp. 22-32.

Heinz, E.A. (2000). *Scalable Search in Computer Chess*. Vieweg Verlag (Morgan Kaufmann), Braunschweig/Wiesbaden. ISBN 3-5280-5732-7.

Herik, H.J. van den and Herschberg, I.S. (1985). The Construction of an Omniscient Endgame Database. *ICCA Journal*, Vol. 8, No. 2, pp. 66-87.

Herik, H.J. van den and Herschberg, I.S. (1986). A Data Base on Data Bases. *ICCA Journal*, Vol. 9, No. 1, pp. 29-34.

Hyatt, R. (2000). ftp://ftp.cis.uab.edu/pub/hyatt/TB/. Server providing CRAFTY and Nalimov's EGTs and statistics.

ICCA J. Editors (1992). Thompson: All About Five Men. *ICCA Journal*, Vol. 15, No. 3, pp. 140-143.

ICCA J. Editors (1993). Thompson: Quintets with Variations. *ICCA Journal*, Vol. 16, No. 2, pp. 86-90.

Karrer, P. (2000). KQQKQP and KQPKQP≈. *ICCA Journal*, Vol. 23, No. 2, pp. 75-84.

Nalimov, E.V., Wirth, C., and Haworth, G.M[c]C. (1999). KQQKQQ and the Kasparov-World Game. *ICCA Journal*, Vol. 22, No. 4, pp. 195-212.

Savin, P.I. and Plaksin, N.M. (1987). *The Art of Chess Composition*. Kishinev.

Stiller, L.B. (1989). Parallel Analysis of Certain Endgames. *ICCA Journal*, Vol. 12, No. 2, pp. 55-64.

Stiller, L.B. (1991). Group Graphs and Computational Symmetry on Massively Parallel Architecture. *The Journal of Supercomputing*, Vol. 5, No. 2, pp. 99-117.

Stiller, L.B. (1994). Multilinear Algebra and Chess Endgames. *Games of No Chance* (ed. R.J. Nowakowski), pp. 151-192. MSRI Publications, v29, CUP, Cambridge, England. ISBN 0-5215-7411-0. Reprinted in paperback (1996). ISBN 0-5216-4652-9.

Stiller, L.B. (1995). *Exploiting Symmetry of Parallel Architectures.* Ph.D. Thesis, Department of Computer Sciences, The John Hopkins University, Baltimore, Md.

Ströhlein, T. (1970). *Untersuchungen über kombinatorische Spiele*. Dissertation, Fakultät für Allgemeine Wissenschaften der Technischen Hochschule München.

Tamplin, J. (2000). http://chess.liveonthenet.com/chess/endings/index.shtml. Access to Thompson's 5-man EGTs and maxDTC positions, and to Nalimov's 3- to 6-man EGTs.

Thompson, K. (1986). Retrograde Analysis of Certain Endgames. *ICCA Journal*, Vol. 9, No. 3, pp. 131-139.

Thompson, K. (1991). Chess Endgames Vol. 1. *ICCA Journal*, Vol. 14, No. 1, p. 22.

Thompson, K. (1996). 6-Piece Endgames. *ICCA Journal*, Vol. 19, No. 4, pp. 215-226.

Troitzkiĭ, A.A. (1912). KNNNNKQ Study. *Deutsche Schachzeitung*.

Troitzkiĭ, A.A. (1934). Два коня против пешек (теоретический очерк). Сборник шахматных этюдов, pp. 248-288. Leningrad. [Dva Konya protiv pešek. Sbornik šakhmatnykh étyudov.] Partly republished (1937) in *Collection of Chess Studies, with a Supplement on the Theory of the End-Game of Two Knights against Pawns.* (trans. A.D. Pritzson), David McKay Company, the latter again re-published (1985) by Olms, Zürich.

Wirth, C. and Nievergelt, J. (1999). Exhaustive and Heuristic Retrograde Analysis of the KPPKP Endgame. *ICCA Journal*, Vol. 22, No. 2, pp. 67-80.

APPENDIX

This appendix provides complete data covering all 2-to-5-man endgames. Tables 9-13 compare the index sizes of Thompson's, Edwards' and Heinz' EGTs with the index size of Nalimov's EGTs as follows:

3-man endgames (Table 9), 4-man endgames (Table 10), 3-2-man pawnless endgames (Table 11), 3-2-man endgames with Pawns (Table 12) and 4-1 man endgames (Table 13).

Tables 14a and 14b give the number and % of residual *broken* positions per endgame in Nalimov's EGTs.

Three studies with unlikely numbers of like men are featured here, *q.v.* Figures 6-8 below.

	wtm	KT	SE	EH	btm	KT	SE	EH
Endgame	**# Elements**	+Δ%	+Δ%	+Δ%	**# Elements**	+Δ%	+Δ%	+Δ%
KBK	27,243	8.53	50.35	5.14	28,644	3.23	43.00	0.00
KNK	26,282	12.50	55.85	8.99	28,644	3.23	43.00	0.00
KPK	81,664	20.38	60.50	6.15	84,012	17.01	56.02	3.19
KQK	25,629	15.37	59.82	11.76	28,644	3.23	43.00	0.00
KRK	27,030	9.39	51.54	5.97	28,644	3.23	43.00	0.00
Aggregate	187,848	15.29	57.00	7.14	198,588	9.06	48.50	1.35

Table 9: Comparison of index sizes for 3-man endgames.

	wtm	KT	SE	EH	btm	KT	SE	EH
Endgame	**# Elements**	+Δ%	+Δ%	+Δ%	**# Elements**	+Δ%	+Δ%	+Δ%
KBBK	789,885	139.57	231.88	10.60	873,642	116.60	200.06	0.00
KBKB	1,661,823	13.87	57.74	5.14	1,661,823	13.87	57.74	5.14
KBKN	1,661,823	13.87	57.74	5.14	1,603,202	18.04	63.51	8.99
KBKP	5,112,000	23.07	64.10	3.44	4,981,504	26.30	68.40	6.15
KBNK	1,550,620	22.04	69.06	12.68	1,747,284	8.30	50.03	0.00
KBPK	4,817,128	30.61	74.14	9.77	5,124,732	22.77	63.69	3.19
KNKN	1,603,202	18.04	63.51	8.99	1,603,202	18.04	63.51	8.99
KNKP	4,931,904	27.57	70.09	7.22	4,981,504	26.30	68.40	6.15
KNNK	735,304	157.36	256.51	18.81	873,642	116.60	200.06	0.00
KNPK	4,648,581	35.34	80.46	13.75	5,124,732	22.77	63.69	3.19
KPKP	3,863,492	22.13	117.13	5.46	3,863,492	22.13	117.13	5.46
KPPK	1,806,671	161.18	364.31	15.16	1,912,372	146.74	338.65	8.79
KQBK	1,512,507	25.11	73.32	15.52	1,747,284	8.30	50.03	0.00
KQKB	1,563,735	21.01	67.64	11.74	1,661,823	13.87	57.74	5.14
KQKN	1,563,735	21.01	67.64	11.74	1,603,202	18.04	63.51	8.99
KQKP	4,810,080	30.80	74.40	9.94	4,981,504	26.30	68.40	6.15
KQKQ	1,563,735	21.01	67.64	11.74	1,563,735	21.01	67.64	11.74
KQKR	1,563,735	21.01	67.64	11.74	1,649,196	14.74	58.95	5.95
KQNK	1,459,616	29.65	79.60	19.71	1,747,284	8.30	50.03	0.00
KQPK	4,533,490	38.78	85.04	16.64	5,124,732	22.77	63.69	3.19
KQQK	698,739	170.82	275.17	25.03	873,642	116.60	200.06	0.00
KQRK	1,500,276	26.13	74.73	16.46	1,747,284	8.30	50.03	0.00
KRBK	1,594,560	18.68	64.40	9.58	1,747,284	8.30	50.03	0.00
KRKB	1,649,196	14.74	58.95	5.95	1,661,823	13.87	57.74	5.14
KRKN	1,649,196	14.74	58.95	5.95	1,603,202	18.04	63.51	8.99
KRKP	5,072,736	24.02	65.37	4.24	4,981,504	26.30	68.40	6.15
KRKR	1,649,196	14.74	58.95	5.95	1,649,196	14.74	58.95	5.95
KRNK	1,538,479	23.00	70.39	13.57	1,747,284	8.30	50.03	0.00
KRPK	4,779,530	31.63	75.51	10.64	5,124,732	22.77	63.69	3.19
KRRK	777,300	143.45	237.25	12.39	873,642	116.60	200.06	0.00
Aggregate	72,662,274	34.34	87.60	9.97	76,439,484	27.70	78.33	4.54

Table 10: Comparison of index sizes for 4-man endgames.

	wtm	**KT**	**SE**	**EH**	**btm**	**KT**	**SE**	**EH**
Endgame	**# Elements**	**+Δ%**	**+Δ%**	**+Δ%**	**# Elements**	**+Δ%**	**+Δ%**	**+Δ%**
KBBKB	47,393,100	155.54	254.00	10.60	49,854,690	142.93	236.52	5.14
KBBKN	47,393,100	155.54	254.00	10.60	48,096,060	151.81	248.83	8.99
KBBKQ	47,393,100	155.54	254.00	10.60	46,912,050	158.17	257.63	11.74
KBBKR	47,393,100	155.54	254.00	10.60	49,475,880	144.79	239.10	5.95
KBNKB	93,037,200	30.17	80.33	12.68	99,709,380	21.46	68.26	5.14
KBNKN	93,037,200	30.17	80.33	12.68	96,192,120	25.90	74.41	8.99
KBNKQ	93,037,200	30.17	80.33	12.68	93,824,100	29.08	78.82	11.74
KBNKR	93,037,200	30.17	80.33	12.68	98,951,760	22.39	69.55	5.95
KNNKB	44,118,240	174.51	280.28	18.81	49,854,690	142.93	236.52	5.14
KNNKN	44,118,240	174.51	280.28	18.81	48,096,060	151.81	248.83	8.99
KNNKQ	44,118,240	174.51	280.28	18.81	46,912,050	158.17	257.63	11.74
KNNKR	44,118,240	174.51	280.28	18.81	49,475,880	144.79	239.10	5.95
KQBKB	90,750,420	33.45	84.87	15.52	99,709,380	21.46	68.26	5.14
KQBKN	90,750,420	33.45	84.87	15.52	96,192,120	25.90	74.41	8.99
KQBKQ	90,750,420	33.45	84.87	15.52	93,824,100	29.08	78.82	11.74
KQBKR	90,750,420	33.45	84.87	15.52	98,951,760	22.39	69.55	5.95
KQNKB	87,576,960	38.29	91.57	19.71	99,709,380	21.46	68.26	5.14
KQNKN	87,576,960	38.29	91.57	19.71	96,192,120	25.90	74.41	8.99
KQNKQ	87,576,960	38.29	91.57	19.71	93,824,100	29.08	78.82	11.74
KQNKR	87,576,960	38.29	91.57	19.71	98,951,760	22.39	69.55	5.95
KQQKB	41,944,320	188.74	299.99	24.97	49,854,690	142.93	236.52	5.14
KQQKN	41,944,320	188.74	299.99	24.97	48,096,060	151.81	248.83	8.99
KQQKQ	41,944,320	188.74	299.99	24.97	46,912,050	158.17	257.63	11.74
KQQKR	41,944,320	188.74	299.99	24.97	49,475,880	144.79	239.10	5.95
KQRKB	90,038,460	34.51	86.33	16.44	99,709,380	21.46	68.26	5.14
KQRKN	90,038,460	34.51	86.33	16.44	96,192,120	25.90	74.41	8.99
KQRKQ	90,038,460	34.51	86.33	16.44	93,824,100	29.08	78.82	11.74
KQRKR	90,038,460	34.51	86.33	16.44	98,951,760	22.39	69.55	5.95
KRBKB	95,673,600	26.59	75.36	9.58	99,709,380	21.46	68.26	5.14
KRBKN	95,673,600	26.59	75.36	9.58	96,192,120	25.90	74.41	8.99
KRBKQ	95,673,600	26.59	75.36	9.58	93,824,100	29.08	78.82	11.74
KRBKR	95,673,600	26.59	75.36	9.58	98,951,760	22.39	69.55	5.95
KRNKB	92,308,740	31.20	81.75	13.57	99,709,380	21.46	68.26	5.14
KRNKN	92,308,740	31.20	81.75	13.57	96,192,120	25.90	74.41	8.99
KRNKQ	92,308,740	31.20	81.75	13.57	93,824,100	29.08	78.82	11.74
KRNKR	92,308,740	31.20	81.75	13.57	98,951,760	22.39	69.55	5.95
KRRKB	46,658,340	159.57	259.58	12.35	49,854,690	142.93	236.52	5.14
KRRKN	46,658,340	159.57	259.58	12.35	48,096,060	151.81	248.83	8.99
KRRKQ	46,658,340	159.57	259.58	12.35	46,912,050	158.17	257.63	11.74
KRRKR	46,658,340	159.57	259.58	12.35	49,475,880	144.79	239.10	5.95
Aggregate	2,917,997,520	66.02	129.98	14.97	3,109,418,880	55.80	115.82	7.89

Table 11: Comparison of index sizes for pawnless 3-2 endgames.

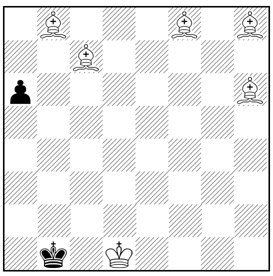

Figure 6: Troitzkiĭ (1905).

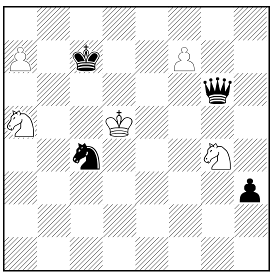

Figure 7: Troitzkiĭ (1912).

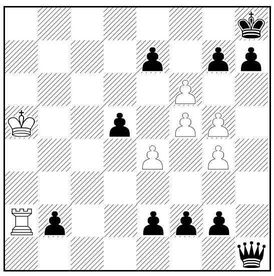

Figure 8: Bondar (1998).

Beasley and Whitworth (1996), referring also to Savin and Plaksin (1987), cite the Troitzkiĭ study of 1905, q.v. Figure 6, with 5 black-square Bishops as follows:

> "1. Bce5 a5 2. Ba1 a4 3. Bbe5 Ka2 4. Kc2 a3 5. Kc3 K×a1 6. Kb3+ Kb1 7. Ba1 a2 8. Kc3 K×a1 Kc2#."

Elkies (2000) recalls the Troitzkiĭ (1912) study, q.v. Figure 7, noting that Troitzkiĭ (1934) includes four pages of analysis proving the 4N-win:

> "1. a8N+ Kd7 2. f8N+ Kc8 3. N×g6 Ne3+ 4. N×e3 (not 4. Ke4 N×g4 5. Kf3 Ne3 6. Kg3 Nd5 =) h2 5. Nb6+ Kc7 6. Kc5 h1Q 7. Ned5+ Kd8 (7. ... Kb8 8. Nge7 Qg1+ 9. Kb5 Qf1+ 10. Nbc4 wins) 8. Nc6+ Ke8 9. Nce5 winning as Black is pushed slowly off the board, e.g. 9. ... Qc1+ 10. Kd6 Qa3+ 11. Kc6 Qc1+ (11. ... Qb3 12. Ngf4 Qc2+ 13. Kd6 Qd2 14. Ne6 Qb4+ 15. Nc5 Qa3 16. Nbd7 Qg3 17. Nc7+ Kd8 18. N5e6+ Kc8 19. Nd5 Qa3+ 20. Ndc5 wins[10]) 12. Nbc4 Kd8 13. Ngf4 Kc8 14. Ne6 Kb8 15. Kd7 wins[11]."

Bondar (1998) composed the study of Figure 8 featuring four losing black Queens:

> 1. f7 e1Q+ 2. Ka6 f1Q+ 3. Ka7 g1Q+ 4. Ka8 Q×f5 5. g×f5 h5 6. g6 Q×g6 7. f8Q+ Kh7 8. f×g6+ Kh6 9. Qf4+ K×g6 10. Qf5+ Kh6 11. Ra6+ wins.

	wtm	KT	SE	EH	btm	KT	SE	EH
Endgame	**# Elements**	**+Δ%**	**+Δ%**	**+Δ%**	**# Elements**	**+Δ%**	**+Δ%**	**+Δ%**
KBBKP	148,223,520	171.65	262.20	7.03	149,445,120	169.43	259.24	6.15
KBNKP	290,989,584	38.37	84.50	9.03	298,890,240	34.72	79.62	6.15
KBPKB	289,027,680	39.31	85.75	9.77	306,720,000	31.28	75.04	3.44
KBPKN	289,027,680	39.31	85.75	9.77	295,914,240	36.07	81.43	7.22
KBPKP	227,896,016	32.51	135.58	7.27	231,758,952	30.30	131.65	5.48
KBPKQ	289,027,680	39.31	85.75	9.77	288,610,560	39.51	86.02	9.93
KBPKR	289,027,680	39.31	85.75	9.77	304,369,920	32.29	76.39	4.24
KNNKP	137,991,648	191.80	289.06	14.96	149,445,120	169.43	259.24	6.15
KNPKB	278,914,860	44.36	92.49	13.75	306,720,000	31.28	75.04	3.44
KNPKN	278,914,860	44.36	92.49	13.75	295,914,240	36.07	81.43	7.22
KNPKP	219,921,779	37.32	144.12	11.16	231,758,952	30.30	131.65	5.48
KNPKQ	278,914,860	44.36	92.49	13.75	288,610,560	39.51	86.02	9.93
KNPKR	278,914,860	44.36	92.49	13.75	304,369,920	32.29	76.39	4.24
KPPKB	108,400,260	178.59	395.27	15.16	120,132,000	151.38	346.90	3.91
KPPKN	108,400,260	178.59	395.27	15.16	115,899,744	160.56	363.22	7.71
KPPKP	84,219,361	168.93	537.47	11.27	89,391,280	153.37	500.59	4.83
KPPKQ	108,400,260	178.59	395.27	15.16	113,036,880	167.16	374.95	10.43
KPPKR	108,400,260	178.59	395.27	15.16	119,209,296	153.33	350.36	4,72
KQBKP	283,818,240	41.87	89.16	11.79	298,890,240	34.72	79.62	6.15
KQNKP	273,904,512	47.00	96.01	15.84	298,890,240	34.72	79.62	6.15
KQPKB	272,015,040	48.03	97.37	16.64	306,720,000	31.28	75.04	3.44
KQPKN	272,015,040	48.03	97.37	16.64	295,914,240	36.07	81.43	7.22
KQPKP	214,481,388	40.80	150.31	13.98	231,758,952	30.30	131.65	5.48
KQPKQ	272,015,040	48.03	97.37	16.64	288,610,560	39.51	86.02	9.93
KQPKR	272,015,040	48.03	97.37	16.64	304,369,920	32.29	76.39	4.24
KQQKP	131,170,128	206.97	309.29	20.94	149,445,120	169.43	259.24	6.15
KQRKP	281,568,240	43.00	90.67	12.68	298,890,240	34.72	79.62	6.15
KRBKP	299,203,200	34.58	79.43	6.04	298,890,240	34.72	79.62	6.15
KRNKP	288,692,928	39.47	85.97	9.90	298,890,240	34.72	79.62	6.15
KRPKB	286,777,440	40.41	87.21	10.64	306,720,000	31.28	75.04	3.44
KRPKN	286,777,440	40.41	87.21	10.64	295,914,240	36.07	81.43	7.22
KRPKP	226,121,876	33.55	137.43	8.11	231,758,952	30.30	131.65	5.48
KRPKQ	286,777,440	40.41	87.21	10.64	288,610,560	39.51	86.02	9.93
KRPKR	286,777,440	40.41	87.21	10.64	304,369,920	32.29	76.39	4.24
KRRKP	145,901,232	175.98	267.97	8.73	149,445,120	169.43	259.24	6.15
Aggregate	8,194,644,772	60.00	129.30	12.09	8,658,285,808	51.43	117.02	6.09

Table 12: Comparison of index sizes over 3-2 endgames with Pawns.

[10] 20. ... Qg3 21. Ne7+ Kb8 22. Ncd7+ Ka8 23. Nec5 Qf4 24. Kc7 Qb4 25. Nb6+ Q×b6+ 26. K×b6 Kb8 27. Na6+ Ka8 etc.

[11] 15. ... Q×c4 16. N×c4 Kb7 17. Nc5+ Ka7 18. Kc7 Ka8 19. Ncb6+ Ka7 20. Nc8+ Ka8 21. Ndb6#.

	wtm	KT	SE	EH	btm	KT	SE	EH
Endgame	# Elements	+Δ%	+Δ%	+Δ%	# Elements	+Δ%	+Δ%	+Δ%
KBBBK	15,010,230	706.85	1017.72	16.41	17,472,840	593.14	860.19	0.00
KBBNK	44,983,618	169.23	272.96	16.53	52,418,520	131.05	220.06	0.00
KBBPK	139,715,040	188.20	284.26	13.54	153,741,960	161.90	249.20	3.19
KBNNK	43,406,294	179.02	286.52	20.76	52,418,520	131.05	220.06	0.00
KBNPK	274,352,939	46.76	95.69	15.65	307,483,920	30.95	74.60	3.19
KBPPK	106,602,156	183.29	403.62	17.10	114,742,320	163.19	367.89	8.79
KNNNK	13,486,227	798.03	1144.03	29.56	17,472,840	593.14	860.19	0.00
KNNPK	130,135,501	209.41	312.55	21.90	153,741,960	161.90	249.20	3.19
KNPPK	102,898,651	193.48	421.75	21.31	114,742,320	163.19	367.89	8.79
KPPPK	26,061,704	769.06	1960.00	20.36	28,388,716	697.83	1791.14	10.49
KQBBK	43,879,679	176.01	282.35	19.46	52,418,520	131.05	220.06	0.00
KQBNK	86,166,717	40.55	94.71	21.67	104,837,040	15.52	60.03	0.00
KQBPK	267,576,632	50.48	100.64	18.57	307,483,920	30.95	74.60	3.19
KQNNK	40,873,646	196.30	310.47	28.25	52,418,520	131.05	220.06	0.00
KQNPK	258,294,639	55.89	107.85	22.84	307,483,920	30.95	74.60	3.19
KQPPK	100,347,220	200.94	435.01	24.40	114,742,320	163.19	367.89	8.79
KQQBK	41,270,973	193.45	306.51	27.01	52,418,520	131.05	220.06	0.00
KQQNK	39,840,787	203.99	321.11	31.57	52,418,520	131.05	220.06	0.00
KQQPK	123,688,859	225.54	334.05	28.26	153,741,960	161.90	249.20	3.19
KQQQK	12,479,974	870.44	1244.33	40.01	17,472,840	593.14	860.19	0.00
KQQRK	40,916,820	195.99	310.03	28.11	52,418,520	131.05	220.06	0.00
KQRBK	88,557,959	36.76	89.45	18.38	104,837,040	15.52	60.03	0.00
KQRNK	85,470,603	41.70	96.29	22.66	104,837,040	15.52	60.03	0.00
KQRPK	265,421,907	51.70	102.27	19.54	307,483,920	30.95	74.60	3.19
KQRRK	43,157,690	180.62	288.74	21.46	52,418,520	131.05	220.06	0.00
KRBBK	46,242,089	161.91	262.81	13.36	52,418,520	131.05	220.06	0.00
KRBNK	90,787,358	33.40	84.80	15.48	104,837,040	15.52	60.03	0.00
KRBPK	281,991,360	42.79	90.39	12.51	307,483,920	30.95	74.60	3.19
KRNNK	43,056,198	181.28	289.66	21.74	52,418,520	131.05	220.06	0.00
KRNPK	272,153,675	47.95	97.27	16.58	307,483,920	30.95	74.60	3.19
KRPPK	105,758,666	185.55	407.64	18.03	114,742,320	163.19	367.89	8.79
KRRBK	45,873,720	164.01	265.73	14.27	52,418,520	131.05	220.06	0.00
KRRNK	44,265,261	173.60	279.02	18.42	52,418,520	131.05	220.06	0.00
KRRPK	137,491,197	192.86	290.48	15.38	153,741,960	161.90	249.20	3.19
KRRRK	14,644,690	726.99	1045.62	19.31	17,472,840	593.14	860.19	0.00
Aggregate	3,516,860,679	124.15	224.39	19.06	4,065,491,116	93.91	180.62	2.99

Table 13: Comparison of index ranges over 4-1 endgames.

	Broken Positions					Broken Positions			
	wtm		btm			wtm		btm	
Endgame	#	%	#	%	Endgame	#	%	#	%
KBBBK	3,795,425	25.29	0	0.00	**KBNKN**	9,252,139	9.94	0	0.00
KBBK	139,093	17.61	0	0.00	**KBNKP**	44,907,128	15.43	0	0.00
KBBKB	8,055,627	17.00	4,272,301	8.57	**KBNKQ**	9,252,139	9.94	24,074,338	25.66
KBBKN	8,055,627	17.00	0	0.00	**KBNKR**	9,252,139	9.94	15,529,736	15.69
KBBKP	32,609,914	22.00	0	0.00	**KBNNK**	4,915,218	11.32	0	0.00
KBBKQ	8,055,627	17.00	12,037,169	25.66	**KBNPK**	35,301,529	12.87	0	0.00
KBBKR	8,055,627	17.00	7,764,868	15.69	**KBPK**	500,513	10.39	0	0.00
KBBNK	8,769,335	19.49	0	0.00	**KBPKB**	29,140,721	10.08	39,073,198	12.74
KBBPK	27,592,969	19.75	0	0.00	**KBPKN**	29,140,721	10.08	13,658,280	4.62
KBK	2,507	9.20	0	0.00	**KBPKP**	32,514,553	14.27	7,406,518	3.20
KBKB	147,587	8.88	147,587	8.88	**KBPKQ**	29,140,721	10.08	83,399,904	28.90
KBKN	147,587	8.88	0	0.00	**KBPKR**	29,140,721	10.08	59,322,146	19.49
KBKP	666,320	13.03	0	0.00	**KBPPK**	12,305,285	11.54	0	0.00
KBNK	158,939	10.25	0	0.00	**KK**	0	0.00	0	0.00
KBNKB	9,252,139	9.94	8,544,602	8.57	**KNK**	0	0.00	0	0.00

Table 14a: Numbers and Percentages of Broken Positions in Nalimov's EGTs.

	Broken Positions						Broken Positions			
	wtm		btm				wtm		btm	
Endgame	#	%	#	%		Endgame	#	%	#	%
KNKN	0	0.00	0	0.00		**KQQKB**	19,489,387	46.46	4,272,301	8.57
KNKP	227,638	4.62	0	0.00		**KQQKN**	19,489,387	46.46	0	0.00
KNNK	0	0.00	0	0.00		**KQQKP**	64,878,086	49.46	0	0.00
KNNKB	0	0.00	616,152	1.24		**KQQKQ**	19,489,387	46.46	12,037,169	25.66
KNNKN	0	0.00	0	0.00		**KQQKR**	19,489,387	46.46	7,764,868	15.69
KNNKP	8,479,456	6.14	0	0.00		**KQQNK**	19,083,485	47.90	0	0.00
KNNKQ	0	0.00	12,037,169	25.66		**KQQPK**	59,373,739	48.00	0	0.00
KNNKR	0	0.00	7,764,868	15.69		**KQQQK**	7,854,527	62.94	0	0.00
KNNNK	0	0.00	0	0.00		**KQQRK**	23,835,461	58.25	0	0.00
KNNPK	4,136,099	3.18	0	0.00		**KQRBK**	41,394,865	46.74	0	0.00
KNPK	73,856	1.59	0	0.00		**KQRK**	616,152	41.07	0	0.00
KNPKB	4,431,360	1.59	39,073,198	12.74		**KQRKB**	35,638,322	39.58	8,544,602	8.57
KNPKN	4,431,360	1.59	13,658,280	4.62		**KQRKN**	35,638,322	39.58	0	0.00
KNPKP	13,811,226	6.28	7,406,518	3.20		**KQRKP**	121,235,002	43.06	0	0.00
KNPKQ	4,431,360	1.59	83,399,904	28.90		**KQRKQ**	35,638,322	39.58	24,074,338	25.66
KNPKR	4,431,360	1.59	59,322,146	19.49		**KQRKR**	35,638,322	39.58	15,529,736	15.69
KNPPK	3,270,048	3.18	0	0.00		**KQRNK**	35,307,376	41.31	0	0.00
KPK	0	0.00	0	0.00		**KQRPK**	109,627,138	41.30	0	0.00
KPKP	123,555	3.20	123,555	3.20		**KQRRK**	22,457,809	52.04	0	0.00
KPPK	0	0.00	0	0.00		**KRBBK**	14,750,918	31.90	0	0.00
KPPKB	0	0.00	20,104,876	16.74		**KRBK**	396,136	24.84	0	0.00
KPPKN	0	0.00	10,532,252	9.09		**KRBKB**	22,924,278	23.96	8,544,602	8.57
KPPKP	2,854,365	3.39	5,664,886	6.34		**KRBKN**	22,924,278	23.96	0	0.00
KPPKQ	0	0.00	36,200,376	32.03		**KRBKP**	85,322,108	28.52	0	0.00
KPPKR	0	0.00	27,657,596	23.20		**KRBKQ**	22,924,278	23.96	24,074,338	25.66
KPPPK	0	0.00	0	0.00		**KRBKR**	22,924,278	23.96	15,529,736	15.69
KQBBK	18,081,566	41.21	0	0.00		**KRBNK**	23,847,355	26.27	0	0.00
KQBK	526,735	34.83	0	0.00		**KRBPK**	74,211,659	26.32	0	0.00
KQBKB	30,490,930	33.60	8,544,602	8.57		**KRK**	4,630	17.13	0	0.00
KQBKN	30,490,930	33.60	0	0.00		**KRKB**	270,560	16.41	147,587	8.88
KQBKP	106,356,738	37.47	0	0.00		**KRKN**	270,560	16.41	0	0.00
KQBKQ	30,490,930	33.60	24,074,338	25.66		**KRKP**	1,022,716	20.16	0	0.00
KQBKR	30,490,930	33.60	15,529,736	15.69		**KRKR**	270,560	16.41	270,560	16.41
KQBNK	30,583,209	35.49	0	0.00		**KRNK**	271,935	17.68	0	0.00
KQBPK	95,439,748	35.67	0	0.00		**KRNKB**	15,669,550	16.98	8,544,602	8.57
KQK	7,137	27.85	0	0.00		**KRNKN**	15,669,550	16.98	0	0.00
KQKB	418,147	26.74	147,587	8.88		**KRNKP**	63,487,156	21.99	0	0.00
KQKN	418,147	26.74	0	0.00		**KRNKQ**	15,669,550	16.98	24,074,338	25.66
KQKP	1,439,112	29.92	0	0.00		**KRNKR**	15,669,550	16.98	15,529,736	15.69
KQKQ	418,147	26.74	418,147	26.74		**KRNNK**	7,861,335	18.26	0	0.00
KQKR	418,147	26.74	270,560	16.41		**KRNPK**	53,055,381	19.49	0	0.00
KQNK	404,593	27.72	0	0.00		**KRPK**	840,944	17.59	0	0.00
KQNKB	23,344,829	26.66	8,544,602	8.57		**KRPKB**	48,472,746	16.90	39,073,198	12.74
KQNKN	23,344,829	26.66	0	0.00		**KRPKN**	48,472,746	16.90	13,658,280	4.62
KQNKP	84,872,244	30.99	0	0.00		**KRPKP**	47,046,257	20.81	7,406,518	3.20
KQNKQ	23,344,829	26.66	24,074,338	25.66		**KRPKQ**	48,472,746	16.90	83,399,904	28.90
KQNKR	23,344,829	26.66	15,529,736	15.69		**KRPKR**	48,472,746	16.90	59,322,146	19.49
KQNNK	11,305,947	27.66	0	0.00		**KRPPK**	19,194,662	18.15	0	0.00
KQNPK	74,628,435	28.89	0	0.00		**KRRBK**	17,408,683	37.95	0	0.00
KQPK	1,259,793	27.79	0	0.00		**KRRK**	245,132	31.54	0	0.00
KQPKB	72,713,627	26.73	39,073,198	12.74		**KRRKB**	14,121,920	30.27	4,272,301	8.57
KQPKN	72,713,627	26.73	13,658,280	4.62		**KRRKN**	14,121,920	30.27	0	0.00
KQPKP	64,376,740	30.02	7,406,518	3.20		**KRRKP**	50,151,272	34.37	0	0.00
KQPKQ	72,713,627	26.73	83,399,904	28.90		**KRRKQ**	14,121,920	30.27	12,037,169	25.66
KQPKR	72,713,627	26.73	59,322,146	19.49		**KRRKR**	14,121,920	30.27	7,764,868	15.69
KQPPK	27,886,605	27.79	0	0.00		**KRRNK**	14,334,054	32.38	0	0.00
KQQBK	22,021,058	53.36	0	0.00		**KRRPK**	44,331,316	32.24	0	0.00
KQQK	336,585	48.17	0	0.00		**KRRRK**	6,387,602	43.62	0	0.00

Table 14b: Numbers and Percentages of Broken Positions in Nalimov's EGTs.