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GENERALIZED GOLDEN RATIOS OVER INTEGER ALPHABETS

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Abstract

It is a well known result that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ there exists uncountably many $(\epsilon_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} \epsilon_i \beta^{-i}$. When $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$ there exists $x \in (0, \frac{1}{\beta-1})$ for which there exists a unique $(\epsilon_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} \epsilon_i \beta^{-i}$. In this paper we consider the more general case when our sequences are elements of $\{0, \dots, m\}^{\mathbb{N}}$. We show that an analogue of the golden ratio exists and give an explicit formula for it.

1. Introduction

Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $I_{\beta, m} = [0, \frac{m}{\beta-1}]$. Each $x \in I_{\beta, m}$ has an expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i},$$

for some $(\epsilon_i)_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}}$. We call such a sequence a β -expansion for x . For $x \in I_{\beta, m}$ we denote the set of β -expansions for x by $\Sigma_{\beta, m}(x)$, i.e.,

$$\Sigma_{\beta, m}(x) = \left\{ (\epsilon_i)_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x \right\}.$$

In [6] the authors consider the case when $m = 1$, they show that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ the set $\Sigma_{\beta, 1}(x)$ is uncountable for every $x \in (0, \frac{1}{\beta-1})$. The endpoints of $[0, \frac{1}{\beta-1}]$ trivially have a unique β -expansion. In [5] it is shown that for $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$ there exists $x \in (0, \frac{1}{\beta-1})$ with a unique β -expansion.

For $m \in \mathbb{N}$ we define $\mathcal{G}(m) \in \mathbb{R}$ to be a *generalized golden ratio* for m if for $\beta \in (1, \mathcal{G}(m))$ the set $\Sigma_{\beta, m}(x)$ is uncountable for every $x \in (0, \frac{m}{\beta-1})$, and for every $\beta \in (\mathcal{G}(m), m+1]$ there exists $x \in (0, \frac{m}{\beta-1})$ for which $|\Sigma_{\beta, m}(x)| = 1$.

In [11] the authors consider a similar setup. They consider the case where β -expansions are elements of $\{a_1, a_2, a_3\}^{\mathbb{N}}$, for some $a_1, a_2, a_3 \in \mathbb{R}$. They show that for each ternary alphabet there exists a constant $G \in \mathbb{R}$ such that, there exists nontrivial unique β -expansions if and only if $\beta > G$. Moreover they give an explicit formula for G .

Our main result is the following.

Theorem 1.1. *For each $m \in \mathbb{N}$ a generalized golden ratio exists and is equal to:*

$$\mathcal{G}(m) = \begin{cases} k + 1 & \text{if } m = 2k \\ \frac{k+1+\sqrt{k^2+6k+5}}{2} & \text{if } m = 2k + 1. \end{cases} \tag{1}$$

Remark 1.2. $\mathcal{G}(m)$ is a Pisot number for all $m \in \mathbb{N}$. Recall a Pisot number is a real algebraic integer greater than 1 whose Galois conjugates are of modulus strictly less than 1.

In section 6 we include a table of values for $\mathcal{G}(m)$. We prove Theorem 1.1 in section 3. In section 4 we consider the set of points with unique β -expansion for $\beta \in (\mathcal{G}(m), m + 1]$, and in section 5 we study the growth rate and dimension theory of the set of β -expansions for $\beta \in (1, \mathcal{G}(m))$.

2. Preliminaries

Before proving Theorem 1.1 we require the following preliminary results and theory. Let $m \in \mathbb{N}$ be fixed and $\beta \in (1, m + 1]$. For each $i \in \{0, \dots, m\}$ we fix $T_{\beta,i}(x) = \beta x - i$. The proof of the following lemma is trivial and therefore omitted.

Lemma 2.1. *The map $T_{\beta,i}$ satisfies the following:*

- $T_{\beta,i}$ has a unique fixed point equal to $\frac{i}{\beta-1}$.
- $T_{\beta,i}(x) > x$ for all $x > \frac{i}{\beta-1}$,
- $T_{\beta,i}(x) < x$ for all $x < \frac{i}{\beta-1}$,
- $|T_{\beta,i}(x) - T_{\beta,i}(\frac{i}{\beta-1})| = \beta|x - \frac{i}{\beta-1}|$, for all $x \in \mathbb{R}$, that is $T_{\beta,i}$ scales the distance between the fixed point $\frac{i}{\beta-1}$ and an arbitrary point by a factor β .

Understanding where in $I_{\beta,m}$ these fixed points are will be important in our later analysis.

We let

$$\Omega_{\beta,m}(x) = \left\{ (a_i)_{i=1}^{\infty} \in \{T_{\beta,0}, \dots, T_{\beta,m}\}^{\mathbb{N}} : (a_n \circ a_{n-1} \circ \dots \circ a_1)(x) \in I_{\beta,m} \text{ for all } n \in \mathbb{N} \right\}.$$

Similarly we define

$$\Omega_{\beta,m,n}(x) = \left\{ (a_i)_{i=1}^n \in \{T_{\beta,0}, \dots, T_{\beta,m}\}^n : (a_n \circ a_{n-1} \circ \dots \circ a_1)(x) \in I_{\beta,m} \right\}.$$

Typically we will denote an element of $\Omega_{\beta,m,n}(x)$ or any finite sequence of maps by a . When we want to emphasise the length of a we will use the notation $a^{(n)}$. We also adopt the notation $a^{(n)}(x)$ to mean $(a_n \circ a_{n-1} \circ \dots \circ a_1)(x)$.

Remark 2.2. *It is important to note that if for some finite sequence of maps a , $a(x) \notin I_{\beta,m}$ then we cannot concatenate a by any finite sequence of maps b , such that $b(a(x)) \in I_{\beta,m}$.*

Remark 2.3. *Let $\beta \in (1, m + 1]$. For any $x \in I_{\beta,m}$ there always exists $i \in \{0, \dots, m\}$ such that $T_{\beta,i}(x) \in I_{\beta,m}$. For $\beta > m + 1$ such an i does not always exist.*

Lemma 2.4. $|\Sigma_{\beta,m}(x)| = |\Omega_{\beta,m}(x)|$.

Proof. It is a simple exercise to show that

$$\Sigma_{\beta,m}(x) = \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \dots, m\}^\mathbb{N} : x - \sum_{i=1}^n \frac{\epsilon_i}{\beta^i} \in \left[0, \frac{m}{\beta^n(\beta-1)}\right] \text{ for all } n \in \mathbb{N} \right\}.$$

Following [8] we observe that

$$\begin{aligned} \Sigma_{\beta,m}(x) &= \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \dots, m\}^\mathbb{N} : x - \sum_{i=1}^n \frac{\epsilon_i}{\beta^i} \in \left[0, \frac{m}{\beta^n(\beta-1)}\right] \text{ for all } n \in \mathbb{N} \right\} \\ &= \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \dots, m\}^\mathbb{N} : \beta^n x - \sum_{i=1}^n \epsilon_i \beta^{n-i} \in I_{\beta,m} \text{ for all } n \in \mathbb{N} \right\} \\ &= \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \dots, m\}^\mathbb{N} : (T_{\beta,\epsilon_n} \circ \dots \circ T_{\beta,\epsilon_1})(x) \in I_{\beta,m} \text{ for all } n \in \mathbb{N} \right\}. \end{aligned}$$

Our result follows immediately. □

By Lemma 2.4 we can rephrase the definition of a generalized golden ratio in terms of the set $\Omega_{\beta,m}(x)$. This equivalent definition will be more suitable for our purposes. The set $\Omega_{\beta,m,n}(x)$ will be useful when we study the growth rate and dimension theory of the set of β -expansions.

For a point $x \in I_{\beta,m}$ we can take i to be the first digit in a β -expansion for x if and only if $\beta x - i \in I_{\beta,m}$. This is equivalent to

$$x \in \left[\frac{i}{\beta}, \frac{i\beta + m - i}{\beta(\beta - 1)} \right],$$

as such we refer to the interval $\left[\frac{i}{\beta}, \frac{i\beta + m - i}{\beta(\beta - 1)} \right]$ as the i -th digit interval. Generally speaking we can take i to be the j -th digit in a β -expansion for x if and only if there exists $a \in \Omega_{\beta,m,j-1}(x)$ such that, $a(x) \in \left[\frac{i}{\beta}, \frac{i\beta + m - i}{\beta(\beta - 1)} \right]$. When x or an image of x is contained in

the intersection of two digit intervals we have a choice of digit in our β -expansion for x . Generally speaking any two digit intervals may intersect for β sufficiently small, however for our purposes we need only consider the case when the i -th digit interval intersects the adjacent $(i - 1)$ -th or $(i + 1)$ -th digit intervals, for some $i \in \{0, \dots, m\}$. Any intersection of this type is of the form

$$\left[\frac{i}{\beta}, \frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)} \right],$$

for some $i \in \{1, \dots, m\}$. In what follows we refer to the interval $\left[\frac{i}{\beta}, \frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)} \right]$ as the i -th choice interval. Both $T_{\beta, i-1}$ and $T_{\beta, i}$ map the i -th choice interval into $I_{\beta, m}$. These intervals always exist and are nontrivial for $\beta \in (1, m + 1)$.

Proposition 2.5. *Suppose for any $x \in (0, \frac{m}{\beta-1})$ there always exists a finite sequence of maps that map x into the interior of a choice interval, then $\Omega_{\beta, m}(x)$ is uncountable.*

The proof of this proposition is essentially contained in the proof of Theorem 1 in [17].

Proof. Let $x \in (0, \frac{m}{\beta-1})$. Suppose there exists $n \in \mathbb{N}$ and $a \in \Omega_{\beta, m, n}(x)$ such that $a(x) \in (\frac{i}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)})$, for some $i \in \{1, \dots, m\}$. As $a(x)$ is an element of the interior of a choice interval both $T_{\beta, i-1}(a(x)) \in (0, \frac{m}{\beta-1})$ and $T_{\beta, i}(a(x)) \in (0, \frac{m}{\beta-1})$. As such our hypothesis applies to both $T_{\beta, i-1}(a(x))$ and $T_{\beta, i}(a(x))$, and we can assert that there exists a finite sequence of maps that map these two distinct images of x into the interior of another choice interval. Repeating this procedure arbitrarily many times it is clear that $\Omega_{\beta, m}(x)$ is uncountable. \square

By Proposition 2.5, to prove Theorem 1.1 it suffices to show that for $\beta \in (1, \mathcal{G}(m))$ every $x \in (0, \frac{m}{\beta-1})$ can be mapped into the interior of a choice interval, and for $\beta \in (\mathcal{G}(m), m + 1]$ there exists $x \in (0, \frac{m}{\beta-1})$ that never maps into a choice interval.

We define the *switch region* to be the interval

$$\left[\frac{1}{\beta}, \frac{(m - 1)\beta + 1}{\beta(\beta - 1)} \right].$$

The significance of this interval is that if a point x has a choice of digit in the j -th entry of a β -expansion, then there exists $a \in \Omega_{\beta, m, j-1}(x)$ such that $a(x) \in [\frac{1}{\beta}, \frac{(m-1)\beta+1}{\beta(\beta-1)}]$. The following lemmas are useful in understanding the dynamics of the maps $T_{\beta, i}$ around the switch region, understanding these dynamics will be important in our proof of Theorem 1.1.

Lemma 2.6. *For $\beta \in (1, \frac{m+\sqrt{m^2+4}}{2})$ and $x \in (0, \frac{m}{\beta-1})$ there exists a finite sequence of maps that map x into the interior of our switch region.*

Proof. If x is contained within the interior of the switch region we are done, let us suppose otherwise. By the monotonicity of the maps $T_{\beta, 0}$ and $T_{\beta, m}$ it suffices to show that

$$T_{\beta, 0}\left(\frac{1}{\beta}\right) < \frac{(m - 1)\beta + 1}{\beta(\beta - 1)} \text{ and } T_{\beta, m}\left(\frac{(m - 1)\beta + 1}{\beta(\beta - 1)}\right) > \frac{1}{\beta}.$$

Both of these inequalities are equivalent to $\beta^2 - m\beta - 1 < 0$, applying the quadratic formula we can conclude our result. \square

Remark 2.7. When $m = 1$ the switch region is a choice interval. An application of Lemma 2.4, Proposition 2.5 and Lemma 2.6 yields the result stated in [6], i.e, for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ the set $\Sigma_{\beta,1}(x)$ is uncountable.

Lemma 2.8. For $\beta \in (1, \frac{m+2}{2})$ every x in the interior of the switch region is contained in the interior of a choice interval.

Proof. It suffices to show that for each $i \in \{1, 2, \dots, m - 1\}$ the $(i - 1)$ -th and $(i + 1)$ -th digit intervals intersect in a nontrivial interval. This is equivalent to

$$\frac{i + 1}{\beta} < \frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)},$$

a simple manipulation yields that this is equivalent to $\beta < \frac{m+2}{2}$. \square

We refer the reader to Figure 1 for a diagram depicting the case where $\beta < \frac{m+2}{2}$. For $i \in \{1, 2, \dots, m - 1\}$ and $\beta \geq \frac{m+2}{2}$ the interval

$$\left[\frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)}, \frac{i + 1}{\beta} \right]$$

is well defined. We refer to this interval as the i -th fixed digit interval. The significance of this interval is that if a point x is contained in the interior of the i -th fixed digit interval only $T_{\beta,i}$ maps x into $I_{\beta,m}$. Similarly we define the 0-th fixed digit interval to be $[0, \frac{1}{\beta}]$ and the m -th fixed digit interval to be $[\frac{(m-1)\beta+1}{\beta(\beta-1)}, \frac{m}{\beta-1}]$. Understanding how the different $T_{\beta,i}$'s behave on these intervals will be important when it comes to constructing generalized golden ratios in the case where m is odd.

3. Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1, for ease of exposition we reduce our analysis to two cases, when m is even and when m is odd.

3.1. Case where m is even

In what follows we assume $m = 2k$ for some $k \in \mathbb{N}$.

Proposition 3.1. For $\beta \in (1, k + 1)$ every $x \in (0, \frac{m}{\beta-1})$ has uncountably many β -expansions.



Figure 1: The case where $\beta \in (1, \frac{m+2}{2})$

Proof. By Lemma 2.4 and Proposition 2.5 it suffices to show that every $x \in (0, \frac{m}{\beta-1})$ can be mapped into the interior of a choice interval. It is a simple exercise to show that $\frac{m+2}{2} < \frac{m+\sqrt{m^2+4}}{2}$ for all $m \in \mathbb{N}$, therefore for $\beta \in (1, k + 1)$ we can apply Lemma 2.6, therefore there exists a sequence of maps that map x into the interior of the switch region. By Lemma 2.8 every point in the interior of our switch region is contained in the interior of a choice interval. \square

Proposition 3.2. For $\beta \in (k + 1, m + 1]$ there exists $x \in (0, \frac{m}{\beta-1})$ with a unique β -expansion.

Proof. It suffices to show that there exists $x \in (0, \frac{m}{\beta-1})$ that never maps into a choice interval. We consider the point $\frac{k}{\beta-1}$, we will show that this point has a unique β -expansion. This point is contained in the k -th digit interval and is the fixed point under the map $T_{\beta,k}$. To show that it has a unique β -expansion it suffices to show that it is not contained within the $(k - 1)$ -th or $(k + 1)$ -th digit intervals, this is equivalent to

$$\frac{(k - 1)\beta + m - (k - 1)}{\beta(\beta - 1)} < \frac{k}{\beta - 1} < \frac{k + 1}{\beta}.$$

Both of these inequalities are equivalent to $\beta > k + 1$. \square

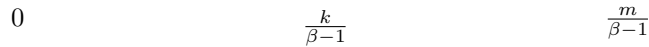


Figure 2: A point with unique β -expansion for $\beta \in (k + 1, m + 1]$.

Figure 2 describes the construction of our point with unique β -expansion for $\beta \in (k + 1, m + 1]$. By Proposition 3.1 and Proposition 3.2 we can conclude Theorem 1.1 in the case where m is even.

3.2. Case where m is odd

The analysis of the case where m is odd is somewhat more intricate. In what follows we assume $m = 2k + 1$ for some $k \in \mathbb{N}$. Before finishing our proof of Theorem 1.1 we require the following technical results.

Lemma 3.3. *For $\beta \in (1, k + 2)$ the fixed point of $T_{\beta,i}$ is contained in the interior of the choice interval $[\frac{i}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ for $i \in \{1, \dots, k\}$, and in the interior of the choice interval $[\frac{i+1}{\beta}, \frac{i\beta+m-i}{\beta(\beta-1)}]$ for $i \in \{k + 1, \dots, m - 1\}$.*

Proof. Let $i \in \{1, \dots, k\}$. To show that the fixed point $\frac{i}{\beta-1}$ is contained in the interior of the interval $[\frac{i}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ it suffices to show that

$$\frac{i}{\beta - 1} < \frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)}.$$

This is equivalent to $\beta < m + 1 - i$, which for $\beta \in (1, k + 2)$ is true for all $i \in \{1, \dots, k\}$. The case where $i \in \{k + 1, \dots, m - 1\}$ is proved similarly. \square

Corollary 3.4. For $\beta \in [\frac{2k+3}{2}, k + 2)$ the map $T_{\beta,i}$ satisfies $T_{\beta,i}(x) - \frac{i}{\beta-1} = \beta(x - \frac{i}{\beta-1})$ for all x contained in the i -th fixed digit interval for $i \in \{1, \dots, k\}$, and $\frac{i}{\beta-1} - T_{\beta,i}(x) = \beta(\frac{i}{\beta-1} - x)$ for all x contained in the i -th fixed digit interval for $i \in \{k + 1, \dots, m - 1\}$.

Proof. Let $i \in \{1, \dots, k\}$. By Lemma 3.3 the i -th fixed digit interval is to the right of the fixed point of $T_{\beta,i}$, our result follows from Lemma 2.1. The case where $i \in \{k + 1, \dots, m - 1\}$ is proved similarly. \square

Lemma 3.5. Suppose $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ and x is an element of the i -th fixed digit interval for some $i \in \{1, \dots, m - 1\}$. For $i \in \{1, \dots, k\}$

$$T_{\beta,i}(x) < \frac{k\beta + m - k}{\beta(\beta - 1)}$$

and for $i \in \{k + 1, \dots, m - 1\}$

$$T_{\beta,i}(x) > \frac{k + 1}{\beta}.$$

Proof. By the monotonicity of the maps $T_{\beta,i}$ it is sufficient to show that

$$T_{\beta,i}\left(\frac{i + 1}{\beta}\right) < \frac{k\beta + m - k}{\beta(\beta - 1)}$$

for $i \in \{1, \dots, k\}$, and

$$T_{\beta,i}\left(\frac{(i - 1)\beta + m - (i - 1)}{\beta(\beta - 1)}\right) > \frac{k + 1}{\beta},$$

for $i \in \{k + 1, \dots, m - 1\}$. Each of these inequalities are equivalent to $\beta^2 - (k + 1)\beta - (k + 1) < 0$. Our result follows by an application of the quadratic formula. \square

Proposition 3.6. For $\beta \in (1, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ every $x \in (0, \frac{m}{\beta-1})$ has uncountably many β -expansions.

Proof. The proof where $\beta \in (1, \frac{2k+3}{2})$ is analogous to that given in the even case. As such, in what follows we assume $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. We remark that

$$\frac{k + 1 + \sqrt{k^2 + 6k + 5}}{2} \leq \frac{m + \sqrt{m^2 + 4}}{2}$$

and

$$\frac{k + 1 + \sqrt{k^2 + 6k + 5}}{2} < k + 2,$$

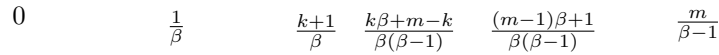


Figure 3: A diagram of the case where $m = 2k + 1$ and $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$

for all $k \in \mathbb{N}$. We can therefore use Lemma 2.6 and Corollary 3.4. Let $x \in (0, \frac{m}{\beta-1})$. We will show that there exists a sequence of maps that map x into the interior of a choice interval, by Lemma 2.4 and Proposition 2.5 our result follows. By Lemma 2.6 there exist a finite sequence of maps that map x into the interior of the switch region. Suppose the image of x is not contained in the interior of a choice interval, then it must be contained in the i -th fixed digit interval for some $i \in \{1, \dots, m - 1\}$. By repeatedly applying Corollary 3.4 and Lemma 3.5 the image of x must eventually be mapped into the interior of a choice interval. \square

We refer the reader to Figure 3 for a diagram illustrating the case where $m = 2k + 1$ and $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$.

Proposition 3.7. For $\beta \in (\frac{k+1+\sqrt{k^2+6k+5}}{2}, m + 1]$ there exists $x \in (0, \frac{m}{\beta-1})$ that has a unique β -expansion.

Proof. We will show that the points

$$\frac{k\beta + (k + 1)}{\beta^2 - 1} \text{ and } \frac{(k + 1)\beta + k}{\beta^2 - 1}$$

have a unique β -expansion. The significance of these points is that

$$T_{\beta,k}\left(\frac{k\beta + (k + 1)}{\beta^2 - 1}\right) = \frac{(k + 1)\beta + k}{\beta^2 - 1}$$

and

$$T_{\beta,k+1}\left(\frac{(k + 1)\beta + k}{\beta^2 - 1}\right) = \frac{k\beta + (k + 1)}{\beta^2 - 1}.$$

To show that these points have a unique β -expansion it suffices to show that $\frac{k\beta+(k+1)}{\beta^2-1}$ and $\frac{(k+1)\beta+k}{\beta^2-1}$ belong to the k -th and $(k + 1)$ -th fixed digit intervals, respectively, i.e.

$$\frac{(k - 1)\beta + m - (k - 1)}{\beta(\beta - 1)} < \frac{k\beta + (k + 1)}{\beta^2 - 1} < \frac{k + 1}{\beta}, \tag{2}$$

and

$$\frac{k\beta + (m - k)}{\beta(\beta - 1)} < \frac{(k + 1)\beta + k}{\beta^2 - 1} < \frac{k + 2}{\beta}. \tag{3}$$

The left hand side of (2) is equivalent to $0 < \beta^2 - k\beta - (k + 2)$ which is equivalent to

$$\frac{k + \sqrt{k^2 + 4k + 8}}{2} < \beta,$$

in the same time

$$\frac{k + \sqrt{k^2 + 4k + 8}}{2} < \frac{k + 1 + \sqrt{k^2 + 6k + 5}}{2}$$

for all $k \in \mathbb{N}$, therefore the left hand side of (2) holds. The right hand side of (2) is equivalent to $0 < \beta^2 - (k + 1)\beta - (k + 1)$. So (2) holds by the quadratic formula.

The right hand side of (3) is equivalent to $0 < \beta^2 - k\beta - (k + 2)$ which we know to be true by the above. Similarly the left hand side of (3) is equivalent to $0 < \beta^2 - (k + 1)\beta - (k + 1)$, which we also know to be true. It follows that both $\frac{k\beta+(k+1)}{\beta^2-1}$ and $\frac{(k+1)\beta+k}{\beta^2-1}$ are never mapped into a choice interval and have a unique β -expansion for $\beta \in (\frac{k+1+\sqrt{k^2+6k+5}}{2}, m + 1]$. \square

We refer the reader to Figure 4 for a diagram describing the points we constructed with unique β -expansion for $\beta \in (\frac{k+1+\sqrt{k^2+6k+5}}{2}, m + 1]$. By Proposition 3.6 and Proposition 3.7 we can conclude Theorem 1.1.

4. The set of points with unique β -expansion

In this section we study the set of points whose β -expansion is unique for $\beta \in (\mathcal{G}(m), m + 1]$. Let

$$U_{\beta,m} = \left\{ x \in I_{\beta,m} \mid |\Sigma_{\beta,m}(x)| = 1 \right\}$$

$$0 \qquad \frac{k\beta+(k+1)}{\beta^2-1} \quad \frac{(k+1)\beta+k}{\beta^2-1} \qquad \frac{m}{\beta-1}$$

Figure 4: A point with unique β -expansion for $\beta \in (\frac{k+1+\sqrt{k^2+6k+5}}{2}, m + 1]$.

and

$$W_{\beta,m} = \left\{ x \in \left(\frac{m+1-\beta}{\beta-1}, 1 \right) \mid |\Sigma_{\beta,m}(x)| = 1 \right\}.$$

The significance of the set $W_{\beta,m}$ is that if $x \in U_{\beta,m}$, then it maps to $W_{\beta,m}$ under a finite sequence of $T_{\beta,i}$'s. In [9] the authors study the case where $m = 1$, they show that the following theorems hold.

Theorem 4.1. *The set $U_{\beta,1}$ satisfies the following:*

1. $|U_{\beta,1}| = \aleph_0$ for $\beta \in (\frac{1+\sqrt{5}}{2}, \beta_c)$
2. $|U_{\beta,1}| = 2^{\aleph_0}$ for $\beta = \beta_c$
3. $U_{\beta,1}$ is a set of positive Hausdorff dimension for $\beta \in (\beta_c, 2]$.

Theorem 4.2. *The set $W_{\beta,1}$ satisfies the following:*

1. $|W_{\beta,1}| = 2$ for $\beta \in (\frac{1+\sqrt{5}}{2}, \beta_f]$, where β_f is the root of the equation $x^3 - 2x^2 + x - 1 = 0$, $\beta_f = 1.75487\dots$

2. $|W_{\beta,1}| = \aleph_0$ for $\beta \in (\beta_f, \beta_c)$
3. $|W_{\beta,1}| = 2^{\aleph_0}$ for $\beta = \beta_c$
4. $W_{\beta,1}$ is a set of positive Hausdorff dimension for $\beta \in (\beta_c, 2]$.

Here $\beta_c \approx 1.78723$ is the Komornik-Loreti constant introduced in [12]. It is the smallest value of β for which $1 \in U_{\beta,1}$. Moreover β_c is the unique solution of the equation

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\beta^i} = 1,$$

where $(\lambda_i)_{i=0}^{\infty}$ is the Thue-Morse sequence (see [3]), i.e. $\lambda_0 = 0$ and if λ_i is already defined for some $i \geq 0$ then $\lambda_{2i} = \lambda_i$ and $\lambda_{2i+1} = 1 - \lambda_i$. The sequence $(\lambda_i)_{i=0}^{\infty}$ begins

$$(\lambda_i)_{i=0}^{\infty} = 0110\ 1001\ 1001\ 0110\ 1001\ \dots$$

In [2] it was shown that β_c is transcendental. For $m \geq 2$ we define the sequence $(\lambda_i(m))_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}}$ as follows:

$$\lambda_i(m) = \begin{cases} k + \lambda_i - \lambda_{i-1} & \text{if } m = 2k \\ k + \lambda_i & \text{if } m = 2k + 1. \end{cases}$$

We define $\beta_c(m)$ to be the unique solution of

$$\sum_{i=1}^{\infty} \frac{\lambda_i(m)}{\beta^i} = 1.$$

In [13] the authors proved that $\beta_c(m)$ is transcendental and the smallest value of β for which $1 \in U_{\beta,m}$. In section 6 we include a table of values for $\beta_c(m)$. We begin our study of the sets $U_{\beta,m}$ and $W_{\beta,m}$ by showing that the following proposition holds.

Proposition 4.3. *If $m \geq 2$, then $|U_{\beta,m}| \geq \aleph_0$ for $\beta \in (\mathcal{G}(m), m + 1]$.*

In [14] the following statements were shown to hold: if $\beta \in (1, \beta_c(m))$ then $U_{\beta,m}$ is countable, $U_{\beta_c(m),m}$ has cardinality equal to that of the continuum, and for $\beta \in (\beta_c(m), m + 1]$ the Hausdorff dimension of $U_{\beta,m}$ is strictly positive. Combining these results with Proposition 4.3 the following analogue of Theorem 4.1 is immediate.

Theorem 4.4. *For $m \geq 2$ the set $U_{\beta,m}$ satisfies the following:*

1. $|U_{\beta,m}| = \aleph_0$ for $\beta \in (\mathcal{G}(m), \beta_c(m))$
2. $|U_{\beta,m}| = 2^{\aleph_0}$ for $\beta = \beta_c(m)$
3. $U_{\beta,m}$ is a set of positive Hausdorff dimension for $\beta \in (\beta_c(m), m + 1]$.

Proof of Proposition 4.3. To begin with let us assume $m = 2k$ for some $k \in \mathbb{N}$, in this case $\mathcal{G}(m) = k + 1$. It is a simple exercise to show that for $\beta \in (k + 1, m + 1]$

$$T_{\beta,0}^{-n}\left(\frac{k}{\beta-1}\right) = \frac{k}{\beta^n(\beta-1)} < \frac{1}{\beta} \tag{4}$$

for all $n \in \mathbb{N}$. By the proof of Proposition 3.2 we know that $\frac{k}{\beta-1}$ has a unique β -expansion. It follows from (4) that $T_{\beta,0}^{-n}\left(\frac{k}{\beta-1}\right)$ is never mapped into a choice interval and therefore has a unique β -expansion. As n was arbitrary we can conclude our result. The case where $m = 2k + 1$ is proved similarly, in this case we can consider preimages of $\frac{k\beta+(k+1)}{\beta^2-1}$. \square

We also show that the following analogue of Theorem 4.2 holds.

Theorem 4.5. *If $m = 2k$ the set $W_{\beta,m}$ satisfies the following:*

1. $|W_{\beta,m}| = 1$ for $\beta \in (\mathcal{G}(m), \beta_f(m)]$, where $\beta_f(m)$ is the root of the equation

$$x^2 - (k + 1)x - k = 0, \beta_f(m) = \frac{k + 1 + \sqrt{k^2 + 6k + 1}}{2}$$

2. $|W_{\beta,m}| = \aleph_0$ for $\beta \in (\beta_f(m), \beta_c(m))$
3. $|W_{\beta,m}| = 2^{\aleph_0}$ for $\beta = \beta_c(m)$
4. $W_{\beta,m}$ is a set of positive Hausdorff dimension for $\beta \in (\beta_c(m), m + 1]$.

If $m = 2k + 1$ the set $W_{\beta,m}$ satisfies the following:

1. $|W_{\beta,m}| = 2$ for $\beta \in (\mathcal{G}(m), \beta_f(m)]$, where $\beta_f(m)$ is the root of the equation

$$x^3 - (k + 2)x^2 + x - (k + 1) = 0$$

2. $|W_{\beta,m}| = \aleph_0$ for $\beta \in (\beta_f(m), \beta_c(m))$
3. $|W_{\beta,m}| = 2^{\aleph_0}$ for $\beta = \beta_c(m)$
4. $W_{\beta,m}$ is a set of positive Hausdorff dimension for $\beta \in (\beta_c(m), m + 1]$.

Remark 4.6. $\beta_f(m)$ is a Pisot number for all $m \in \mathbb{N}$.

Using Theorem 4.4, to prove Theorem 4.5 it suffices to show that statement 1 holds in both the odd and even cases and $|W_{\beta,m}| \geq \aleph_0$ for $\beta > \beta_f(m)$ in both the odd and even cases. In section 6 we include a table of values for $\beta_f(m)$.

4.1. Proof of Theorem 4.5

The proof of Theorem 4.5 is more involved than Theorem 4.4 and as we will see requires more technical results. The following is taken from [14]. Firstly let us define the lexicographic order on $\{0, \dots, m\}^{\mathbb{N}}$: we say that $(x_i)_{i=1}^{\infty} < (y_i)_{i=1}^{\infty}$ with respect to the lexicographic order if there exists $n \in \mathbb{N}$ such that $x_i = y_i$ for all $i < n$ and $x_n < y_n$ or if $x_1 < y_1$. For a sequence $(x_i)_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}}$ we define $(\bar{x}_i)_{i=1}^{\infty} = (m - x_i)_{i=1}^{\infty}$. We also adopt the notation $(\epsilon_1, \dots, \epsilon_j)^{\infty}$ to denote the element of $\{0, \dots, m\}^{\mathbb{N}}$ obtained by the infinite concatenation of the finite sequence $(\epsilon_1, \dots, \epsilon_j)$. Let the sequence $(d_i(m))_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}}$ be defined as follows: let $d_1(m)$ be the largest element of $\{0, \dots, m\}$ such that $\frac{d_1(m)}{\beta} < 1$, and if $d_i(m)$ is defined for $i < n$ then $d_n(m)$ is defined to be the largest element of $\{0, \dots, m\}$ such that $\sum_{i=1}^n \frac{d_i(m)}{\beta^i} < 1$. The sequence $(d_i(m))_{i=1}^{\infty}$ is called the quasi-greedy expansion of 1 with respect to β ; it is trivially a β -expansion for 1 and the largest infinite β -expansion of 1 with respect to the lexicographic order not ending with $(0)^{\infty}$. We let

$$S_{\beta,m} = \left\{ (\epsilon_i)_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} \in W_{\beta,m} \right\},$$

it follows from the definition of $W_{\beta,m}$ that $|W_{\beta,m}| = |S_{\beta,m}|$ and to prove Theorem 4.5 it suffices to show that equivalent statements hold for $S_{\beta,m}$. The following lemma which is essentially due to Parry [15] provides a useful characterisation of $S_{\beta,m}$.

Lemma 4.7.

$$S_{\beta,m} = \left\{ (\epsilon_i)_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}} : (\epsilon_i, \epsilon_{i+1}, \dots) < (d_{1,m}, d_{2,m}, \dots) \text{ and } (\bar{d}_{1,m}, \bar{d}_{2,m}, \dots) < (\epsilon_i, \epsilon_{i+1}, \dots) \text{ for all } i \in \mathbb{N} \right\}$$

Remark 4.8. *If $\beta < \beta'$ then the quasi-greedy expansion of 1 with respect to β is lexicographically strictly less than the quasi-greedy expansion of 1 with respect to β' . As a corollary of this we have $S_{\beta,m} \subseteq S_{\beta',m}$ for $\beta < \beta'$.*

Proposition 4.9. *For $\beta \in (\mathcal{G}(m), \beta_f(m)]$ $|S_{\beta,m}| = 1$ when m is even, $|S_{\beta,m}| = 2$ when m is odd and $|S_{\beta,m}| \geq \aleph_0$ for $\beta \in (\beta_f(m), m + 1]$.*

By the remarks following Theorem 4.5 this will allow us to conclude our result.

Proof. We begin by considering the case where $m = 2k$. When $\beta = \beta_f(m)$ we have $(d_i(m))_{i=1}^{\infty} = (k + 1, k - 1)^{\infty}$ and by Lemma 4.7

$$S_{\beta_f(m),m} = \left\{ (\epsilon_i)_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}} : (\epsilon_i, \epsilon_{i+1}, \dots) < (k + 1, k - 1)^{\infty} \text{ and } (k - 1, k + 1)^{\infty} < (\epsilon_i, \epsilon_{i+1}, \dots) \text{ for all } i \in \mathbb{N} \right\}.$$

By our previous analysis we know that for $\beta \in (\mathcal{G}(m), m + 1]$ the point $\frac{k}{\beta - 1}$ has a unique β -expansion, the β -expansion of this point is the sequence $(k)^{\infty}$. By Remark 4.8, to prove

$|S_{\beta,m}| = 1$ for $\beta \in (\mathcal{G}(m), \beta_f(m)]$ it suffices to show that $S_{\beta_f(m),m} = \{(k)^\infty\}$. If $(\epsilon_i)_{i=1}^\infty \in S_{\beta_f(m),m}$, then clearly ϵ_i must equal $k - 1, k$ or $k + 1$. If $\epsilon_i = k + 1$ then by Lemma 4.7 $\epsilon_{i+1} = k - 1$, similarly if $\epsilon_i = k - 1$ then $\epsilon_{i+1} = k + 1$. Therefore if $\epsilon_i \neq k$ for some i , then $(\epsilon_i, \epsilon_{i+1}, \dots)$ must equal $(k - 1, k + 1)^\infty$ or $(k + 1, k - 1)^\infty$. By Lemma 4.7 this cannot happen and we can conclude that $S_{\beta_f(m),m} = \{(k)^\infty\}$. For $\beta \in (\beta_f(m), m + 1]$, we can construct a countable subset of $S_{\beta,m}$; for example all sequences of the form $(k)^j(k + 1, k - 1)^\infty$ where $j \in \mathbb{N}$.

We now consider the case where $m = 2k + 1$, when $\beta = \beta_f(m)$ we have $(d_i(m))_{i=1}^\infty = (k + 1, k + 1, k, k)^\infty$ and

$$S_{\beta_f(m),m} = \left\{ (\epsilon_i)_{i=1}^\infty \in \{0, \dots, m\}^\mathbb{N} : (\epsilon_i, \epsilon_{i+1}, \dots) < (k + 1, k + 1, k, k)^\infty \text{ and } (k, k, k + 1, k + 1)^\infty < (\epsilon_i, \epsilon_{i+1}, \dots) \text{ for all } i \in \mathbb{N} \right\}.$$

By our earlier analysis we know that $\{(k, k + 1)^\infty, (k + 1, k)^\infty\} \subset S_{\beta,m}$ for $\beta \in (\mathcal{G}(m), m + 1]$. By Remark 4.8 to prove $|S_{\beta,m}| = 2$ for $\beta \in (\mathcal{G}(m), \beta_f(m)]$ it suffices to show that $S_{\beta_f(m),m} = \{(k, k + 1)^\infty, (k + 1, k)^\infty\}$. By an analogous argument to that given in [9] we can show that if $(\epsilon_i)_{i=1}^\infty \in S_{\beta_f(m),m}$ then $\epsilon_i = k$ implies $\epsilon_{i+1} = k + 1$, and $\epsilon_i = k + 1$ implies $\epsilon_{i+1} = k$. Clearly any element of $S_{\beta_f(m),m}$ must begin with k or $k + 1$ and we may therefore conclude that $S_{\beta_f(m),m} = \{(k, k + 1)^\infty, (k + 1, k)^\infty\}$. To see that $|W_{\beta,m}| \geq \aleph_0$ for $\beta > \beta_f(m)$ we observe that $(k + 1, k)^j(k + 1, k + 1, k, k)^\infty \in S_{\beta,m}$ for all $j \in \mathbb{N}$, for $\beta > \beta_f(m)$. \square

4.2. The growth rate of $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$

In this section we study the growth rate of the sequences $(\mathcal{G}(m))_{m=1}^\infty$, $(\beta_f(m))_{m=1}^\infty$ and $(\beta_c(m))_{m=1}^\infty$. The following theorem summarises the growth rate of each of these sequences.

Theorem 4.10. 1. $\mathcal{G}(2k) = k + 1$ for all $k \in \mathbb{N}$.

2. $\beta_f(2k) - (k + 2) = O(\frac{1}{k})$.

3. $\beta_c(2k) - (k + 2) \rightarrow 0$ as $k \rightarrow \infty$.

4. $\mathcal{G}(2k + 1) - (k + 2) = O(\frac{1}{k})$.

5. $\beta_f(2k + 1) - (k + 2) \rightarrow 0$ as $k \rightarrow \infty$.

6. $\beta_c(2k + 1) - (k + 2) \rightarrow 0$ as $k \rightarrow \infty$.

The proof of this theorem is somewhat trivial but we include it for completion. To prove this result we firstly require the following lemma.

Lemma 4.11. The sequence $\beta_c(m)$ is asymptotic to $\frac{m}{2}$, i.e., $\lim_{m \rightarrow \infty} \frac{\beta_c(m)}{m/2} = 1$.

Proof. Suppose $m = 2k$. It is a direct consequence of the definition of $\lambda_i(m)$ and $\beta_c(m)$ that the following inequalities hold

$$\sum_{i=0}^{\infty} \frac{k-1}{\beta_c(m)^i} \leq \beta_c(m) \leq \sum_{i=0}^{\infty} \frac{k+1}{\beta_c(m)^i},$$

which is equivalent to

$$\frac{k-1}{1 - \frac{1}{\beta_c(m)}} \leq \beta_c(m) \leq \frac{k+1}{1 - \frac{1}{\beta_c(m)}}.$$

Dividing through by $m/2$ and using the fact that $\beta_c(m) \rightarrow \infty$ we can conclude our result. The case where $m = 2k + 1$ is proved similarly. \square

We are now in a position to prove Theorem 4.10.

Proof of Theorem 4.10. Statements 1, 2 and 4 are an immediate consequence of Theorem 1.1 and Theorem 4.5. It remains to show statements 3 and 6 hold; statement 5 will follow from the fact that $\mathcal{G}(2k+1) < \beta_f(2k+1) < \beta_c(2k+1)$. It is immediate from the definition of $\lambda_i(m)$ that if $m = 2k$ then

$$\beta_c(m) = k + 1 + \frac{k}{\beta_c(m)} + \sum_{i=2}^{\infty} \frac{\lambda_{i+1}(m)}{\beta_c(m)^i}.$$

It is a straightforward consequence of $1 \in U_{\beta_c(m),m}$ that $|\sum_{i=1}^{\infty} \frac{\lambda_{i+j}(m)}{\beta_c(m)^i}| \leq 1$, for all $j, m \in \mathbb{N}$. Therefore $\sum_{i=2}^{\infty} \frac{\lambda_{i+1}(m)}{\beta_c(m)^i} \rightarrow 0$ as $m \rightarrow \infty$, combining this statement with Lemma 4.11 we may conclude our result when $m = 2k$. The case where $m = 2k + 1$ is proved similarly. \square

5. The growth rate and dimension theory of $\Sigma_{\beta,m}(x)$

To describe the growth rate of β -expansions we consider the following. Let

$$\mathcal{E}_{\beta,m,n}(x) = \left\{ (\epsilon_1, \dots, \epsilon_n) \in \{0, \dots, m\}^n \mid \exists (\epsilon_{n+1}, \epsilon_{n+2}, \dots) \in \{0, \dots, m\}^{\mathbb{N}} \right. \\ \left. : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x \right\}.$$

We define an element of $\mathcal{E}_{\beta,m,n}(x)$ to be a n -prefix for x . Moreover, we let

$$\mathcal{N}_{\beta,m,n}(x) = |\mathcal{E}_{\beta,m,n}(x)|$$

and define the growth rate of $\mathcal{N}_{\beta,m,n}(x)$ to be

$$\lim_{n \rightarrow \infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n},$$

when this limit exists. When this limit does not exist we can consider the lower and upper growth rates of $\mathcal{N}_{\beta,m,n}(x)$, these are defined to be

$$\liminf_{n \rightarrow \infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n} \text{ and } \limsup_{n \rightarrow \infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n}$$

respectively.

In this paper we also consider $\Sigma_{\beta,m}(x)$ from a dimension theory perspective. We endow $\{0, \dots, m\}^{\mathbb{N}}$ with the metric $d(\cdot, \cdot)$ defined as follows:

$$d(x, y) = \begin{cases} (m + 1)^{-n(x,y)} & \text{if } x \neq y, \text{ where } n(x, y) = \inf\{i : x_i \neq y_i\} \\ 0 & \text{if } x = y. \end{cases}$$

We will consider the Hausdorff dimension of $\Sigma_{\beta,m}(x)$ with respect to this metric. It is a simple exercise to show that the following inequalities hold:

$$\dim_H(\Sigma_{\beta,m}(x)) \leq \liminf_{n \rightarrow \infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n} \leq \limsup_{n \rightarrow \infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n}. \quad (5)$$

The case where $m = 1$ is studied in [4], [8] and [10]. In [4] and [8] the authors show that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ we can bound the lower growth rate and Hausdorff dimension of $\Sigma_{\beta,1}(x)$ below by some strictly positive function depending only on β , in [10] the growth rate is studied from a measure theoretic perspective. Our main result is the following.

Theorem 5.1. For $\beta \in (1, \mathcal{G}(m))$ and $x \in (0, \frac{m}{\beta-1})$ the Hausdorff dimension of $\Sigma_{\beta,m}(x)$ can be bounded below by some strictly positive constant depending only on β .

By (5) a similar statement holds for both the lower and upper growth rates of $\mathcal{N}_{\beta,m,n}(x)$. Replicating the proof of Lemma 2.4 it can be shown that the following result holds.

Proposition 5.2. $\mathcal{N}_{\beta,m,n}(x) = |\Omega_{\beta,m,n}(x)|$

By Proposition 5.2 we can identify elements of $\Omega_{\beta,m,n}(x)$ with elements of $\mathcal{E}_{\beta,m,n}(x)$, as such we also define an element of $\Omega_{\beta,m,n}(x)$ to be a n -prefix for x . To prove Theorem 5.1 we will use a method analogous to that given in [4]. We construct an interval $\mathcal{I}_\beta \subset I_{\beta,m}$ such that, for each $x \in \mathcal{I}_\beta$ we can generate multiple prefixes for x of a fixed length depending on β that map x back into \mathcal{I}_β . As we will see Theorem 5.1 will then follow by a counting argument. As was the case in our previous analysis we reduce the proof of Theorem 5.1 to two cases.

5.1. Case where m is even

In what follows we assume $m = 2k$ for some $k \in \mathbb{N}$. To prove Theorem 5.1 we require the following technical lemma.

Lemma 5.3. *For each $\beta \in (1, k + 1)$ there exists $\epsilon_0(\beta) > 0$ such that, if $x \in [\frac{1}{\beta}, \frac{1}{\beta} + \epsilon_0(\beta)]$ then $T_{\beta,0}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$, and similarly if $x \in (\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)}]$ then $T_{\beta,m}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$.*

Proof. This follows from Lemma 2.6 and a continuity argument. □

For each $i \in \{1, \dots, m-1\}$ we let $\epsilon_i(\beta) = \frac{1}{2}(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \frac{i+1}{\beta})$. If $\beta \in (1, k+1)$ then $\epsilon_i(\beta) > 0$ for each $i \in \{1, \dots, m-1\}$. We define the interval $\mathcal{I}_\beta = [L(\beta), R(\beta)]$ where $L(\beta)$ and $R(\beta)$ are defined as follows:

$$L(\beta) = \min \left\{ T_{\beta,1} \left(\frac{1}{\beta} + \epsilon_0(\beta) \right), \min_{i \in \{1, \dots, m-1\}} T_{\beta,i+1} \left(\frac{i+1}{\beta} + \epsilon_i(\beta) \right) \right\}$$

and

$$R(\beta) = \max \left\{ T_{\beta,m-1} \left(\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right), \max_{i \in \{1, \dots, m-1\}} T_{\beta,i-1} \left(\frac{i+1}{\beta} + \epsilon_i(\beta) \right) \right\}.$$

We refer to Figure 5 for a diagram illustrating the interval \mathcal{I}_β in the case where $m = 2$ and $\beta \in (1, 2)$.

Proposition 5.4. *Let $\beta \in (1, k + 1)$. There exists $n(\beta) \in \mathbb{N}$ such that, for each $x \in \mathcal{I}_\beta$ there exists two elements $a, b \in \Omega_{\beta,m,n(\beta)}(x)$ such that $a(x) \in \mathcal{I}_\beta$ and $b(x) \in \mathcal{I}_\beta$.*

Proof. Let $x \in \mathcal{I}_\beta$. Without loss of generality we may assume that $\epsilon_0(\beta)$ is sufficiently small such that \mathcal{I}_β contains the switch region. By Lemma 2.6 there exists a sequence of maps a that map x into the interior of our switch region. By Lemma 5.3 we may assume that $a(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$.

The distance between the endpoints of \mathcal{I}_β and the endpoints of $I_{\beta,m}$ (the fixed points of the maps $T_{\beta,0}$ and $T_{\beta,m}$) can be bounded below by some positive constant, by Lemma 2.1 $T_{\beta,0}$ and $T_{\beta,m}$ both scale the distance between their fixed points and a general point by a factor β , therefore we can bound the length of our sequence a above by some constant $n_s(\beta) \in \mathbb{N}$ that does not depend on x . We will show that we can take $n(\beta) = n_s(\beta) + 1$.

We remark that:

$$L(\beta) \qquad \frac{1}{\beta} + \epsilon_0(\beta) \qquad \frac{2}{\beta} + \epsilon_1(\beta) \qquad \frac{\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \qquad R(\beta)$$

Figure 5: The interval \mathcal{I}_β in the case where $m = 2$ and $\beta \in (1, 2)$.

$$\begin{aligned} \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right] &= \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{2}{\beta} \right] \\ &\cup \left[\frac{(m-2)\beta+2}{\beta(\beta-1)}, \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right] \\ &\cup_{i=1}^{m-2} \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}, \frac{i+2}{\beta} \right] \\ &\cup_{i=1}^{m-1} \left[\frac{i+1}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} \right]. \end{aligned}$$

We now proceed via a case analysis.

- If $a(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{2}{\beta}]$ then $T_{\beta,0}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,1}(a(x)) \in \mathcal{I}_\beta$.
- If $a(x) \in [\frac{(m-2)\beta+2}{\beta(\beta-1)}, \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$ then $T_{\beta,m-1}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,m}(a(x)) \in$

\mathcal{I}_β .

- If $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}, \frac{i+2}{\beta}]$ for some $i \in \{1, \dots, m-2\}$ then $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i+1}(a(x)) \in \mathcal{I}_\beta$.
- We reduce the case where $a(x) \in [\frac{i+1}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ for some $i \in \{1, \dots, m-1\}$ to two subcases. If $a(x) \in [\frac{i+1}{\beta}, \frac{i+1}{\beta} + \epsilon_i(\beta)]$ then by the monotonicity of our maps, both $T_{\beta,i-1}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$. Similarly, in the case where $a(x) \in [\frac{i+1}{\beta} + \epsilon_i(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ both $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i+1}(a(x)) \in \mathcal{I}_\beta$.

We have shown that for any $x \in \mathcal{I}_\beta$ there exists $n(x) \leq n_s(\beta) + 1$ such that two distinct elements of $\Omega_{\beta,m,n(x)}(x)$ map x into \mathcal{I}_β . If $n(x) < n_s(\beta) + 1$ then we can concatenate our two elements of $\Omega_{\beta,m,n(x)}(x)$ by an arbitrary choice of maps of length $n_s(\beta) + 1 - n(x)$ that map the image of x into \mathcal{I}_β . This ensures that we can take our sequences of maps to be of length $n_s(\beta) + 1$. \square

For $\beta \in (1, k + 1)$ and $x \in (0, \frac{m}{\beta-1})$ we may assume that there exists a sequence of maps a that maps x into \mathcal{I}_β . We denote the minimum number of maps required to do this by $j(x)$. Replicating arguments given in [4] we can use Proposition 5.4 to construct an algorithm by which we can generate two prefixes of length $n(\beta)$ for $a^{(j(x))}$. Repeatedly applying this algorithm to successive images of $a^{(j(x))}$ we can generate a closed subset of $\Sigma_{\beta,m}(x)$. We denote this set by $\sigma_{\beta,m}(x)$ and the set of n -prefixes for x generated by this algorithm by $\omega_{\beta,m,n}(x)$. Replicating the proofs given in [4] we can show that the following lemmas hold.

Lemma 5.5. *Let $x \in (0, \frac{m}{\beta-1})$. Assume $n \geq j(x)$ then*

$$|\omega_{\beta,m,n}(x)| \geq 2^{\frac{n-j(x)}{n(\beta)}-1}.$$

Lemma 5.6. *Let $x \in (0, \frac{m}{\beta-1})$. Assume $l \geq j(x)$ and $b \in \omega_{\beta,m,l}(x)$, then for $n \geq l$*

$$|\{a = (a_i)_{i=1}^n \in \omega_{\beta,m,n}(x) : a_i = b_i \text{ for } 1 \leq i \leq l\}| \leq 2^{\frac{n-l}{n(\beta)}+2}.$$

With these lemmas we are now in a position to prove Theorem 5.1 in the case where m is even. The argument used is analogous to the one given in [4], which is based upon Example 2.7 of [7].

Proof of Theorem 5.1 when $m = 2k$. By the monotonicity of Hausdorff dimension with respect to inclusion it suffices to show that $\dim_H(\sigma_{\beta,m}(x))$ can be bounded below by a strictly positive constant depending only on β . It is a simple exercise to show that $\sigma_{\beta,m}(x)$ is a compact set; by this result we may restrict to finite covers of $\sigma_{\beta,m}(x)$. Let $\{U_n\}_{n=1}^N$ be a finite cover of $\sigma_{\beta,m}(x)$. Without loss of generality we may assume that all elements of our cover satisfy $\text{Diam}(U_n) < (m + 1)^{-j(x)}$. For each U_n there exists $l(n) \in \mathbb{N}$ such that

$$(m + 1)^{-(l(n)+1)} \leq \text{Diam}(U_n) < (m + 1)^{-l(n)}.$$

It follows that there exists $z^{(n)} \in \{0, \dots, m\}^{l(n)}$ such that, $y_i = z_i^{(n)}$ for $1 \leq i \leq l(n)$, for all $y \in U_n$. We may assume that $z^{(n)} \in \omega_{\beta, m, l(n)}(x)$, if we supposed otherwise then $\sigma_{\beta, m}(x) \cap U_n = \emptyset$ and we can remove U_n from our cover. We denote by C_n the set of sequences in $\{0, \dots, m\}^{\mathbb{N}}$ whose first $l(n)$ entries agree with $z^{(n)}$, i.e.

$$C_n = \left\{ (\epsilon_i)_{i=1}^{\infty} \in \{0, \dots, m\}^{\mathbb{N}} : \epsilon_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n) \right\}.$$

Clearly $U_n \subset C_n$ and therefore the set $\{C_n\}_{n=1}^N$ is a cover of $\sigma_{\beta, m}(x)$.

Since there are only finitely many elements in our cover there exists J such that $(m + 1)^{-J} \leq \text{Diam}(U_n)$ for all n . We consider the set $\omega_{\beta, m, J}(x)$. Since $\{C_n\}_{n=1}^N$ is a cover of $\sigma_{\beta, m}(x)$ each $a \in \omega_{\beta, m, J}(x)$ satisfies $a_i = z_i^{(n)}$ for $1 \leq i \leq l(n)$, for some n . Therefore

$$|\omega_{\beta, m, J}(x)| \leq \sum_{n=1}^N \left| \{a \in \omega_{\beta, m, J}(x) : a_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n)\} \right|.$$

By counting elements of $\omega_{\beta, m, J}(x)$ and Lemmas 5.5 and 5.6 we observe the following;

$$\begin{aligned} 2^{\frac{J-j(x)}{n(\beta)}-1} &\leq |\omega_{\beta, m, J}(x)| \\ &\leq \sum_{n=1}^N \left| \{a \in \omega_{\beta, m, J}(x) : a_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n)\} \right| \\ &\leq \sum_{n=1}^N 2^{\frac{J-l(n)}{n(\beta)}+2} \\ &= 2^{\frac{J+1}{n(\beta)}+2} \sum_{n=1}^N 2^{-\frac{l(n)+1}{n(\beta)}} \\ &\leq 2^{\frac{J+1}{n(\beta)}+2} \sum_{n=1}^N \text{Diam}(U_n)^{\frac{\log_{m+1} 2}{n(\beta)}}. \end{aligned}$$

Dividing through by $2^{\frac{J+1}{n(\beta)}+2}$ yields

$$\sum_{n=1}^N \text{Diam}(U_n)^{\frac{\log_{m+1} 2}{n(\beta)}} \geq 2^{\frac{-j(x)-3n(\beta)-1}{n(\beta)}},$$

the right hand side is a positive constant greater than zero that does not depend on our choice of cover. It follows that $\dim_H(\sigma_{\beta, m}(x)) \geq \frac{\log_{m+1} 2}{n(\beta)}$, our result follows. \square

5.2. Case where m is odd

In what follows we assume $m = 2k + 1$ for some $k \in \mathbb{N}$. For $\beta \in (1, \frac{2k+3}{2})$ the proof of Theorem 5.1 is analogous to the even case for $\beta \in (1, k + 1)$. As such, in what follows

we assume $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. The significance of $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ is that for $i \in \{1, \dots, m-1\}$ the i -th fixed digit interval is well defined.

Before defining the interval \mathcal{I}_β we require the following. We let

$$\epsilon_i(\beta) = \begin{cases} \frac{1}{2} \left(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \frac{i}{\beta-1} \right) & \text{if } i \in \{1, \dots, k\} \\ \frac{1}{2} \left(\frac{i}{\beta-1} - \frac{i+1}{\beta} \right) & \text{if } i \in \{k+1, \dots, m-1\} \end{cases}$$

By Lemma 3.3, $\epsilon_i(\beta) > 0$ for all $i \in \{1, \dots, m-1\}$ for $\beta \in (1, k+2)$. Before proving an analogue of Proposition 5.4 we require the following technical lemmas. It is a simple exercise to show that the following analogue of Lemma 5.3 holds.

Lemma 5.7. *For each $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ there exists $\epsilon_0(\beta) > 0$ such that, if $x \in [\frac{1}{\beta}, \frac{1}{\beta} + \epsilon_0(\beta))$ then $T_{\beta,0}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$, and similarly if $x \in (\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)})$ then $T_{\beta,m}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$.*

Lemma 5.8. *Let $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. For each $i \in \{1, \dots, k-1\}$ there exists $\epsilon_i^*(\beta) > 0$ such that, if $x \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ then $T_{\beta,i}(x) < \frac{k+2}{\beta} + \epsilon_{k+1}$. Similarly for $i \in \{k+2, \dots, m-1\}$ there exists $\epsilon_i^*(\beta) > 0$ such that, if $x \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta), \frac{i+1}{\beta} + \epsilon_i(\beta)]$ then $T_{\beta,i}(x) > \frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k$.*

Proof. By the analysis given in the proof of Lemma 3.5 for $i \in \{1, \dots, k-1\}$, we have $T_{\beta,i}(\frac{i+1}{\beta}) < \frac{k\beta+m-k}{\beta(\beta-1)}$ for $\beta \in (1, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. However, for $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ $\frac{k\beta+m-k}{\beta(\beta-1)} \leq \frac{k+2}{\beta}$. The existence of $\epsilon_i^*(\beta)$ then follows by a continuity argument and the monotonicity of the maps $T_{\beta,i}$. The case where $i \in \{k+2, \dots, m-1\}$ is proved similarly. □

We are now in a position to define the interval \mathcal{I}_β . Let $\mathcal{I}_\beta = [L(\beta), R(\beta)]$ where

$$L(\beta) = \min \left\{ T_{\beta,1} \left(\frac{1}{\beta} + \epsilon_0(\beta) \right), T_{\beta,k+1} \left(\frac{k\beta+k+1}{\beta^2-1} \right), \min_{i \in \{2, \dots, k\}} \left\{ T_{\beta,i} \left(\frac{i}{\beta} + \epsilon_{i-1}^*(\beta) \right) \right\}, \min_{i \in \{k+2, \dots, m\}} \left\{ T_{\beta,i} \left(\frac{i}{\beta} + \epsilon_{i-1}(\beta) \right) \right\} \right\}$$

and

$$R(\beta) = \max \left\{ T_{\beta,k} \left(\frac{(k+1)\beta+k}{\beta^2-1} \right), T_{\beta,m-1} \left(\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta) \right), \max_{i \in \{1, \dots, k\}} \left\{ T_{\beta,i-1} \left(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta) \right) \right\}, \max_{i \in \{k+2, \dots, m-1\}} \left\{ T_{\beta,i-1} \left(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta) \right) \right\} \right\}.$$

$$\begin{array}{ccc}
 L(\beta) & & R(\beta) \\
 \\
 \frac{1}{\beta} + \epsilon_0(\beta) & \frac{3}{\beta(\beta-1)} - \epsilon_1(\beta) & \frac{\beta+2}{\beta^2-1} \quad \frac{2\beta+1}{\beta^2-1} \quad \frac{3}{\beta} + \epsilon_2(\beta) \quad \frac{2\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)
 \end{array}$$

Figure 6: The interval \mathcal{I}_β in the case where $m = 3$ and $\beta \in [\frac{5}{2}, 1 + \sqrt{3})$.

For ease of exposition in Figure 6 we give a diagram illustrating the interval \mathcal{I}_β , in the case where $m = 3$ and $\beta \in [\frac{5}{2}, 1 + \sqrt{3})$.

Proposition 5.9. *Let $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. There exists $n(\beta) \in \mathbb{N}$ such that, for each $x \in \mathcal{I}_\beta$ there exists two elements $a, b \in \Omega_{\beta,m,n(\beta)}(x)$ such that $a(x) \in \mathcal{I}_\beta$ and $b(x) \in \mathcal{I}_\beta$.*

Proof. Without loss of generality we may assume that $\epsilon_0(\beta)$ is sufficiently small such that \mathcal{I}_β contains the switch region. By Lemma 2.6 there exists a sequence of maps a that map x into the switch region. As the endpoints of \mathcal{I}_β are bounded away from the endpoints of $I_{\beta,m}$ we can bound the length of a above by some $n_s(\beta) \in \mathbb{N}$. Moreover, by Lemma 5.7 we may assume that $a(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$. As in the even case it is useful

to treat $[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$ as the union of subintervals. We observe that

$$\begin{aligned} \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)\right] &= \left[\frac{1}{\beta} + \epsilon_0(\beta), \frac{m}{\beta(\beta-1)} - \epsilon_1(\beta)\right] \\ &\cup \left[\frac{m}{\beta} + \epsilon_{m-1}(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)\right] \\ &\cup \left[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k(\beta), \frac{k+2}{\beta} + \epsilon_{k+1}(\beta)\right] \\ &\cup_{i=2}^k \left[\frac{i}{\beta} + \epsilon_{i-1}^*(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta)\right] \\ &\cup_{i=k+2}^{m-1} \left[\frac{i}{\beta} + \epsilon_{i-1}(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta)\right] \\ &\cup_{i=1}^{k-1} \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)\right] \\ &\cup_{i=k+2}^{m-1} \left[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta), \frac{i+1}{\beta} + \epsilon_i(\beta)\right]. \end{aligned}$$

Without loss of generality we may assume that $\epsilon_0(\beta), \epsilon_i(\beta), \epsilon_i^*(\beta)$ are all sufficiently small such that each of the above intervals in our union are well defined and nontrivial. We now proceed via a case analysis.

- If $a(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{m}{\beta(\beta-1)} - \epsilon_1(\beta)]$ then $T_{\beta,0}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,1}(a(x)) \in \mathcal{I}_\beta$.
- If $a(x) \in [\frac{m}{\beta} + \epsilon_{m-1}(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$ then $T_{\beta,m-1}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,m}(a(x)) \in \mathcal{I}_\beta$.
- Suppose $a(x) \in [\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k(\beta), \frac{k+2}{\beta} + \epsilon_{k+1}(\beta)]$. If $a(x) \in [\frac{k\beta+k+1}{\beta^2-1}, \frac{(k+1)\beta+k}{\beta^2-1}]$ then $T_{\beta,k}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,k+1}(a(x)) \in \mathcal{I}_\beta$. If $a(x) \in [\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k(\beta), \frac{k\beta+k+1}{\beta^2-1}]$ then we are a bounded distance away from the fixed point of the map $T_{\beta,k}$, by Lemma 2.1 we know that $T_{\beta,k}$ scales the distance between $a(x)$ and the fixed point of $T_{\beta,k}$ by a factor β , therefore we can bound the number of maps required to map $a(x)$ into $[\frac{k\beta+k+1}{\beta^2-1}, \frac{(k+1)\beta+k}{\beta^2-1}]$. By a similar argument, if $a(x) \in [\frac{(k+1)\beta+k}{\beta^2-1}, \frac{k+2}{\beta} + \epsilon_{k+1}(\beta)]$ we can bound the number of maps required to map $a(x)$ into $[\frac{k\beta+k+1}{\beta^2-1}, \frac{(k+1)\beta+k}{\beta^2-1}]$. By the above we can assert that when $a(x) \in [\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k(\beta), \frac{k+2}{\beta} + \epsilon_{k+1}(\beta)]$ there exists two distinct sequences of maps whose length we can bound above by some $n_c(\beta) \in \mathbb{N}$ that map $a(x)$ into \mathcal{I}_β .
- If $a(x) \in [\frac{i}{\beta} + \epsilon_{i-1}^*(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta)]$ for some $i \in \{2, \dots, k-1\}$ then $T_{\beta,i-1}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$.

- If $a(x) \in [\frac{i}{\beta} + \epsilon_i(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta)]$ for some $i \in \{k+2, \dots, m-1\}$ then $T_{\beta,i-1}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$.
- If $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ for some $i \in \{1, \dots, k-1\}$ then $a(x)$ is a bounded distance away from the fixed point of the map $T_{\beta,i}$, by Lemma 2.1 we know that $T_{\beta,i}$ scales the distance between $a(x)$ and its fixed point by a factor β , therefore we can bound the number of maps required to map $a(x)$ outside of the interval $[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ by some $n_i(\beta) \in \mathbb{N}$. If $a(x)$ has been mapped into an interval covered by one of the above cases we are done, if not it has to be mapped into another interval of the form $[\frac{(j-1)\beta+m-(j-1)}{\beta(\beta-1)} - \epsilon_j(\beta), \frac{j+1}{\beta} + \epsilon_j^*(\beta)]$. By Corollary 3.4 and Lemma 5.8 we know that $i < j \leq k+1$. Repeating the previous step as many times as is necessary we can ensure that within $\sum_{i=1}^{k-1} n_i(\beta)$ maps, $a(x)$ has to be mapped into an interval that was addressed in one of our previous cases.
- The case where $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta), \frac{i+1}{\beta} + \epsilon_i(\beta)]$ for some $i \in \{k+2, \dots, m-1\}$ is analogous to the case where $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ for some $i \in \{1, \dots, k-1\}$.

We have shown that for any $x \in \mathcal{I}_\beta$ there exists $n(x) \in \mathbb{N}$ such that, two distinct elements of $\Omega_{\beta,m,n(x)}(x)$ map x into \mathcal{I}_β , moreover $n(x) \leq n_s(\beta) + n_c(\beta) + \sum_{i=1}^{k-1} n_i(\beta)$. We take $n(\beta)$ to equal $n_s(\beta) + n_c(\beta) + \sum_{i=1}^{k-1} n_i(\beta)$. If $n(x) < n(\beta)$ then as in the even case we concatenate our image of x by an arbitrary sequence of maps of length $n(\beta) - n(x)$ that map x into \mathcal{I}_β , this ensures our sequences of maps are of length $n(\beta)$. □

Repeating the analysis given in the case where m is even we can conclude Theorem 5.1 in the case where m is odd.

6. Open questions and a table of values for $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$

We conclude with a few open questions and a table of values for $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$.

- In [1] the authors study the order in which periodic orbits appear in the set of uniqueness. When $m = 1$ they show that as $\beta \nearrow 2$ the order in which periodic orbits appear in the set of uniqueness is intimately related to the classical Sharkovskii ordering. It is natural to ask whether a similar result holds in our general case.
- In [18] it is shown that when $m = 1$ and $\beta = \frac{1+\sqrt{5}}{2}$ the set of numbers: $x = \frac{(1+\sqrt{5})n}{2} \pmod{1}$ for some $n \in \mathbb{N}$ have countably many β -expansions, while the other elements of $(0, \frac{1}{\beta-1})$ have uncountably many β -expansions. Does an analogue of this statement hold in the case of general m ?

- Let p_1, \dots, p_k be points in \mathbb{R}^d such that the polyhedra Π with these vertices is convex. Let $\{f_i\}_{i=1}^k$ be the one parameter family of maps given by

$$f_i(x) = \lambda x + (1 - \lambda)p_i,$$

where $\lambda \in (0, 1)$ is our parameter. As is well know there exists a unique compact non-empty S_λ such that $S_\lambda = \cup_{i=1}^k f_i(S_\lambda)$. We say that $(\epsilon_i)_{i=1}^\infty \in \{1, \dots, k\}^\mathbb{N}$ is an address for $x \in S_\lambda$ if $\lim_{n \rightarrow \infty} (f_{\epsilon_n} \circ \dots \circ f_{\epsilon_1})(\mathbf{0}) = x$. We ask whether an analogue of the golden ratio exists in this case, i.e, does there exists λ^* such that for $\lambda \in (\lambda^*, 1)$ every $x \in S_\lambda \setminus \{p_1, \dots, p_k\}$ has uncountably many addresses, but for $\lambda \in (0, \lambda^*)$ there exists $x \in S_\lambda \setminus \{p_1, \dots, p_k\}$ with a unique address. In [16] the author shows that an analogue of the golden ratio exists in the case when $d = 2$ and $k = 3$.

Table 1: Table of values for $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$

m	$\mathcal{G}(m)$	$\beta_f(m)$	$\beta_c(m)$
1	$\frac{1+\sqrt{5}}{2} \approx 1.61803 \dots$	1.75488...	1.78723...
2	2	$1 + \sqrt{2} = 2.41421 \dots$	2.47098...
3	$1 + \sqrt{3} \approx 2.73205 \dots$	2.89329...	2.90330...
4	3	$\frac{3+\sqrt{17}}{2} = 3.56155 \dots$	3.66607...
5	$\frac{3+\sqrt{21}}{2} \approx 3.79129 \dots$	3.93947	3.94583...
6	4	$2 + \frac{\sqrt{28}}{2} = 4.64575 \dots$	4.75180...
7	$2 + 2\sqrt{2} \approx 4.82843 \dots$	4.96095...	4.96496...
8	5	$\frac{5+\sqrt{41}}{2} = 5.70156 \dots$	5.80171...
9	$\frac{5+\sqrt{45}}{2} \approx 5.85410 \dots$	5.97273...	5.97537...
10	6	$3 + \sqrt{14} = 6.74166 \dots$	6.83469...

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References

- [1] J.-P. Allouche, M. Clarke and N. Sidorov, Periodic unique beta-expansions: the Sharkovskii ordering, *Ergodic Theory Dynam. Systems* 29 (2009), no. 4, 1055–1074.
- [2] J.-P. Allouche and M. Cosnard, The Komornik-Loreti constant is transcendental, *Amer. Math. Monthly* 107 (2000), no. 5, 448–449.
- [3] J.-P. Allouche and J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in C. Ding, T. Hellese, and H. Niederreiter, eds., *Sequences and their applications: Proceedings of SETA '98*, Springer-Verlag, 1999, pp. 1–16.
- [4] S. Baker, The growth rate and dimension theory of beta-expansions, *Fund. Math.* 219 (2012), 271–285
- [5] Z. Daróczy and I. Katai, Univoque sequences, *Publ. Math. Debrecen* 42 (1993), no. 3–4, 397–407.
- [6] P. Erdős, I. Joó and V. Komornik, Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems, *Bull. Soc. Math. Fr.* 118 (1990), 377–390.
- [7] K. Falconer. *Fractal Geometry: Mathematical Foundation and Applications*. John Wiley, Chichester, 1990.
- [8] D. J. Feng, N. Sidorov, Growth rate for beta-expansions, *Monatsh. Math.* 162 (2011), no. 1, 41–60.
- [9] P. Glendinning and N. Sidorov, Unique representations of real numbers in non-integer bases, *Math. Res. Letters* 8 (2001), 535–543.
- [10] T. Kempton, Counting β -expansions and the absolute continuity of Bernoulli convolutions, Preprint.
- [11] V. Komornik, A. Lai, M. Pedicini, Generalized golden ratios of ternary alphabets, *J. Eur. Math. Soc. (JEMS)* 13 (2011), no. 4, 1113–1146.
- [12] V. Komornik and P. Loreti, Unique developments in non-integer bases, *Amer. Math. Monthly* 105 (1998), no. 7, 636–639.
- [13] V. Komornik and P. Loreti, Subexpansions, superexpansions and uniqueness properties in non-integer bases, *Period. Math. Hungar.* 44 (2002), no. 2, 197–218.
- [14] D. Kong, W. Li, F. Dekking, Intersections of homogeneous Cantor sets and beta-expansions, *Nonlinearity* 23 (2010), no. 11, 2815–2834.
- [15] W. Parry, On the β -expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* 11 (1960) 401–416.

- [16] N. Sidorov, Combinatorics of linear iterated function systems with overlaps. *Non-linearity* 20 (2007), no. 5, 1299–1312.
- [17] N. Sidorov, Expansions in noninteger bases, Lecture notes of a graduate course at the summer school at Queen Mary, University of London, July, 2010.
- [18] N. Sidorov and A. Vershik, Ergodic properties of the Erdos measure, the entropy of the golden shift, and related problems. *Monatsh. Math.* 126 (1998), no. 3, 215–261.