# *Uniform factorial decay estimate for the remainder of rough Taylor expansion*

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# Uniform factorial decay estimates for controlled differential equations*

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**Abstract**

We establish a uniform factorial decay estimate for the Taylor approximation of solutions to controlled differential equations in the $p$-variation metric. As part of the proof, we also obtain a factorial decay estimate for controlled paths which is interesting in its own right.

**Keywords:** Controlled differential equation ; Rough paths ; Taylor expansion ; Factorial Decay.
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## 1 Introduction

For a controlled differential equation of the form

$$\begin{aligned} \mathrm{d}Y_t &= f(Y_t)\,\mathrm{d}X_t \\ Y_0 &= y_0. \end{aligned} \tag{1.1}$$

where $X : [0,T] \to \mathbb{R}^d$ is a path with finite 1-variation and $f : \mathbb{R}^e \to L(\mathbb{R}^d, \mathbb{R}^e)$ is a smooth vector field, we are interested in estimating the Taylor remainder

$$Y_t - Y_s - \sum_{k=1}^{N} f^{\circ k}(Y_s) \int_{s<s_1<\ldots<s_k<t} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_k} \tag{1.2}$$

$$\equiv \int_{s<s_1<\ldots<s_N<t} f^{\circ N}(Y_{s_1}) - f^{\circ N}(Y_s)\,\mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_N}, \tag{1.3}$$

where $f^{\circ m} : \mathbb{R}^e \to L\left(\left(\mathbb{R}^d\right)^{\otimes m}, \mathbb{R}^e\right)$ is defined inductively by

$$\begin{aligned} f^{\circ 1} &= f \\ f^{\circ k+1} &= D\left(f^{\circ k}\right) f. \end{aligned}$$

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The functions $f^{\circ k}$ can also be expressed in terms of iterative applications of the vector field $f$ as differential operators [3]. The iterated integrals in (1.2) will appear numerous times and we shall use the shorthand

$$X_{s,t}^{k} := \int_{s<s_1<\ldots<s_k<t} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_k}. \tag{1.4}$$

Since the $1-$variation norm of $X$ equals to the $L^1$ norm of the derivative of $X$, we have (see for example [4])

$$\left| Y_t - Y_s - \sum_{k=1}^{N} f^{\circ k}\left(Y_s\right) X_{s,t}^{k} \right| \leq \left\| f^{\circ(N+1)} \right\|_{\infty} \frac{|X|_{1-var;[s,t]}^{N+1}}{N!} \tag{1.5}$$

where

$$|X|_{1-var;[s,t]} = \sup_{s<t_1<\ldots<t_n<t} \sum_{i=0}^{n-1} \left| X_{t_{i+1}} - X_{t_i} \right|$$

and $\left\| f^{\circ N} \right\|_{\infty}$ denotes $\sup_{x\in\mathbb{R}^e} \left| f^{\circ N}(x) \right|$ with $|\cdot|$ being the operator norm

$$\left| f^{\circ N}(x) \right| = \sup_{v\in(\mathbb{R}^d)^{\otimes N}} \frac{\left| f^{\circ N}(x)(v) \right|}{\|v\|}.$$

Estimates of the form (1.5) have application both as a theoretical tool for analysing the equation (1.1) and as a practical numerical scheme for constructing the solution. The estimate (1.5), when the $1$-variation metric is replaced by the $p$-variation metric, has been shown in [2] ($p<3$), [5] ($p<3$) and [4] (all $p \geq 1$) without the factorial decay factor. We shall prove such estimate *with* the factorial decay factor. The estimates of Davie [2], Gubinelli [5], Friz and Victoir [4] as well as our estimates below gives a numerical scheme for approximating a solution to (1.1) in $O(1)$ time steps.

**Theorem 1.1.** *Let $p \geq 1$. Let $X = \left(1, X^1, \ldots, X^{\lfloor p \rfloor}\right)$ be a $p$-weak geometric rough path. Let $f$ be a Lip($\gamma - 1$) vector field where $\gamma > p$. Let $Y$ be a solution to the differential equation*

$$\mathrm{d}Y_t = f\left(Y_t\right) \mathrm{d}X_t \tag{1.6}$$

*defined in the sense of [3]. Then there exists a constant $C_p$ depending only on $p$ such that*

$$\left| Y_t - Y_s - \sum_{k=1}^{\lfloor \gamma \rfloor} f^{\circ k}\left(Y_s\right) X_{s,t}^{k} \right| \leq \frac{1}{\left(\frac{\lfloor \gamma \rfloor}{p}\right)!} \beta^{\lfloor \gamma \rfloor} M_{p,\gamma} \|f\|_{\circ\gamma} \|X\|_{p-var,[s,t]}^{\gamma}, \tag{1.7}$$

*where*

$$M_{p,\gamma} = 2C_p \left( |f|_{Lip((\gamma-1)\wedge\lfloor p \rfloor)} \vee 1 \right)^{\lfloor p \rfloor+1} \left( |X|_{p-var} \vee 1 \right)^{\lfloor p \rfloor+1};$$

$$\|f\|_{\circ\gamma} = \max_{\lfloor \gamma \rfloor - \lfloor p \rfloor + 1 \leq m \leq \lfloor \gamma \rfloor} \left| f^{\circ m} \right|_{Lip(\min(\gamma-m,1))}^{\min(\gamma-m,1)}; \tag{1.8}$$

$$\beta = p\left( 1 + \sum_{r=2}^{\infty} \left( \frac{2}{r-1} \wedge 1 \right)^{\frac{\lfloor p \rfloor+1}{p}} \right). \tag{1.9}$$

We refer the readers to Definition 9.16 and Definition 10.2 in [3] for the definition of Lip ($\gamma$) vector fields and weak geometric rough paths respectively. We shall however recall the definition of $p$-variation and some basic notations in Section 2.

**Remark 1.2.** If the equation (1.6) has more than one solution, then any solution must satisfy (1.7).

**Remark 1.3.** Taking the biggest $\gamma$ may not yield the best estimate for the left hand side of (1.7). In general the term $\|f\|_{\circ\gamma}$ could grow factorially fast in $\gamma$. Since a Lip($\gamma$) function is also Lip($\gamma'$) for all $\gamma' < \gamma$, we may choose $\gamma'$ which optimises the estimate (1.7).

The proof for (1.5) relies heavily on the relation between the 1-variation of the path and the $L^1$ norm of its derivative. Proving an estimate of the form (1.5) for the $p$-variation metric, even without the factorial decay factor, requires the clever idea of Young[9]. The integration with respect to a path can be expressed in terms of the limit of a Riemann sum as the size of partition converges to zero. Young's idea was to estimate the Riemann sum with respect to a partition by removing points from the partition successively. This idea had been used in [6] to show that, for $p < 2$, the $n$-th order iterated integral of a path $X$ is uniformly bounded by

$$\left(1 + 4^{\frac{1}{p}}\zeta\left(2/p\right)\right)^{n}\left(\frac{1}{n!}\right)^{\frac{1}{p}}\|X\|^{n}_{p-var,[0,T]}. \tag{1.10}$$

where $\zeta$ is the classical zeta function. T. Lyons' proof for the $p \geq 2$ case in [7] is slightly different and used the neoclassical inequality ([7],[1])

$$\sum_{k=0}^{N}\frac{1}{\Gamma\left(k/p+1\right)\Gamma\left(\left(n-k\right)/p+1\right)}a^{k/p}b^{(n-k)/p} \leq p\frac{1}{\Gamma\left(n/p+1\right)}\left(a+b\right)^{n/p} \tag{1.11}$$

to obtain an uniform bound of the form

$$\beta^{n-1}\frac{1}{\Gamma\left(n/p+1\right)}\|X\|^{n}_{p-var,[0,T]}$$

where $\Gamma$ is the Gamma function and $\beta$ is as defined in (1.9).

## 2 The Proof

### 2.1 Notations and basic definitions

For each $k \in \mathbb{N}$, we equip a norm on $\left(\mathbb{R}^d\right)^{\otimes k}$ by identifying it with $\mathbb{R}^{d^k}$. Let

$$T_1^N\left(\mathbb{R}^d\right) = 1 \oplus \mathbb{R}^d \oplus \ldots \oplus \left(\mathbb{R}^d\right)^N.$$

If $\pi_k$ denotes the projection operator $T_1^N\left(\mathbb{R}^d\right) \to \left(\mathbb{R}^d\right)^{\otimes k}$, then we define a norm on $T_1^N\left(\mathbb{R}^d\right)$ by

$$\|x\| = \max_{1\leq k\leq N}\|\pi_k\left(x\right)\|^{\frac{1}{k}}.$$

**Definition 2.1.** *Let $T > 0$ and $p \geq 1$. A path $X : [0,T] \to T_1^{\lfloor p\rfloor}\left(\mathbb{R}^d\right)$ has finite $p$-variation if for all $0 < s < t < T$,*

$$\|X\|_{p-var,[s,t]} := \sup_{s<t_1<\ldots<t_n<t}\max_{1\leq k\leq\lfloor p\rfloor}\left(\sum_{i=0}^{n-1}\left\|\pi_k\left(X_{t_i}^{-1}X_{t_{i+1}}\right)\right\|^{\frac{p}{k}}\right)^{\frac{1}{p}} < \infty \tag{2.1}$$

*where $X^{-1}$ denote the unique multiplicative inverse of $X \in T_1^{\lfloor p\rfloor}\left(\mathbb{R}^d\right)$. We will denote $\|X\|_{p-var,[0,T]}$ by $\|X\|_{p-var}$.*

We first recall Lyons' extension theorem, which will be used repeatedly in the following form:

**Fact 2.2.** *(Theorem 2.2.1 in [7]) Let $p \geq 1$ and $X = \left(1, X^1, \ldots, X^{\lfloor p\rfloor}\right)$ be a $p$-weak geometric rough path. Then for all $N \geq \lfloor p\rfloor + 1$, there exists a unique continuous* *path* $\mathbf{X} = \left(1, X^1, \ldots, X^N\right) \in T_1^N\left(\mathbb{R}^d\right)$ *which extends* $X$*,* $\mathbf{X}_0 = (1, 0 \ldots, 0)$ *and for all* $\lfloor p \rfloor \leq l \leq N$*,*

$$\left\|\pi_l\left(\mathbf{X}_{t_i}^{-1}\mathbf{X}_{t_{i+1}}\right)\right\| \leq \frac{\beta^{l-1}}{\left(\frac{l}{p}\right)!}\|X\|_{p-var,[s,t]}^l. \tag{2.2}$$

**Remark 2.3.** We will denote $\mathbf{X}_s^{-1}\mathbf{X}_t$ by $\mathbf{X}_{s,t}$ and $\pi_l\left(\mathbf{X}_{s,t}\right)$ by $X_{s,t}^l$. In particular, $\mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} = \mathbf{X}_{s,t}$ and so, for any $s < u < t$,

$$X_{s,t}^m = \sum_{l=0}^m X_{s,u}^{m-l} \otimes X_{u,t}^l. \tag{2.3}$$

Note that for paths with finite 1-variation, the $\left(X^k\right)_{k \geq 1}$ defined in this theorem are exactly the iterated integrals of $X$. Hence no confusion will arise by using the same notation as in (1.4).

**Remark 2.4.** If $r \geq \lfloor p \rfloor$, then for any $m \geq 0$,

$$X_{s,t}^m = \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{n-1} \sum_{k=1}^r X_{s,t_i}^{m-k} \otimes X_{t_i,t_{i+1}}^k \tag{2.4}$$

where the limit is taken as the mesh size of the partition $\mathcal{P} = (s < t_1 < \ldots < t_{n-1} < t)$ goes to zero. By convention, for any $s < t$, $X_{s,t}^0 = 1$ and $X_{s,t}^m = 0$ if $m < 0$. In the case $r = m$, (2.4) follows directly from (2.3). For $r < m$, note that the sum over $k$ from $r+1$ to $m$ in (2.4) vanishes after the taking of limit, due to (2.2). See [5] for details.

## 2.2 The proof

The following lemma is a factorial decay estimate for the Taylor remainder of a controlled path in the sense of Gubinelli [5]. This lemma is interesting in its own right. We interpret it as the dual counterpart of Fact 2.2.

**Lemma 2.5.** *Let* $p \geq 1$ *and* $\gamma > p$*. Let* $\left(1, X^1, \ldots, X^{\lfloor p \rfloor}\right)$ *be a* $p$*-weak geometric rough path. Let* $Y^{(i)}$ *be a function* $[0, T] \to L\left(\left(\mathbb{R}^d\right)^{\otimes i}, \mathbb{R}^e\right)$ *and* $\left(Y^{(0)}, Y^{(1)}, \ldots, Y^{(\lfloor \gamma \rfloor)}\right)$ *satisfies, for* $\lceil \gamma - p \rceil \leq m \leq \lfloor \gamma \rfloor$*,*

$$\left|Y_t^{(m)} - \sum_{l=0}^{\lfloor \gamma \rfloor - m} Y_s^{(l+m)} X_{s,t}^l\right| \leq \frac{1}{\left(\frac{\lfloor \gamma \rfloor - m}{p}\right)!} M \beta^{\lfloor \gamma \rfloor - m}\|X\|_{p-var,[s,t]}^{\gamma - m}, \tag{2.5}$$

*for all* $s \leq t$ *and for* $0 \leq m \leq \lceil \gamma - p \rceil - 1$*, the limit*

$$\lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{n-1} \sum_{l=1}^{\lfloor \gamma \rfloor - m} Y_{t_i}^{(m+l)} X_{t_i,t_{i+1}}^l, \tag{2.6}$$

*where* $|\mathcal{P}| \to 0$ *denotes the limit as the mesh size of a partition* $\mathcal{P}$ *on* $[s,t]$ *goes to zero, exists and equals*

$$Y_t^{(m)} - Y_s^{(m)}. \tag{2.7}$$

*For* $l \geq \lfloor p \rfloor + 1$*, let* $X^l$ *denote the projection to* $\left(\mathbb{R}^d\right)^{\otimes l}$ *of the unique extension of* $\left(1, X^1, \ldots, X^{\lfloor p \rfloor}\right)$ *given in Fact 2.2. Then (2.5) holds for all* $0 \leq m \leq \lfloor \gamma \rfloor$*.*

*Proof.* We will carry out backward induction on $k$ starting from $\lceil \gamma - p \rceil$ and moving down to $0$.

The base induction step of $k = \lceil \gamma - p \rceil$ holds because of the assumption. We will assume from now onwards that $k \leq \lceil \gamma - p \rceil - 1$. It is useful to bear in mind that

$$\lfloor \gamma \rfloor - \lfloor p \rfloor \leq \lceil \gamma - p \rceil \leq \lfloor \gamma \rfloor - \lfloor p \rfloor + 1.$$

For the induction step, note that by (2.4) and the equality of (2.6) and (2.7),

$$\begin{aligned} & Y_t^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+l)} X_{s,t}^l && (2.8) \\ = \; & \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n} \sum_{l_2=1}^{\lfloor \gamma \rfloor - k} \left( Y_{t_i}^{(k+l_2)} - \sum_{l_1=0}^{\lfloor \gamma \rfloor - k - l_2} Y_s^{(k+l_1+l_2)} X_{s,t_i}^{l_1} \right) X_{t_i,t_{i+1}}^{l_2}, && (2.9) \end{aligned}$$

where the limit is taken as the mesh size of the partition $\mathcal{P} = (s < t_1 < \ldots < t_{n-1} < t)$ goes to zero.

We first show that the term

$$\sum_{i=0}^{n-1} \sum_{l_2=1}^{\lfloor \gamma \rfloor - k} \sum_{l_1=0}^{\lfloor \gamma \rfloor - k - l_2} Y_s^{(k+l_1+l_2)} X_{s,t_i}^{l_1} X_{t_i,t_{i+1}}^{l_2}. \tag{2.10}$$

is in fact independent of the partition $\mathcal{P}$.

$$\begin{aligned} & \sum_{i=0}^{n-1} \sum_{l_2=1}^{\lfloor \gamma \rfloor - k} \sum_{l_1=0}^{\lfloor \gamma \rfloor - k - l_2} Y_s^{(k+l_1+l_2)} X_{s,t_i}^{l_1} X_{t_i,t_{i+1}}^{l_2} \\ = \; & \sum_{i=0}^{n-1} \left[ \sum_{0 \leq l_1 + l_2 \leq \lfloor \gamma \rfloor - k} Y_s^{(k+l_1+l_2)} X_{s,t_i}^{l_1} X_{t_i,t_{i+1}}^{l_2} - \sum_{l_1=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+l_1)} X_{s,t_i}^{l_1} \right] \\ = \; & \sum_{i=0}^{n-1} \left[ \sum_{r=0}^{\lfloor \gamma \rfloor - k} \sum_{l_1+l_2=r} Y_s^{(k+r)} X_{s,t_i}^{l_1} X_{t_i,t_{i+1}}^{l_2} - \sum_{l_1=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+l_1)} X_{s,t_i}^{l_1} \right] \\ = \; & \sum_{i=0}^{n-1} \left[ \sum_{r=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+r)} X_{s,t_{i+1}}^{r} - \sum_{r=0}^{\lfloor \gamma \rfloor - k} Y_s^{(k+r)} X_{s,t_i}^{r} \right] \\ = \; & \sum_{r=1}^{\lfloor \gamma \rfloor - k} Y_s^{(k+r)} X_{s,t}^{r} \end{aligned}$$

where we have used (2.3) in the third line. Let

$$\left( Y_s^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(l)} X_{s,t}^l \right)^{\mathcal{P}} = \sum_{i=0}^{n-1} \sum_{l_2=1}^{\lfloor \gamma \rfloor - k} \left( Y_{t_i}^{(k+l_2)} - \sum_{l_1=0}^{\lfloor \gamma \rfloor - k - l} Y_s^{(k+l+l_1)} X_{s,t_i}^{l_1} \right) X_{t_i,t_{i+1}}^{l_2}.$$

Since (2.10) is independent of the partition,

$$\begin{aligned} & \left( Y_s^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(l)} X_{s,t}^l \right)^{\mathcal{P}} - \left( Y_s^{(k)} - \sum_{l=0}^{\lfloor \gamma \rfloor - k} Y_s^{(l)} X_{s,t}^l \right)^{\mathcal{P}\setminus\{t_j\}} && (2.11) \\ = \; & \sum_{l'=1}^{\lfloor \gamma \rfloor - k} Y_{t_{j-1}}^{(k+l')} X_{t_{j-1},t_j}^{l'} + \sum_{l'=1}^{\lfloor \gamma \rfloor - k} Y_{t_j}^{(k+l')} X_{t_j,t_{j+1}}^{l'} - \sum_{l'=1}^{\lfloor \gamma \rfloor - k} Y_{t_{j-1}}^{(k+l')} X_{t_{j-1},t_{j+1}}^{l'} \\ = \; & \sum_{l_2=1}^{\lfloor \gamma \rfloor - k} \left( Y_{t_j}^{(k+l_2)} - \sum_{l_1=0}^{\lfloor \gamma \rfloor - k - l_2} Y_{t_{j-1}}^{(k+l_1+l_2)} X_{t_{j-1},t_j}^{l_1} \right) X_{t_j,t_{j+1}}^{l_2}. && (2.12) \end{aligned}$$

By induction hypothesis, (2.5) which holds for $m > k$ and Theorem 2.2.1 in [7],

$$\begin{aligned}
&\left|\sum_{l_2=1}^{\lfloor\gamma\rfloor-k}\left(Y_{t_j}^{(k+l_2)}-\sum_{l_1=0}^{\lfloor\gamma\rfloor-k-l}Y_{t_{j-1}}^{(k+l_1+l_2)}X_{t_{j-1},t_j}^{l_1}\right)X_{t_j,t_{j+1}}^{l_2}\right|\\
\leq\quad&\sum_{l_2=1}^{\lfloor\gamma\rfloor-k}\left[\frac{1}{\left(\frac{\lfloor\gamma\rfloor-k-l_2}{p}\right)!\left(\frac{l_2}{p}\right)!}M\beta^{\lfloor\gamma\rfloor-k-l_2}\left\|X\right\|_{p-var,[t_{j-1},t_j]}^{\gamma-k-l_2}\right.\\
&\left.\times\beta^{l_2-1}\left\|X\right\|_{p-var,[t_j,t_{j+1}]}^{l_2}\right] \qquad (2.13)\\
\leq\quad&\frac{1}{\left(\frac{\lfloor\gamma\rfloor-k}{p}\right)!}\frac{p}{\beta}M\beta^{\lfloor\gamma\rfloor-k}\left\|X\right\|_{p-var,[t_{j-1},t_{j+1}]}^{\gamma-k}, \qquad (2.14)
\end{aligned}$$

where the final line is obtained by the neoclassical inequality (1.11), proved in [1].

Let $\omega(s,t)=\|X\|_{p-var,[s,t]}^{p}$. We now choose $j$ such that, for $|\mathcal{P}|\geq 2$,

$$\omega(t_{j-1},t_{j+1})\leq\left(\frac{2}{|\mathcal{P}|-1}\wedge 1\right)\omega(s,t)$$

which exists since

$$\sum_{i=1}^{n-1}\omega(t_{i-1},t_{i+1})\leq 2\omega(s,t)$$

and also that

$$\omega(t_{j-1},t_{j+1})\leq\omega(s,t)$$

for all $j$. Then as $\gamma-k\geq\lfloor p\rfloor+1$, (2.14) is less than or equal to

$$\frac{1}{\left(\frac{\lfloor\gamma\rfloor-k}{p}!\right)}\frac{p}{\beta}M\beta^{\lfloor\gamma\rfloor-k}\left(\frac{2}{n-1}\wedge 1\right)^{\frac{\lfloor p\rfloor+1}{p}}\|X\|_{p-var,[s,t]}^{\gamma-k}.$$

By removing points successively from $\mathcal{P}$ and using that $\left(Y_s^{(k)}-\sum_{l=0}^{\lfloor\gamma\rfloor-k}Y_s^{(k+l)}X_{s,t}^{l}\right)^{\{s,t\}}=0$, we have

$$\begin{aligned}
\left|\left(Y_s^{(k)}-\sum_{l=0}^{\lfloor\gamma\rfloor-k}Y_s^{(k+l)}X_{s,t}^{l}\right)^{\mathcal{P}}\right| &\leq \frac{1}{\left(\frac{\lfloor\gamma\rfloor-k}{p}!\right)}\frac{p}{\beta}M\beta^{\lfloor\gamma\rfloor-k}\sum_{n=2}^{\infty}\left(\frac{2}{n-1}\wedge 1\right)^{\frac{\lfloor p\rfloor+1}{p}}\|X\|_{p-var,[s,t]}^{\gamma-k}\\
&\leq \frac{1}{\left(\frac{\lfloor\gamma\rfloor-k}{p}!\right)}M\beta^{\lfloor\gamma\rfloor-k}\|X\|_{p-var,[s,t]}^{\gamma-k},
\end{aligned}$$

where the final line follows from (1.9).

By taking limit as $|\mathcal{P}|\to 0$, (2.5) follows for $m=k$. □

For the differential equation

$$\mathrm{d}Y_t=f(Y_t)\,\mathrm{d}X_t \qquad (2.15)$$

we wish to apply Lemma 2.5 to $\left(Y,f^{\circ 1}(Y),\ldots,f^{\circ(\lfloor\gamma\rfloor)}(Y)\right)$. Using the standard estimates for rough differential equations, it turns out that it suffices to verify the assumption of Lemma 2.5 for paths with finite 1-variation. To do so, we need the following lemma.

**Lemma 2.6.** *Let $X : [0,T] \to \mathbb{R}^d$ be a path with finite $1$-variation. Let $f$ be a Lip($\gamma - 1$) vector field. Let $Y_t$ be a solution to the differential equation (2.15). Then*

$$\begin{aligned} & f^{\circ m}\left(Y_t\right) - f^{\circ m}\left(Y_s\right) - \sum_{k=1}^{\lfloor\gamma\rfloor - m} f^{\circ(m+k)}\left(Y_s\right) X^k_{s,t} \\ = & \begin{cases} \int_{s\le s_1\le\ldots\le s_{\lfloor\gamma\rfloor-m}\le t} f^{\circ\lfloor\gamma\rfloor}\left(Y_{s_1}\right) - f^{\circ\lfloor\gamma\rfloor}\left(Y_s\right) \mathrm{d}X_{s_1}\otimes\ldots\otimes \mathrm{d}X_{s_{\lfloor\gamma\rfloor-m}} & ,0\le m<\lfloor\gamma\rfloor \\ f^{\circ\lfloor\gamma\rfloor}\left(Y_t\right) - f^{\circ\lfloor\gamma\rfloor}\left(Y_s\right) & ,m=\lfloor\gamma\rfloor. \end{cases} \end{aligned}$$

*Proof.* We will prove it by backward induction, starting from $\lfloor\gamma\rfloor$.

The case $m = \lfloor\gamma\rfloor$ is trivially true.

For the induction step, note first that by the fundamental theorem of calculus,

$$\begin{aligned} & \int_s^t f^{\circ(m+1)}\left(Y_u\right) \mathrm{d}X_u \\ = & \int_s^t D\left(f^{\circ m}\right)\left(Y_u\right) f\left(Y_u\right) \mathrm{d}X_u \\ = & \int_s^t D\left(f^{\circ m}\right)\left(Y_u\right) \mathrm{d}Y_u \\ = & f^{\circ m}\left(Y_t\right) - f^{\circ m}\left(Y_s\right). \end{aligned} \tag{2.16}$$

Then by (2.16) and the induction hypothesis,

$$\begin{aligned} & f^{\circ m}\left(Y_t\right) - f^{\circ m}\left(Y_s\right) - \sum_{k=1}^{\lfloor\gamma\rfloor-m} f^{\circ(m+k)}\left(Y_s\right) X^k_{s,t} \\ = & \int_s^t f^{\circ m+1}\left(Y_{s_{\lfloor\gamma\rfloor-m}}\right) \mathrm{d}X_{s_{\lfloor\gamma\rfloor-m}} - \sum_{k=1}^{\lfloor\gamma\rfloor-m} f^{\circ(m+k)}\left(Y_s\right) X^{k-1}_{s,s_{\lfloor\gamma\rfloor-m}} \otimes \mathrm{d}X_{s_{\lfloor\gamma\rfloor-m}} \\ = & \int_{s\le s_1\le\ldots\le s_{\lfloor\gamma\rfloor-m}\le t} f^{\circ\lfloor\gamma\rfloor}\left(Y_{s_1}\right) - f^{\circ\lfloor\gamma\rfloor}\left(Y_s\right) \mathrm{d}X_{s_1}\otimes\ldots\otimes\mathrm{d}X_{s_{\lfloor\gamma\rfloor-m}}. \end{aligned}$$

□

*Proof of Theorem 1.* The only thing to prove is that $\left(Y, f^{\circ 1}\left(Y\right), \ldots, f^{\circ(\lfloor\gamma\rfloor)}\left(Y\right)\right)$ satisfies the assumptions of Lemma 2.5.

For each $s \le t$, let $x^{s,t} : [s,t] \to \mathbb{R}^d$ be a continuous path with finite $1$-variation such that for $1 \le l \le \lfloor p\rfloor$,

$$\left(x^{s,t}\right)^l_{s,t} = X^l_{s,t}, \tag{2.17}$$

where we use the notation from (1.4) and

$$\int_s^t \left|\mathrm{d}x^{s,t}_u\right| \le c_p \left\|X\right\|_{p-var,[s,t]} \tag{2.18}$$

for a function $c_p$ of $p$ which is specified in [3] along with the existence of $x^{s,t}$.

Consider the differential equation

$$\begin{aligned} \mathrm{d}Y^{s,t}_u &= f\left(Y^{s,t}_u\right)\mathrm{d}x^{s,t}_u \\ Y^{s,t}_s &= Y_s. \end{aligned} \tag{2.19}$$

By Theorem 10.16 in [3], there exists a solution $Y^{s,t}$ of (2.19) such that the following estimate holds

$$\left|Y_t - Y^{s,t}_t\right| \le C_p \left|f\right|^{\gamma\wedge(\lfloor p\rfloor+1)}_{Lip((\gamma-1)\wedge\lfloor p\rfloor)} \left\|X\right\|^{\gamma\wedge(\lfloor p\rfloor+1)}_{p-var,[s,t]} \tag{2.20}$$

for some function $C_p$ depending on $p$ only.

Note that by (2.17) and $m \geq \lceil \gamma - p \rceil \geq \lfloor \gamma \rfloor - \lfloor p \rfloor$,

$$\begin{aligned} & \left| f^{\circ(m)}\left(Y_t\right) - \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)}\left(Y_s\right) X_{s,t}^k \right| \\ \leq \quad & \left| f^{\circ m}\left(Y_t\right) - f^{\circ m}\left(Y_t^{s,t}\right)\right| + \left| f^{\circ m}\left(Y_t^{s,t}\right) - \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)}\left(Y_s\right)\left(x^{s,t}\right)_{s,t}^k \right| \end{aligned} \tag{2.21}$$

By (2.20), for $0 \leq m \leq \lfloor \gamma \rfloor - 1$,

$$\begin{aligned} & \left| f^{\circ m}\left(Y_t\right) - f^{\circ m}\left(Y_t^{s,t}\right)\right| \\ \leq \quad & \left| f^{\circ m}\right|_{Lip(1)} \left| Y_t - Y_t^{s,t}\right| \\ \leq \quad & C_p \left| f^{\circ m}\right|_{Lip(1)} \left| f\right|_{Lip((\gamma-1)\wedge\lfloor p \rfloor)}^{\gamma\wedge(\lfloor p \rfloor+1)} \left\| X \right\|_{p-var,[s,t]}^{\gamma\wedge(\lfloor p \rfloor+1)}. \end{aligned} \tag{2.22}$$

If $\lceil \gamma - p \rceil \leq m \leq \lfloor \gamma \rfloor - 1$, then $\gamma - m \leq \lfloor p \rfloor$ and so

$$\left| f^{\circ m}\left(Y_t\right) - f^{\circ m}\left(Y_t^{s,t}\right)\right| \tag{2.23}$$

$$\leq \quad C_p \left| f^{\circ m}\right|_{Lip(1)} \left| f\right|_{Lip((\gamma-1)\wedge\lfloor p \rfloor)}^{\gamma\wedge(\lfloor p \rfloor+1)} \left( \left\| X \right\|_{p-var,[s,t]} \vee 1\right)^{(\lfloor p \rfloor+1)} \left\| X \right\|_{p-var,[s,t]}^{\gamma-m}. \tag{2.24}$$

To estimate (2.23) for $m = \lfloor \gamma \rfloor$, we note that

$$\begin{aligned} & \left| f^{\circ\lfloor \gamma \rfloor}\left(Y_t\right) - f^{\circ\lfloor \gamma \rfloor}\left(Y_t^{s,t}\right)\right| \\ \leq \quad & \left| f^{\circ\lfloor \gamma \rfloor}\right|_{Lip(\gamma-\lfloor \gamma \rfloor)} \left| Y_t - Y_t^{s,t}\right|^{\gamma-\lfloor \gamma \rfloor} \\ \leq \quad & C_p \left| f^{\circ\lfloor \gamma \rfloor}\right|_{Lip(\gamma-\lfloor \gamma \rfloor)} \left| f\right|_{Lip((\gamma-1)\wedge\lfloor p \rfloor)}^{\gamma\wedge(\lfloor p \rfloor+1)(\gamma-\lfloor \gamma \rfloor)} \left\| X \right\|_{p-var,[s,t]}^{\gamma\wedge(\lfloor p \rfloor+1)(\gamma-\lfloor \gamma \rfloor)}. \end{aligned}$$

In particular, we have

$$\begin{aligned} & \left| f^{\circ\lfloor \gamma \rfloor}\left(Y_t\right) - f^{\circ\lfloor \gamma \rfloor}\left(Y_t^{s,t}\right)\right| \\ \leq \quad & C_p \left| f^{\circ\lfloor \gamma \rfloor}\right|_{Lip(\gamma-\lfloor \gamma \rfloor)} \left| f\right|_{Lip((\gamma-1)\wedge\lfloor p \rfloor)}^{\gamma\wedge(\lfloor p \rfloor+1)(\gamma-\lfloor \gamma \rfloor)} \left( \left\| X \right\|_{p-var,[s,t]} \vee 1\right)^{(\lfloor p \rfloor+1)} \left\| X \right\|_{p-var,[s,t]}^{\gamma-\lfloor \gamma \rfloor}. \end{aligned}$$

To estimate the second term in (2.21), we use Lemma 2.6 to see that for $\lceil \gamma - p \rceil \leq m \leq \lfloor \gamma \rfloor$,

$$\left| f^{\circ m}\left(Y_t^{s,t}\right) - \sum_{k=0}^{\lfloor \gamma \rfloor - m} f^{\circ(m+k)}\left(Y_s\right)\left(x^{s,t}\right)_{s,t}^k \right|$$

$$= \quad \left| \int_{s \leq s_1 \leq \ldots \leq s_{\lfloor \gamma \rfloor - m} < t} f^{\circ(\lfloor \gamma \rfloor)}\left(Y_{s_1}^{s,t}\right) - f^{\circ(\lfloor \gamma \rfloor)}\left(Y_s\right) \mathrm{d}x_{s_1}^{s,t} \ldots \mathrm{d}x_{s_{\lfloor \gamma \rfloor - m}}^{s,t} \right| \tag{2.25}$$

$$\leq \quad C_p^{\lfloor \gamma \rfloor - m} \left| f^{\circ\lfloor \gamma \rfloor}\right|_{Lip(\gamma-\lfloor \gamma \rfloor)} \left| Y_\cdot^{s,t}\right|_{p-var,[s,t]}^{\gamma-\lfloor \gamma \rfloor} \left\| X \right\|_{p-var,[s,t]}^{\lfloor \gamma \rfloor - m}$$

$$\leq \quad C_p' \left| f^{\circ\lfloor \gamma \rfloor}\right|_{Lip(\gamma-\lfloor \gamma \rfloor)} \left( \left| f\right|_{Lip((\gamma-1)\wedge\lfloor p \rfloor)} \vee 1\right)^{p(\gamma-\lfloor \gamma \rfloor)} \tag{2.26}$$

$$\times \left( \left\| X \right\|_{p-var,[s,t]} \vee 1\right)^{(p-1)(\gamma-\lfloor \gamma \rfloor)} \left\| X \right\|_{p-var,[s,t]}^{\gamma-m}, \tag{2.27}$$

where in the third line we have used the $\gamma-\lfloor\gamma\rfloor$ Hölder continuity of $f^{\circ(\lfloor\gamma\rfloor)}$ with (2.18) and in the final line we have used Theorem 10.16 in [3].

Combining (2.21), (2.23) and (2.26), we have for $\lceil\gamma-p\rceil\leq m\leq\lfloor\gamma\rfloor$,

$$\begin{aligned}
&\left|f^{\circ(m)}\left(Y_{t}\right)-\sum_{k=0}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{s}\right)X_{s,t}^{k}\right|\\
\leq\;& 2C_{p}\max_{\lfloor\gamma\rfloor-\lfloor p\rfloor+1\leq m\leq\lfloor\gamma\rfloor}\left|f^{\circ m}\right|_{Lip(\min(\gamma-m,1))}^{\min(\gamma-m,1)}\left(\left|f\right|_{Lip((\gamma-1)\wedge\lfloor p\rfloor)}\vee1\right)^{\lfloor p\rfloor+1}\\
&\times\left(\left\|X\right\|_{p-var}\vee1\right)^{\lfloor p\rfloor+1}\left\|X\right\|_{p-var,[s,t]}^{\gamma-m}. \qquad (2.28)
\end{aligned}$$

Here since $\lceil\gamma-p\rceil\leq m\leq\lfloor\gamma\rfloor$ so $\lfloor\gamma\rfloor-m\leq\lfloor p\rfloor$ and

$$(\lfloor\gamma\rfloor-m)!\leq\lfloor p\rfloor!.$$

Therefore, by changing the constant $C_p$, we rewrite (2.28) in the form of the right hand side of (2.5). It now suffices to show (2.7). Note first that for $0\leq m\leq\lceil\gamma-p\rceil-1$ and $s\leq u\leq v\leq t$,

$$\begin{aligned}
&\left|f^{\circ m}\left(Y_{v}\right)-\sum_{k=0}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)X_{u,v}^{k}\right| && (2.29)\\
\leq\;& \left|f^{\circ m}\left(Y_{v}\right)-f^{\circ m}\left(Y_{v}^{u,v}\right)\right|+\left|f\left(Y_{v}^{u,v}\right)-\sum_{k=0}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)\left(x^{u,v}\right)_{u,v}^{k}\right| && (2.30)\\
&+\left|\sum_{k=\lfloor p\rfloor+1}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)\left(x^{u,v}\right)_{u,v}^{k}-\sum_{k=\lfloor p\rfloor+1}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)X_{u,v}^{k}\right|. && (2.31)
\end{aligned}$$

The estimate (2.22) still holds with $(s,t)$ replaced by $(u,v)$ and (2.26) would hold with the constant $C_p$ now depending on $\gamma$ as well. For the final term in (2.31),

$$\begin{aligned}
&\left|\sum_{k=\lfloor p\rfloor+1}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)\left(x^{u,v}\right)_{u,v}^{k}-\sum_{k=\lfloor p\rfloor+1}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)X_{u,v}^{k}\right|\\
\leq\;& \left|\sum_{k=\lfloor p\rfloor+1}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)\left(x^{u,v}\right)_{u,v}^{k}\right|+\left|\sum_{k=\lfloor p\rfloor+1}^{\lfloor\gamma\rfloor-m}f^{\circ(m+k)}\left(Y_{u}\right)X_{u,v}^{k}\right|\\
\leq\;& 2\lfloor\gamma\rfloor c_{p}^{\lfloor\gamma\rfloor}\max_{0\leq m\leq\lfloor\gamma\rfloor}\sup_{s\leq u\leq t}\left|f^{\circ m}\left(Y_{u}\right)\right|\left(\left\|X\right\|_{p-var,[s,t]}\vee1\right)^{\lfloor\gamma\rfloor}\left\|X\right\|_{p-var,[u,v]}^{\lfloor p\rfloor+1}
\end{aligned}$$

where we used Fact 2.2 and

$$\begin{aligned}
\left|\left(x^{u,v}\right)_{u,v}^{k}\right| &\leq c_{p}^{k}\left(\int_{u}^{v}\left|\mathrm{d}x_{r}^{u,v}\right|\right)^{k}\\
&\leq C_{p}^{k}\left\|X\right\|_{p-var,[u,v]}^{k}.
\end{aligned}$$

Therefore, combining with (2.22) and (2.26), we have for some constants $C_{f,p,X,s,t\gamma}, C'_{f,p,X,s,t\gamma}$

independent of $u,v$ such that when $|u-v|$ is sufficiently small,

$$\begin{aligned}
&\left| f^{\circ m}\left(Y_v\right) - \sum_{k=0}^{\lfloor\gamma\rfloor-m} f^{\circ(m+k)}\left(Y_u\right) X_{u,v}^{k} \right| \\
\leq\ & C_{f,p,X,s,t\gamma}\left( \|X\|_{p-var,[u,v]}^{\gamma\wedge(\lfloor p\rfloor+1)} + \|X\|_{p-var,[u,v]}^{\gamma-m} + \|X\|_{p-var,[u,v]}^{\lfloor p\rfloor+1} \right) \\
\leq\ & C'_{f,p,X,s,t\gamma} \|X\|_{p-var,[u,v]}^{\gamma\wedge(\lfloor p\rfloor+1)}
\end{aligned}$$

Denote the expression in (2.29) as $E(u,v)$. Let $\lim_{|\mathcal{P}|\to 0}$ denote the limit as the mesh size of a partition $\mathcal{P}$ on $[s,t]$ goes to zero. Then for $m\leq\lceil\gamma-p\rceil-1$,

$$\begin{aligned}
&\lim_{|\mathcal{P}|\to 0}\sum_{i=0}^{n-1}\sum_{l=1}^{\lfloor\gamma\rfloor-m} E\left(t_i,t_{i+1}\right) \\
\leq\ & C'_{f,p,X,s,t\gamma}\lim_{|\mathcal{P}|\to 0}\sum_{i=0}^{n-1}\|X\|_{p-var,[t_i,t_{i+1}]}^{\gamma\wedge(\lfloor p\rfloor+1)} && (2.32)\\
\leq\ & C'_{f,p,X,\gamma}\lim_{|\mathcal{P}|\to 0}\max_i\|X\|_{p-var,[t_i,t_{i+1}]}^{\gamma\wedge(\lfloor p\rfloor+1)-p}\sum_{i=0}^{n-1}\|X\|_{p-var,[t_i,t_{i+1}]}^{p} && (2.33)
\end{aligned}$$

Since for $s<u<t$ ,

$$\|X\|_{p-var,[s,u]}^{p}+\|X\|_{p-var,[u,t]}^{p}\leq\|X\|_{p-var,[s,t]}^{p},$$

(2.33) is bounded by

$$C_{f,p,X,\gamma}\lim_{|\mathcal{P}|\to 0}\max_i\|X\|_{p-var,[t_i,t_{i+1}]}^{\gamma\wedge(\lfloor p\rfloor+1)-p}\|X\|_{p-var,[s,t]}^{p},$$

which equals $0$ by the uniform continuity of the map $(u,v)\to\|X\|_{p-var,[u,v]}^{p}$ (See [8]). Finally,

$$\begin{aligned}
&\lim_{|\mathcal{P}|\to 0}\sum_{i=0}^{n-1}\sum_{l=1}^{\lfloor\gamma\rfloor-m} f^{\circ(m+l)}\left(Y_{t_i}\right)X_{t_i,t_{i+1}}^{l} \\
=\ & \lim_{|\mathcal{P}|\to 0}\sum_{i=0}^{n-1} f^{\circ m}\left(Y_{t_{i+1}}\right)-f^{\circ m}\left(Y_{t_i}\right)+E\left(t_i,t_{i+1}\right) \\
=\ & f^{\circ m}\left(Y_t\right)-f^{\circ m}\left(Y_s\right).
\end{aligned}$$

□

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