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Baker, S. (2015) On small bases which admit countably many expansions. Journal of Number Theory, 147. pp. 515-532. ISSN 0022-314X doi: https://doi.org/10.1016/j.jnt.2014.08.003 Available at https://centaur.reading.ac.uk/46863/

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To link to this article DOI: http://dx.doi.org/10.1016/j.jnt.2014.08.003

Publisher: Elsevier

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ON SMALL BASES WHICH ADMIT COUNTABLY MANY EXPANSIONS

SIMON BAKER

Dedicated to P. Erdős on the 100th anniversary of his birth

ABSTRACT. Let $q \in (1,2)$ and $x \in [0, \frac{1}{q-1}]$. We say that a sequence $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ is an expansion of x in base q (or a q-expansion) if

$$x = \sum_{i=1}^{\infty} \epsilon_i q^{-i}.$$

Let $\mathcal{B}_{\aleph_0}$ denote the set of q for which there exists x with exactly $\aleph_0$ expansions in base q. In [5] it was shown that $\min \mathcal{B}_{\aleph_0} = \frac{1+\sqrt{5}}{2}$. In this paper we show that the smallest element of $\mathcal{B}_{\aleph_0}$ strictly greater than $\frac{1+\sqrt{5}}{2}$ is $q_{\aleph_0} \approx 1.64541$, the appropriate root of $x^6 = x^4 + x^3 + 2x^2 + x + 1$. This leads to a full dichotomy for the number of possible q-expansions for $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$. We also prove some general results regarding $\mathcal{B}_{\aleph_0} \cap [\frac{1+\sqrt{5}}{2}, q_f]$, where $q_f \approx 1.75488$ is the appropriate root of $x^3 = 2x^2 - x + 1$. Moreover, the techniques developed in this paper imply that if $x \in [0, \frac{1}{q-1}]$ has uncountably many q-expansions then the set of q-expansions for x has cardinality equal to that of the continuum, this proves that the continuum hypothesis holds when restricted to this specific case.

1. INTRODUCTION

Let $q \in (1,2)$ and $I_q = [0, \frac{1}{q-1}]$. Each $x \in I_q$ has an expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{q^i}, \tag{1.1}$$

for some $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$. We call such a sequence a *q-expansion* of x, when (1.1) holds we will adopt the notation $x = (\epsilon_1, \epsilon_2, \ldots)_q$. Expansions in non-integer bases were pioneered in the papers of Rényi [11] and Parry [10].

Given $x \in I_q$ we denote the set of q-expansions of x by $\Sigma_q(x)$, i.e.,

$$\Sigma_q(x) = \Big\{ (\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{q^i} = x \Big\}.$$

The endpoints of I_q always have a unique q-expansion, typically an element of $(0, \frac{1}{q-1})$ will have a nonunique q-expansion. In [7] it was shown that for $q \in (1, \frac{1+\sqrt{5}}{2})$ the set $\Sigma_q(x)$ is uncountable

Date: September 14, 2015.
2010 *Mathematics Subject Classification.* 11A63, 37A45.
Key words and phrases. Beta-expansion, non-integer base.

for all $x \in (0, \frac{1}{q-1})$. When $q = \frac{1+\sqrt{5}}{2}$ it was shown in [15] that every $x \in (0, \frac{1}{q-1})$ has uncountably many q-expansions unless $x = \frac{(1+\sqrt{5})n}{2} \mod 1$, for some $n \in \mathbb{Z}$, in which case $\Sigma_q(x)$ is infinite countable. In [12] it was shown that for $q \in (\frac{1+\sqrt{5}}{2}, 2)$ the set $\Sigma_q(x)$ is uncountable for almost every $x \in (0, \frac{1}{q-1})$. Furthermore, if $q \in (\frac{1+\sqrt{5}}{2}, 2)$ then it was shown in [4] that there always exists $x \in (0, \frac{1}{q-1})$ with a unique q-expansion.

In this paper we will be interested in the set of $q \in (1,2)$ for which there exists $x \in (0, \frac{1}{q-1})$ satisfying card $\Sigma_q(x) = \aleph_0$. More specifically, we will be interested in the set

$$\mathcal{B}_{\aleph_0} := \Big\{ q \in (1,2) \Big| \text{ there exists } x \in \Big(0, \frac{1}{q-1}\Big) \text{ satisfying card } \Sigma_q(x) = \aleph_0 \Big\}.$$

In [5] it was shown $\min \mathcal{B}_{\aleph_0} = \frac{1+\sqrt{5}}{2}$. We can define $\mathcal{B}_k$ in an analogous way for all $k \geq 1$. It was first shown in [6] that $\mathcal{B}_k \neq \emptyset$ for all $k \geq 2$, this was later improved upon in [13] where it was shown that for each $k \in \mathbb{N}$ there exists $\gamma_k > 0$ such that $(2 - \gamma_k, 2) \subset \mathcal{B}_j$ for all $1 \leq j \leq k$. Combining the results stated in [13] and [3] the following theorem is shown to hold.

Theorem 1.1.
- *The smallest element of $\mathcal{B}_2$ is*
$$q_2 \approx 1.71064,$$
the appropriate root of $x^4 = 2x^2 + x + 1$.
- *For $k \geq 3$ the smallest element of $\mathcal{B}_k$ is*
$$q_f \approx 1.75488,$$
the appropriate root of $x^3 = 2x^2 - x + 1$.
- *Moreover, the first element of $\mathcal{B}_2$ strictly greater than q_2 is q_f.*

In this paper we will show that the following theorem holds.

Theorem 1.2. *The smallest element of $\mathcal{B}_{\aleph_0}$ strictly greater than $\frac{1+\sqrt{5}}{2}$ is*
$$q_{\aleph_0} \approx 1.64541,$$
the appropriate root of $x^6 = x^4 + x^3 + 2x^2 + x + 1$.

This answers a question originally posed in [13]. The following corollary is an immediate consequence of Theorem 1.1, Theorem 1.2 and our earlier remarks, it implies a full dichotomy for the number of possible q-expansions for $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$.

Corollary 1.3. *Let $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$, then there exists $x \in (0, \frac{1}{q-1})$ such that $\Sigma_q(x)$ is uncountable and there exists $x \in (0, \frac{1}{q-1})$ with a unique q-expansion, moreover, for any $x \in (0, \frac{1}{q-1})$ the set $\Sigma_q(x)$ is either uncountable or a singleton set.*

Before stating the theory behind Theorem 1.2 we shall outline our method of proof, this will help to motivate the following sections. If $q \in \mathcal{B}_{\aleph_0}$, then as we will see, there must exist $x \in I_q$ for which $\Sigma_q(x)$ takes a highly nontrivial structure, in the following sections we refer to these x as q null infinite points. If I_q contains a q null infinite point and $q \in [\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\}$, then q must satisfy certain strong algebraic properties. Once these properties are appropriately formalised it is apparent that they cannot be satisfied for q' sufficiently close to q, this implies that there exists $\delta > 0$ such that $\mathcal{B}_{\aleph_0} \cap (\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} + \delta) = \emptyset$, and more generally that $\mathcal{B}_{\aleph_0} \cap ([\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\})$ is a discrete set. The δ produced by our method in fact turns out to be optimal. We remark that $q_{\aleph_0} \in \mathcal{B}_{\aleph_0}$ was already known to Hare and Sidorov [9], moreover, numerical experiments done by Hare seemed to suggest $q_{\aleph_0}$ was the smallest element of $\mathcal{B}_{\aleph_0}$ strictly greater than $\frac{1+\sqrt{5}}{2}$.

In Section 2 we establish several technical results that will be used in Section 3 where we prove Theorem 1.2. In Section 4 we prove several results that arose naturally from our proof of Theorem 1.2. In particular, we prove that for all $q \in (1,2)$, if $x \in I_q$ satisfies $\Sigma_q(x)$ is uncountable, then it must have cardinality equal to that of the continuum, as such the continuum hypothesis holds for this specific case, this answers a question attributed to Erdős. We also show that $\mathcal{B}_{\aleph_0} \cap ([\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\})$ is a discrete set, and propose a method by which we can determine whether a typical $q \in [\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\}$ is an element of $\mathcal{B}_{\aleph_0}$. Finally in Section 5 we pose some open questions.

2. Preliminaries

We begin by recalling some standard results. In what follows we fix $T_{q,0}(x) = qx$ and $T_{q,1}(x) = qx - 1$, we typically denote an element of $\bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ by a; here $\{T_{q,0}, T_{q,1}\}^0$ denotes the set consisting of the identity map. Moreover, if $a = (a_1, \ldots, a_n)$ we shall use $a(x)$ to denote $(a_n \circ \cdots \circ a_1)(x)$ and $|a|$ to denote the length of a. Given $a \in \bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ and $q' \neq q$, we can identify a with an element of $\bigcup_{n=0}^{\infty}\{T_{q',0}, T_{q',1}\}^n$ by replacing each $T_{q,i}$ term in a with a $T_{q',i}$ term. By an abuse of notation we also denote the element of $\bigcup_{n=0}^{\infty}\{T_{q',0}, T_{q',1}\}^n$ attained through this identification by a, whether we are interpreting a as an element of $\bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ or $\bigcup_{n=0}^{\infty}\{T_{q',0}, T_{q',1}\}^n$ will be clear from the context. We will make regular use of this identification in Section 4.

We let

$$\Omega_q(x) = \Big\{(a_i)_{i=1}^{\infty} \in \{T_{q,0}, T_{q,1}\}^{\mathbb{N}} : (a_n \circ \cdots \circ a_1)(x) \in I_q \text{ for all } n \in \mathbb{N}\Big\}.$$

The significance of $\Omega_q(x)$ is made clear by the following lemma.

Lemma 2.1. *card* $\Sigma_q(x) =$ *card* $\Omega_q(x)$ *where our bijection identifies* $(\epsilon_i)_{i=1}^{\infty}$ *with* $(T_{q,\epsilon_i})_{i=1}^{\infty}$.

The proof of Lemma 2.1 is contained within [2]. It is an immediate consequence of Lemma 2.1 that we can interpret Theorem 1.2 in terms of $\Omega_q(x)$ rather than $\Sigma_q(x)$. Throughout the course of our proof of Theorem 1.2 we will frequently switch between $\Sigma_q(x)$ and the dynamical interpretation of $\Sigma_q(x)$ provided by Lemma 2.1, often considering $\Omega_q(x)$ will help our exposition.

An element $x \in I_q$ satisfies $T_{q,0}(x) \in I_q$ and $T_{q,1}(x) \in I_q$ if and only if $x \in [\frac{1}{q}, \frac{1}{q(q-1)}]$. Furthermore, if card $\Sigma_q(x) > 1$ or equivalently card $\Omega_q(x) > 1$, then there exists a unique minimal sequence of transformations a such that $a(x) \in [\frac{1}{q}, \frac{1}{q(q-1)}]$. Throughout this paper when we speak of a finite sequence being minimal we mean minimal amongst $\bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ with respect to length. In what follows we let $S_q := [\frac{1}{q}, \frac{1}{q(q-1)}]$; S_q is usually referred to as the *switch* *region*. If $x \in (0, \frac{1}{q-1})$ and a is a finite sequence of transformations satisfying $a(x) \in S_q$, then we say that the sequence a is a *branching sequence for* x and $a(x)$ is a *branching point of* x.

In what follows we denote the set of $x \in I_q$ with unique q-expansion by U_q, i.e.

$$U_q = \Big\{ x \in I_q | \text{ card } \Sigma_q(x) = 1 \Big\}.$$

The following lemma is a consequence of [8, Theorem 2].

Lemma 2.2. *Let* $q \in (\frac{1+\sqrt{5}}{2}, q_f]$*, then*

$$U_q = \Big\{ (0^k(10)^\infty)_q, (1^k(10)^\infty)_q, 0, \frac{1}{q-1} \Big\},$$

where $k \geq 0$.

In Lemma 2.2 we have adopted the notation $(\epsilon_1, \ldots, \epsilon_n)^k$ to denote the concatenation of $(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$ by itself k times and $(\epsilon_1, \ldots, \epsilon_n)^\infty$ to denote the element of $\{0,1\}^{\mathbb{N}}$ obtained by concatenating $(\epsilon_1, \ldots, \epsilon_n)$ by itself infinitely many times, we will use this notation throughout. Lemma 2.2 will be a useful tool when it comes to showing that $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0} = \emptyset$.

2.1. **Branching argument.** To prove Theorem 1.2 we use a variation of the branching argument that first appeared in [14]. Before giving details of our approach we describe the construction given in [14].

2.1.1. *Construction of the branching tree corresponding to* x. We define the *branching tree corresponding to* x as follows. Suppose x satisfies card $\Omega_q(x) = 1$, then we define the branching tree corresponding to x to be an infinite horizontal line. If x satisfies card $\Omega_q(x) > 1$, then there exists a unique minimal branching sequence a, we depict this choice of transformation by a horizontal line of finite length that then bifurcates with an upper and lower branch. The upper branch corresponds to the sequence of transformations obtained by concatenating the branching sequence a by $T_{q,0}$ and the lower branch corresponds to the sequence of transformations obtained by concatenating the branching sequence a by $T_{q,1}$. If $T_{q,i}(a(x))$ satisfies $\Omega_q(T_{q,i}(a(x))) = 1$ then we extend the branch corresponding to $T_{q,i}(a(x))$ by an infinite horizontal line. If $\Omega_q(T_{q,i}(a(x))) > 1$ then there exists a unique minimal branching sequence for $T_{q,i}(a(x))$ that we call a', we depict this choice of transformation by extending the branch corresponding to $T_{q,i}(a(x))$ by a horizontal line of finite length that then bifurcates, again the upper branch corresponds to concatenating a' by $T_{q,0}$, and the lower branch corresponds to concatenating a' by $T_{q,1}$. Repeatedly applying these rules to successive branching points of x we construct an infinite tree which we refer to as the branching tree corresponding to x. We refer the reader to Figure 1 for a diagram illustrating the construction of the branching tree corresponding to x. Where appropriate we denote the branching tree corresponding to x by $\mathcal{T}(x)$. The branching tree corresponding to x is referred to as the branching compactum in [14].

Remark 2.3. It is immediate from the construction of $\mathcal{T}(x)$ that there is a bijection between the space of infinite paths in $\mathcal{T}(x)$ and $\Omega_q(x)$, which by Lemma 2.1 implies there is also a bijection between the space of infinite paths in $\mathcal{T}(x)$ and $\Sigma_q(x)$.

x $\cdots\cdots$ The branching tree corresponding to to x when card $\Omega_q(x) = 1$.

$T_{q,0}(a(x))$

x $a(x)$ The first bifurcation when $a(x) \in S_q$.

$T_{q,1}(a(x))$

x The branching tree corresponding to x when card $\Omega_q(x) > 1$.

FIGURE 1. The construction of the branching tree corresponding to x

2.1.2. *Construction of the infinite branching tree corresponding to x.* We now give details of our variation of the above construction which will be more suited towards our exposition. Suppose $x \in I_q$ satisfies $\Omega_q(x)$ is infinite or equivalently $\Sigma_q(x)$ is infinite, we define the *infinite branching tree corresponding to* x as follows. If for each branching point of x, $a(x)$, we have card $\Omega_q(T_{q,i}(a(x))) < \infty$, for some $i \in \{0, 1\}$, then we define the infinite branching tree corresponding to x to be an infinite horizontal line. If this is not the case then there exists a branching point $a(x)$ such that $\Omega_q(T_{q,0}(a(x)))$ and $\Omega_q(T_{q,1}(a(x)))$ are both infinite. Taking a to be the unique minimal branching sequence of x for which $\Omega_q(T_{q,0}(a(x)))$ and $\Omega_q(T_{q,1}(a(x)))$ are both infinite we draw a horizontal line of finite length which bifurcates, the upper branch corresponds to $T_{q,0}(a(x))$ and the lower branch corresponds to $T_{q,1}(a(x))$. We then extend the branch corresponding to $T_{q,i}(a(x))$ in accordance with the same rules, i.e., if for each branching point of $T_{q,i}(a(x))$, $a'(T_{q,i}(a(x)))$, we have card $\Omega_q(a'(T_{q,i}(a(x)))) < \infty$, for some $i \in \{0, 1\}$, we extend the branch corresponding to $T_{q,i}(a(x))$ by an infinite horizontal line, and if there exists a branching point of $T_{q,i}(a(x))$, $a'(T_{q,i}(a(x)))$, such that both $\Omega_q(T_{q,0}(a'(T_{q,i}(a(x)))))$ and $\Omega_q(T_{q,1}(a'(T_{q,i}(a(x)))))$ are infinite, we extend the branch corresponding to $T_{q,i}(a(x))$ by a finite horizontal line that then bifurcates, with upper branching corresponding to $T_{q,0}(a'(T_{q,i}(a(x))))$, and lower branch corresponding to $T_{q,1}(a'(T_{q,i}(a(x))))$. Repeatedly applying these rules to each

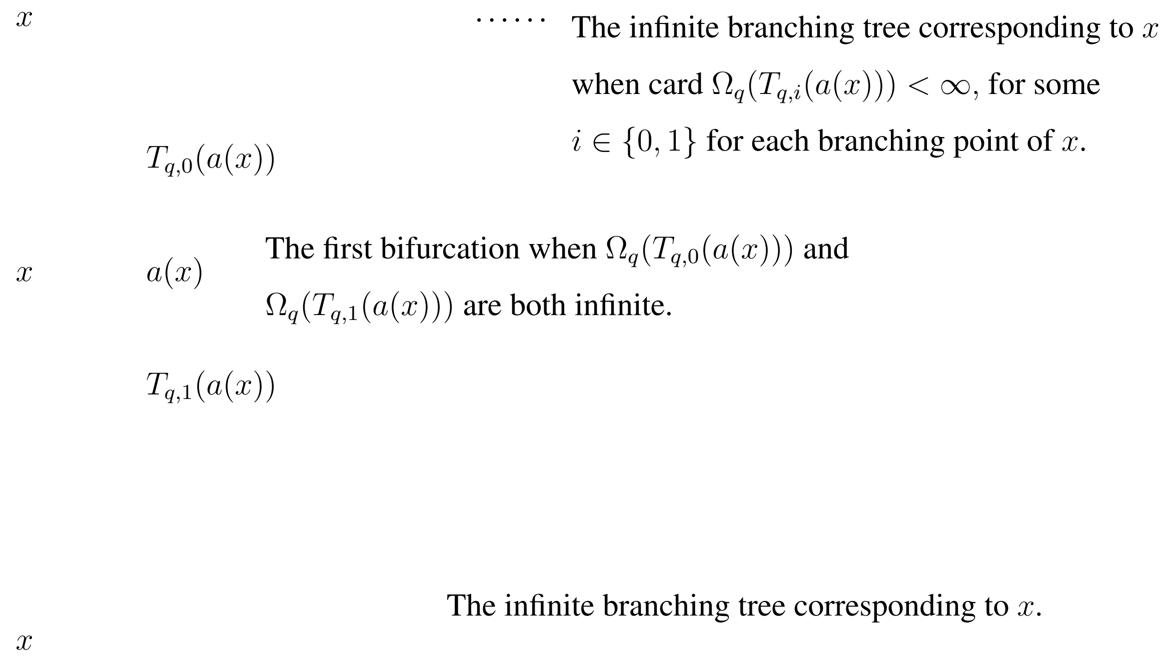

FIGURE 2. The construction of the infinite branching tree corresponding to x

upper and lower branch of our construction we obtain an infinite tree that we refer to as the infinite branching tree corresponding to x. We refer the reader to Figure 2 for a diagram illustrating the construction of the branching tree corresponding to x. Where appropriate we denote the infinite branching tree corresponding to x by $\mathcal{T}_\infty(x)$. If $\mathcal{T}_\infty(x)$ contains at least one bifurcation then every branch except the initial horizontal branch begins at a point where $\mathcal{T}_\infty(x)$ bifurcates, we refer to this point as the *root of the branch*. It is clear from the construction of $\mathcal{T}_\infty(x)$ that the root of a branch can be identified with a branching point of x.

Remark 2.4. Every infinite path in $\mathcal{T}_\infty(x)$ can be identified with a unique element of $\Omega_q(x)$. However, unlike $\mathcal{T}(x)$ not every element of $\Omega_q(x)$ necessarily corresponds to a unique infinite path in $\mathcal{T}_\infty(x)$.

If x satisfies $\Omega_q(x)$ is infinite and for each branching point $a(x)$ we have card $\Omega_q(T_{q,i}(a(x))) < \infty$, for some $i \in \{0,1\}$, i.e., the case where the infinite branching tree is an infinite horizontal line, then we refer to x as a *q null infinite point*. It is an immediate consequence of our definition that if x is a q null infinite point then card $\Omega_q(x) =$ card $\Sigma_q(x) = \aleph_0$.

Remark 2.5. For $q \in (\frac{1+\sqrt{5}}{2}, q_f)\setminus\{q_2\}$, it is a consequence of Theorem 1.1 that if x satisfies $\Omega_q(x)$ is infinite, then at each branching point $a(x)$, either both $\Omega_q(T_{q,0}(a(x)))$ and $\Omega_q(T_{q,1}(a(x)))$ are infinite or one of them is infinite and one of them is a singleton set, i.e, $T_{q,i}(a(x)) \in U_q$ for some $i \in \{0,1\}$. As such, for $q \in (\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\}$ we may interpret $\mathcal{T}_\infty(x)$ as the infinite tree obtained from $\mathcal{T}(x)$ if we remove all branches that end in infinite horizontal lines.

Remark 2.6. The case where x is a q null infinite point is of particular importance. By Theorem 1.1 it follows that for $q \in (\frac{1+\sqrt{5}}{2}, q_2)\cup(q_2, q_f)$, if x is a q null infinite point then for each branching point of x, $a(x)$, we must have card $\Omega_q(T_{q,i}(a(x))) = \aleph_0$ and card $\Omega_q(T_{q,1-i}(a(x))) = 1$, for some $i \in \{0,1\}$.

As the following proposition shows, it is in fact the case that whenever $q \in \mathcal{B}_{\aleph_0}$, then $(0, \frac{1}{q-1})$ contains a q null infinite point.

Proposition 2.7. *Suppose $q \in \mathcal{B}_{\aleph_0}$, then $(0, \frac{1}{q-1})$ contains a q null infinite point.*

Proof. If $q \in \mathcal{B}_{\aleph_0}$ there exists $x \in (0, \frac{1}{q-1})$ satisfying card $\Omega_q(x) = \aleph_0$. If x is a q null infinite point then we are done, let us assume this is not the case and that $\mathcal{T}_\infty(x)$ contains at least one bifurcation. If each branch of $\mathcal{T}_\infty(x)$ was to always bifurcate then $\mathcal{T}_\infty(x)$ would be the full binary tree, as each infinite path in $\mathcal{T}_\infty(x)$ can be identified with a unique element of $\Omega_q(x)$ and the set of infinite paths in the full binary tree has cardinality equal to the continuum we would have card $\Omega_q(x) = 2^{\aleph_0}$, a contradiction. As such there must exist at least one branch that no longer bifurcates, by considering the root of this branch and the corresponding branching point $a(x) \in S_q$, either $T_{q,0}(a(x))$ or $T_{q,1}(a(x))$ must be a q null infinite point. □

To prove Theorem 1.2 we first of all show that $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0} = \emptyset$, we do this by contradiction. By Proposition 2.7 if $q \in \mathcal{B}_{\aleph_0}$ then $(0, \frac{1}{q-1})$ contains a q null infinite point, by studying q null infinite points we will be able to derive our desired contradiction.

3. Proof of Theorem 1.2

Our proof of Theorem 1.2 will be split into two parts, we begin by showing that $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0} = \emptyset$, we then explicitly construct an $x \in I_{q_{\aleph_0}}$ for which card $\Omega_{q_{\aleph_0}}(x) = \aleph_0$.

3.1. Proof that $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0})\cap\mathcal{B}_{\aleph_0} = \emptyset$.

To show that $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0})\cap\mathcal{B}_{\aleph_0} = \emptyset$ it is useful to consider the following interval:

$$J_q := \Big[\frac{q+q^2}{q^4-1}, \frac{1+q^3}{q^4-1}\Big].$$

The following identities hold

$$T_{q,1}\Big(T_{q,0}\Big(\frac{q+q^2}{q^4-1}\Big)\Big) = \frac{1+q^3}{q^4-1} \text{ and } T_{q,0}\Big(T_{q,1}\Big(\frac{1+q^3}{q^4-1}\Big)\Big) = \frac{q+q^2}{q^4-1}, \tag{3.1}$$

it is an immediate consequence of (3.1) that $\frac{q+q^2}{q^4-1} = ((0110)^\infty)_q$ and $\frac{1+q^3}{q^4-1} = ((1001)^\infty)_q$. The endpoints of J_q are contained within a 4-cycle

$$\Big\{((0110)^\infty)_q, ((1100)^\infty)_q, ((1001)^\infty)_q, ((0011)^\infty)_q\Big\}.$$

For $q > q_f$ this cycle it is a subset of U_q, moreover, it is the first 4-cycle to be a subset of U_q, see [1]. The significance of the interval J_q is made clear by the following lemma.

Lemma 3.1. *Let $q \in (\frac{1+\sqrt{5}}{2}, q_f]$. Suppose $x \in I_q$ satisfies card $\Omega_q(x) > 1$, then there exists a finite sequence of transformations a such that $a(x) \in J_q$.*

Proof. It is a simple exercise to show that if $q \in (\frac{1+\sqrt{5}}{2}, q_f]$ then $J_q \subseteq S_q$ with equality if and only if $q = q_f$. Let $x \in I_q$ satisfy card $\Omega_q(x) > 1$, then there exists a finite sequence of transformations a such that $a(x) \in S_q$. If $q = q_f$ then we may immediately conclude our result, as such in what follows we assume $q \in (\frac{1+\sqrt{5}}{2}, q_f)$. If $q \in (\frac{1+\sqrt{5}}{2}, 2)$, then $S_q \subset (\frac{1}{q^2-1}, \frac{q}{q^2-1})$. The significance of the points $\frac{1}{q^2-1}$ and $\frac{q}{q^2-1}$ is that $T_{q,0}(\frac{1}{q^2-1}) = \frac{q}{q^2-1}$ and $T_{q,1}(\frac{q}{q^2-1}) = \frac{1}{q^2-1}$. If $y \in (\frac{1}{q^2-1}, \frac{q}{q^2-1})$, then the following identities hold:

$$T_{q,1}(T_{q,0}(y)) - \frac{1}{q^2-1} = q^2\Big(y - \frac{1}{q^2-1}\Big) \text{ and } \frac{q}{q^2-1} - T_{q,0}(T_{q,1}(y)) = q^2\Big(\frac{q}{q^2-1} - y\Big), \tag{3.2}$$

i.e., $T_{q,1} \circ T_{q,0}$ scales the distance between y and $\frac{1}{q^2-1}$ by a factor q^2 and $T_{q,0} \circ T_{q,1}$ scales the distance between y and $\frac{q}{q^2-1}$ by a factor q^2.

Returning to $a(x) \in S_q$, if $a(x) \in J_q$ then we are done, let us suppose this is not the case and $a(x) \in S_q \setminus J_q = [\frac{1}{q}, \frac{q+q^2}{q^4-1}) \cup (\frac{1+q^3}{q^4-1}, \frac{1}{q(q-1)}]$. If $a(x)$ in $[\frac{1}{q}, \frac{q+q^2}{q^4-1})$ then it follows from (3.1), (3.2) and the monotonicity of the maps $T_{q,0}$ and $T_{q,1}$ that sufficiently many iterates of the map $T_{q,1} \circ T_{q,0}$ will map $a(x)$ into J_q, similarly if $a(x) \in (\frac{1+q^3}{q^4-1}, \frac{1}{q(q-1)}]$ then sufficiently many iterates of the map $T_{q,0} \circ T_{q,1}$ will map $a(x)$ into J_q. □

To prove $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0} = \emptyset$ it is necessary to determine which elements of J_q are preimages of points with unique q-expansion, these points are classified in the following proposition.

Proposition 3.2. *Let $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$, then*

$$(T_{q,0}^{-1}(U_q) \cap J_q) \cup (T_{q,1}^{-1}(U_q) \cap J_q) = \{(01(10)^\infty)_q, (011(10)^\infty)_q, (10(01)^\infty)_q, (100(01)^\infty)_q\}.$$

Proof. By Lemma 2.2 to prove our result it suffices to show that the following identities hold for $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$:

1. $T_{q,0}(\frac{q+q^2}{q^4-1}) \in (((10)^\infty)_q, (1(10)^\infty)_q)$
2. $T_{q,0}(\frac{1+q^3}{q^4-1}) \in ((11(10)^\infty)_q, (111(10)^\infty)_q)$
3. $T_{q,1}(\frac{q+q^2}{q^4-1}) \in ((000(01)^\infty)_q, (00(01)^\infty)_q)$
4. $T_{q,1}(\frac{1+q^3}{q^4-1}) \in ((0(01)^\infty, (01)^\infty)_q)$.

Performing several straightforward calculations we can show that for $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$ each of these identities hold. We remark that the upper bound $q_{\aleph_0}$ is optimal as

$$T_{q_{\aleph_0},0}\Big(\frac{1+q_{\aleph_0}^3}{q_{\aleph_0}^4 - 1}\Big) = (111(10)^\infty)_{q_{\aleph_0}} \text{ and } T_{q_{\aleph_0},1}\Big(\frac{q_{\aleph_0} + q_{\aleph_0}^2}{q_{\aleph_0}^4 - 1}\Big) = (000(01)^\infty)_{q_{\aleph_0}}.$$

□

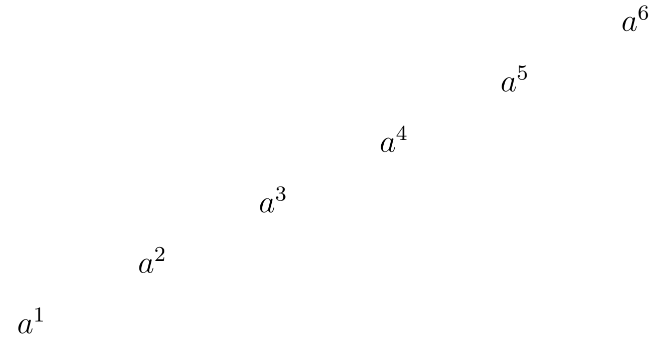

FIGURE 3. The branching tree corresponding to x when x is a q null infinite point and $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$

We are now in a position to prove $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0} = \emptyset$. Suppose $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0}$, then by Proposition 2.7 we may assume $x \in (0, \frac{1}{q-1})$ is a q null infinite point. We let $a^1 = (a^1_1, \ldots, a^1_{k_1})$ denote the unique minimal branching sequence of x, since card $\Omega_q(T_{q,i}(a^1(x))) = 1$ for some $i \in \{0,1\}$, there exists a unique minimal sequence of transformations $a^2 = (a^2_1, \ldots, a^2_{k_2})$ such that $a^2(a^1(x)) \in S_q$. Similarly, for $i \geq 2$ we let $a^i = (a^i_1, \ldots, a^i_{k_i})$ denote the unique minimal sequence of transformations satisfying $a^i(a^{i-1}(\ldots(a^1(x))\ldots)) \in S_q$. We refer the reader to Figure 3 for a diagram depicting the branching tree corresponding to x when x is a q null infinite point and $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$, this diagram illustrates the role the of the sequence $(a^i)_{i=1}^{\infty}$. For ease of exposition we denote the finite concatenation $a^1 a^2 \cdots a^i$ by b^i, therefore $a^i(a^{i-1}(\ldots(a^1(x))\ldots)) = b^i(x)$. For $q \in (\frac{1+\sqrt{5}}{2}, 2)$, if $b^i(x) \in S_q$ then $T_{q,i}(b^i(x)) \notin S_q$ for $i \in \{0,1\}$, this implies $k_i \geq 2$ for all $i \in \mathbb{N}$.

By Lemma 3.1 we can assert that $b^n(x) \in J_q$ for some $n \geq 2$. Since x is a q null infinite point Proposition 3.2 implies $b^n(x) \in \{(01(10)^{\infty})_q, (011(10)^{\infty})_q, (10(01)^{\infty})_q, (100(01)^{\infty})_q\}$. We now show that if $b^n(x) \in \{(01(10)^{\infty})_q, (011(10)^{\infty})_q, (10(01)^{\infty})_q, (100(01)^{\infty})_q\}$, then $T_{q,i}(b^{n-1}(x)) \notin U_q$, for $i \in \{0,1\}$. This will contradict our assumption that x is a q null infinite point and implies $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0} = \emptyset$.

If $a^n = (a^n_1, \ldots, a^n_{k_n})$ then without loss in generality we may assume that $a^n_{k_n} = T_{q,0}$. The following lemma determines a^n_i for $1 \leq i < k_n$.

Lemma 3.3. *Let x and a^n be as above. Suppose $a^n_{k_n} = T_{q,0}$, then $a^n_1 = T_{q,1}$ and $a^n_i = T_{q,0}$ for $1 < i < k_n$.*

Proof. We begin by showing $a^n_1 = T_{q,1}$. We suppose $a^n_1 = T_{q,0}$ and derive a contradiction. If $a^n_1 = T_{q,0}$, then $T_{q,0}(b^{n-1}(x)) \in (\frac{1}{q(q-1)}, \frac{1}{q-1}]$ and $k_n \geq 3$. Letting $a' = (a^n_2, \ldots, a^n_{k_n-1})$ it follows that $a'(T_{q,0}(b^{n-1}(x)))=T_{q,0}^{-1}(a^n(x))$ and by the minimality of a^n we must have

$$(a_i^n\circ\cdots\circ a_2^n)(T_{q,0}(b^{n-1}(x)))\notin S_q$$

for $2\leq i\leq k_n-1$, we now show that this is not possible.

If $a'(T_{q,0}(b^{n-1}(x)))=T_{q,0}^{-1}(a^n(x))$ and $(a_i^n\circ\cdots\circ a_2^n)(T_{q,0}(b^{n-1}(x)))\notin S_q$ for all $2\leq i\leq k_n-1$, then it is a consequence of $T_{q,1}$ being strictly decreasing on $(0,\frac{1}{q-1})$ and $T_{q,0}$ being strictly increasing on $(0,\frac{1}{q-1})$ that we must have

$$T_{q,1}\Big(\frac{1}{q(q-1)}\Big)\leq T_{q,0}^{-1}(b^n(x)). \tag{3.3}$$

By our assumption $b^n(x)\in\{(01(10)^\infty)_q,(011(10)^\infty)_q,(10(01)^\infty)_q,(100(01)^\infty)_q\}$, therefore to derive our contradiction it suffices to show that (3.3) does not hold for each of these four cases. As such, to conclude $a_1^n=T_{q,1}$ we need to show that following inequalities hold:

(1) $T_{q,0}^{-1}((01(10)^\infty)_q)<T_{q,1}\Big(\frac{1}{q(q-1)}\Big)$
(2) $T_{q,0}^{-1}((011(10)^\infty)_q)<T_{q,1}\Big(\frac{1}{q(q-1)}\Big)$
(3) $T_{q,0}^{-1}((10(01)^\infty)_q)<T_{q,1}\Big(\frac{1}{q(q-1)}\Big)$
(4) $T_{q,0}^{-1}((100(01)^\infty)_q)<T_{q,1}\Big(\frac{1}{q(q-1)}\Big)$.

By the monotonicity of the map $T_{q,0}^{-1}$ it suffices to show only (2) and (3) hold. We can show that (2) holds for $q\in(\frac{1+\sqrt{5}}{2},1.69765\ldots)$, here $1.69765\ldots$ is the appropriate root of $x^6=x^5+2x^4-x^3-x^2+1$. Similarly (3) holds for $q\in(\frac{1+\sqrt{5}}{2},1.68042\ldots)$ where $1.68042\ldots$ is the appropriate root of $x^5=x^4+x^3+x-1$. Therefore (3.3) does not hold and we may conclude $a_1^n=T_{q,1}$.

It remains to show $a_i^n=T_{q,0}$ for $1<i<k_n$. Suppose $a_i^n=T_{q,1}$ for some $1<i<k_n$, then either

$$(a_{i-1}^n\circ\cdots\circ a_1^n)(b^{n-1}(x))\in S_q \text{ or } (a_{i-1}^n\circ\cdots\circ a_1^n)(b^{n-1}(x))\in\Big(\frac{1}{q(q-1)},\frac{1}{q-1}\Big].$$

As a consequence of the minimality of a^n we cannot have $(a_{i-1}^n\circ\cdots\circ a_1^n)(b^{n-1}(x))\in S_q$. By analogous reasoning to that stated in the first part of our proof, the minimality of a^n also implies that we cannot have $(a_{i-1}^n\circ\cdots\circ a_1^n)(b^{n-1}(x))\in(\frac{1}{q(q-1)},\frac{1}{q-1}]$. We may therefore conclude $a_i^n=T_{q,0}$ for all $1<i<k_n$. □

By Lemma 3.3 we have $b^n(x)=(T_{q,0}^{k_n}\circ T_{q,1})(b^{n-1}(x))$, since x is a q null infinite point we have $T_{q,0}(b^{n-1}(x))\in U_q$. This is equivalent to

$$T_{q,0}^{-k_n}(b^n(x))+1\in U_q. \tag{3.4}$$

To derive our contradiction we will show that (3.4) cannot occur for all $q\in(\frac{1+\sqrt{5}}{2},q_{\aleph_0})$. Since $b^n(x)\in\{(01(10)^\infty)_q,(011(10)^\infty)_q,(10(01)^\infty)_q,(100(01)^\infty)_q\}$ there are four cases to consider, the analysis of these cases is summarised in the following proposition.

Proposition 3.4. *For $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$ the following inequalities hold:*

(1) $T_{q,0}^{-1}((01(10)^\infty)_q) + 1 \in ((111(10)^\infty)_q, (1111(10)^\infty)_q)$
(2) $T_{q,0}^{-2}((01(10)^\infty)_q) + 1 \in ((1(10)^\infty)_q, (11(10)^\infty)_q)$
(3) $T_{q,0}^{-j}((01(10)^\infty)_q) + 1 \in (((10)^\infty), (1(10)^\infty)_q)$ *for all* $j \geq 3$
(4) $T_{q,0}^{-1}((011(10)^\infty)_q) + 1 \in ((1111(10)^\infty)_q, (11111(10)^\infty)_q)$
(5) $T_{q,0}^{-2}((011(10)^\infty)_q) + 1 \in ((1(10)^\infty)_q, (11(10)^\infty)_q)$
(6) $T_{q,0}^{-j}((011(10)^\infty)_q) + 1 \in (((10)^\infty)_q, (1(10)^\infty)_q)$ *for all* $j \geq 3$
(7) $T_{q,0}^{-1}((100(01)^\infty)_q) + 1 \in ((111(10)^\infty)_q, (1111(10)^\infty)_q)$
(8) $T_{q,0}^{-2}((100(01)^\infty)_q) + 1 \in ((1(10)^\infty)_q, (11(10)^\infty)_q)$
(9) $T_{q,0}^{-j}((100(01)^\infty)_q) + 1 \in (((10)^\infty), (1(10)^\infty)_q)$ *for all* $j \geq 3$
(10) $T_{q,0}^{-1}((10(01)^\infty)_q) + 1 \in ((1111(10)^\infty)_q, (11111(10)^\infty)_q)$
(11) $T_{q,0}^{-2}((10(01)^\infty)_q) + 1 \in ((1(10)^\infty)_q, (11(10)^\infty)_q)$
(12) $T_{q,0}^{-j}((10(01)^\infty)_q) + 1 \in (((10)^\infty)_q, (1(10)^\infty)_q)$ *for all* $j \geq 3$

Proof. Showing that these identities hold is a simple yet time consuming exercise, as such we omit the details. Our calculations yielded the following:

- (1), (2) and (3) hold for $q \in (\frac{1+\sqrt{5}}{2}, 1.67602\ldots)$, where $1.67602\ldots$ is the appropriate root of $x^5 = 2x^3 + x^2 + 1$
- (4), (5) and (6) hold for $q \in (\frac{1+\sqrt{5}}{2}, 1.65462\ldots)$, where $1.65462\ldots$ is the appropriate root of $x^6 = 2x^4 + x^3 + 1$
- (7), (8) and (9) hold for $q \in (\frac{1+\sqrt{5}}{2}, 1.666184\ldots)$, where $1.66184\ldots$ is the appropriate root of $x^5 = x^3 + x^2 + 2x + 2$
- (10), (11) and (12) hold for $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$.

□

By Proposition 3.4 we can deduce that (3.4) does not hold and we have our desired contradiction, we may therefore conclude $(\frac{1+\sqrt{5}}{2}, q_{\aleph_0}) \cap \mathcal{B}_{\aleph_0} = \emptyset$.

3.2. **Proof that $q_{\aleph_0} \in \mathcal{B}_{\aleph_0}$.** By the above remarks to conclude Theorem 1.2 it suffices to show that $q_{\aleph_0} \in \mathcal{B}_{\aleph_0}$. The proof of this statement is contained within the following proposition.

Proposition 3.5. $\frac{q_{\aleph_0} + q_{\aleph_0}^2}{q_{\aleph_0}^4 - 1}$ *and* $\frac{1 + q_{\aleph_0}^3}{q_{\aleph_0}^4 - 1}$ *have countably infinite $q_{\aleph_0}$-expansions.*

Proof. To begin with we recall that

$$T_{q,1}\Big(T_{q,0}\Big(\frac{q+q^2}{q^4-1}\Big)\Big) = \frac{1+q^3}{q^4-1} \text{ and } T_{q,0}\Big(T_{q,1}\Big(\frac{1+q^3}{q^4-1}\Big)\Big) = \frac{q+q^2}{q^4-1},$$

for all $q \in (1,2)$. As stated in the proof of Proposition 3.2

$$T_{q_{\aleph_0},0}\Big(\frac{1+q_{\aleph_0}^3}{q_{\aleph_0}^4-1}\Big) = (111(10)^\infty)_{q_{\aleph_0}} \text{ and } T_{q_{\aleph_0},1}\Big(\frac{q_{\aleph_0}+q_{\aleph_0}^2}{q_{\aleph_0}^4-1}\Big) = (000(01)^\infty)_{q_{\aleph_0}}.$$

Since

$$T_{q_{\aleph_0},0}\Big(\frac{q_{\aleph_0}+q_{\aleph_0}^2}{q_{\aleph_0}^4-1}\Big)\in\Big(\frac{1}{q_{\aleph_0}(q_{\aleph_0}-1)},\frac{1}{q_{\aleph_0}-1}\Big]\text{ and }T_{q_{\aleph_0},1}\Big(\frac{1+q_{\aleph_0}^3}{q_{\aleph_0}^4-1}\Big)\in\Big[0,\frac{1}{q_{\aleph_0}}\Big),$$

it follows that

$$\Sigma_{q_{\aleph_0}}\Big(\frac{1+q_{\aleph_0}^3}{q_{\aleph_0}^4-1}\Big)=\Big\{(1001)^\infty,(1001)^k0111(10)^\infty,(1001)^k101000(01)^\infty|\text{ for some }k\geq 0\Big\}$$

and

$$\Sigma_{q_{\aleph_0}}\Big(\frac{q_{\aleph_0}+q_{\aleph_0}^2}{q_{\aleph_0}^4-1}\Big)=\Big\{(0110)^\infty,(0110)^k1000(01)^\infty,(0110)^k010111(10)^\infty|\text{ for some }k\geq 0\Big\}.$$

□

It is immediate from the proof of Proposition 3.5 that both $\frac{q_{\aleph_0}+q_{\aleph_0}^2}{q_{\aleph_0}^4-1}$ and $\frac{1+q_{\aleph_0}^3}{q_{\aleph_0}^4-1}$ are $q_{\aleph_0}$ null infinite points. By Proposition 3.5 we have $q_{\aleph_0}\in\mathcal{B}_{\aleph_0}$ and by our earlier remarks we may conclude Theorem 1.2.

4. General results

In this section we shall prove some general results that arose from our proof of Theorem 1.2.

4.1. The continuum hypothesis for $\Sigma_q(x)$.

In this section we show that the following theorem holds.

Theorem 4.1. *Let $q\in(1,2)$ and $x\in I_q$, if $\Sigma_q(x)$ is uncountable then card $\Sigma_q(x)=2^{\aleph_0}$.*

To prove this statement we need to construct another infinite tree in a similar way to how we constructed the branching tree corresponding to x and the infinite branching tree corresponding to x. We define the $2^{\aleph_0}$ *branching tree corresponding to* x as follows. Suppose $x\in I_q$ satisfies $\Omega_q(x)$ is uncountable, if for each branching point of x we have card $\Omega_q(T_{q,i}(a(x)))\leq\aleph_0$ for some $i\in\{0,1\}$, then the $2^{\aleph_0}$ branching tree corresponding to x is an infinite horizontal line. If this is not the case then there exists a unique minimal branching sequence a such that $\Omega_q(T_{q,0}(a(x)))$ and $\Omega_q(T_{q,1}(a(x)))$ are both uncountable, in this case we draw a finite horizontal line that then bifurcates with upper branch corresponding to $T_{q,0}(a(x))$ and lower branch corresponding to $T_{q,1}(a(x))$. Applying these rules to the branches corresponding to $T_{q,0}(a(x))$, $T_{q,1}(a(x))$ and all subsequent branches we obtain an infinite tree. We refer to the infinite tree we obtain through this construction as the $2^{\aleph_0}$ branching tree corresponding to x. Where appropriate we denote the $2^{\aleph_0}$ branching tree corresponding to x by $\mathcal{T}_{2^{\aleph_0}}(x)$.

Remark 4.2. As was the case for $\mathcal{T}(x)$ and $\mathcal{T}_\infty(x)$ each infinite path in $\mathcal{T}_{2^{\aleph_0}}(x)$ can be identified with a unique element of $\Omega_q(x)$.

By Remark 4.2 to prove that if $\Sigma_q(x)$ is uncountable then card $\Sigma_q(x)=2^{\aleph_0}$ it suffices to show that $\mathcal{T}_{2^{\aleph_0}}(x)$ is always the full binary tree. We will show that whenever $x\in I_q$ satisfies $\Sigma_q(x)$ is uncountable then there exists a branching sequence for x such that $\Omega_q(T_{q,0}(a(x)))$ and $\Omega_q(T_{q,1}(a(x)))$ are both uncountable. Repeatedly applying this result to successive branches in our construction will imply that every branch bifurcates and that $\mathcal{T}_{2^{\aleph_0}}(x)$ is the full binary tree.

Lemma 4.3. *Let $q \in (1,2)$ and $x \in I_q$. If $\Omega_q(x)$ is uncountable or equivalently $\Sigma_q(x)$ is uncountable, then there exists a branching point of x, $a(x)$, such that $\Omega_q(T_{q,0}(a(x)))$ and $\Omega_q(T_{q,1}(a(x)))$ are both uncountable.*

Proof. Suppose that for every branching point of x we have card $\Omega_q(T_{q,i}(a(x))) \leq \aleph_0$ for some $i \in \{0,1\}$. We let $a^1 = (a_1^1, \ldots, a_{n_1}^1)$ denote the unique minimal branching sequence of x, by our assumption card $\Omega_q(T_{q,i_1}(a^1(x))) \leq \aleph_0$ for some $i_1 \in \{0,1\}$, as $\Omega_q(x)$ is uncountable we must have $\Omega_q(T_{q,1-i_1}(a^1(x)))$ is uncountable. It is a consequence of $\Omega_q(T_{q,1-i_1}(a^1(x)))$ being uncountable and our assumption, that there exists a unique minimal branching sequence $a^2 = (a_1^2, \ldots, a_{n_2}^2)$ and $i_2 \in \{0,1\}$ satisfying: $n_2 > n_1$, $a_j^1 = a_j^2$ for $1 \leq j \leq n_1$, $\Omega_q(a^2(x))$ is uncountable, card $\Omega_q(T_{q,i_2}(a^2(x))) \leq \aleph_0$ and $\Omega_q(T_{q,1-i_2}(a^2(x)))$ is uncountable. Moreover, for $k \geq 2$ we define $a^k = (a_1^k, \ldots, a_{n_k}^k)$ and $i_k \in \{0,1\}$ inductively as follows, let a^k denote the the unique minimal branching sequence of x such that $n_k > n_{k-1}$, $a_j^{k-1} = a_j^k$ for $1 \leq j \leq n_{k-1}$ and $\Omega_q(a^k(x))$ is uncountable, we let i_k denote the unique element of $\{0,1\}$ such that card $\Omega_q(T_{q,i_k}(a^k(x))) \leq \aleph_0$ and $\Omega_q(T_{q,1-i_k}(a^k(x)))$ is uncountable.

To each a^k we associate the set

$$\Omega_{a^k}(x) = \Big\{ a \in \Omega_q(x) | a_j = a_j^k \text{ for } 1 \leq j \leq n_k \text{ and } a_{n_k+1} = T_{q,i_k} \Big\}.$$

Clearly card $\Omega_{a^k}(x) \leq \aleph_0$. Letting $a^\infty \in \{T_{q,0}, T_{q,1}\}^{\mathbb{N}}$ denote the unique infinite sequence obtained as the componentwise limit of $(a^k)_{k=1}^\infty$, it is an immediate consequence of our construction that

$$\Omega_q(x) = \{a^\infty\} \cup (\bigcup_{k=1}^{\infty} \Omega_{a^k(x)})$$

and that card $\Omega_q(x) \leq \aleph_0$, a contradiction. Therefore there must exists a branching point of x such that both $\Omega_q(T_{q,0}(a(x)))$ and $\Omega_q(T_{q,1}(a(x)))$ are uncountable. □

Theorem 4.1 follows from our earlier remarks.

4.2. **Properties of** $\mathcal{B}_{\aleph_0} \cap ([\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\})$**.** It is clear from the proof of Theorem 1.2 that the interval J_q is an appropriate object of study, in particular, we are interested in its subset $(T_{q,0}^{-1}(U_q) \cap J_q) \cup (T_{q,1}^{-1}(U_q) \cap J_q)$. For $k \geq 3$ we let α_k denote the unique $q \in (1,2)$ such that

$$T_{q,0}\Big(\frac{1+q^3}{q^4-1}\Big) = ((1)^k(10)^\infty)_q,$$

the appropriate root of $x^{k+4} = x^{k+3} + x^{k+2} + x^k - x^2 - 1$. In particular $\alpha_3 = q_{\aleph_0}$. It is a simple exercise to show that $\alpha_k \in [q_{\aleph_0}, q_f)$ for all $k \geq 3$ and $\alpha_k \nearrow q_f$. Adapting the proof of Proposition 3.5 it can be shown that $\alpha_k \in \mathcal{B}_{\aleph_0}$, for all $k \geq 3$. The significance of α_k follows from the fact that for $q \in [\alpha_k, \alpha_{k+1})$ we have

$$(T_{q,0}^{-1}(U_q) \cap J_q) \cup (T_{q,1}^{-1}(U_q) \cap J_q) = \Big\{ (1(0)^j(01)^\infty)_q, (0(1)^j(10)^\infty)_q | \text{ for } 1 \leq j \leq k \Big\}. \tag{4.1}$$

In what follows we let

$$P_q = (T_{q,0}^{-1}(U_q) \cap J_q) \cup (T_{q,1}^{-1}(U_q) \cap J_q)$$

and

$$U_{k,q}=\Big\{(1(0)^j(01)^\infty)_q,(0(1)^j(10)^\infty)_q|\text{ for }1\leq j\leq k\Big\}.$$

The following result is implicit in our proof of Theorem 1.2 and therefore stated without proof.

Proposition 4.4. *Let $q\in[q_{\aleph_0},q_f)\setminus\{q_2\}$, then $q\in\mathcal{B}_{\aleph_0}$ if and only if P_q contains a q null infinite point.*

Suppose $q\in[q_{\aleph_0},q_f)\setminus\{q_2\}$, then $q\in[\alpha_k,\alpha_{k+1})$ for some $k\geq 3$, it follows from (4.1) and Proposition 4.4 that to determine whether $q\in\mathcal{B}_{\aleph_0}$ we only have to verify whether $U_{k,q}$ contains a q null infinite point. This statement makes determining whether $q\in\mathcal{B}_{\aleph_0}$ a reasonably straightforward computation as we only have finitely many cases to consider. Proposition 4.4 also yields the following result.

Theorem 4.5. *$\mathcal{B}_{\aleph_0}\cap([\frac{1+\sqrt{5}}{2},q_f)\setminus\{q_2\})$ is a discrete set.*

Proof. As $\mathcal{B}_{\aleph_0}\cap[\frac{1+\sqrt{5}}{2},q_{\aleph_0})=\{\frac{1+\sqrt{5}}{2}\}$ it suffices to show that $\mathcal{B}_{\aleph_0}\cap([q_{\aleph_0},q_f)\setminus\{q_2\})$ is a discrete set. For each $q^*\in\mathcal{B}_{\aleph_0}\cap([q_{\aleph_0},q_f)\setminus\{q_2\})$, we shall construct an open interval I_{q^*} satisfying: $q^*\in I_{q^*}$ and $(I_{q^*}\setminus\{q^*\})\cap\mathcal{B}_{\aleph_0}=\emptyset$, this will imply $\mathcal{B}_{\aleph_0}\cap([q_{\aleph_0},q_f)\setminus\{q_2\})$ is a discrete set.

Suppose $q^*\in\mathcal{B}_{\aleph_0}\cap([q_{\aleph_0},q_f)\setminus\{q_2\})$, then $q^*\in[\alpha_k,\alpha_{k+1})$ for some $k\geq 3$ and $P_{q^*}=U_{k,q^*}$. By a continuity argument there exists an open interval I_1 satisfying: $q^*\in I_1$ and $P_q\subseteq U_{k,q}$, for all $q\in I_1$. We let

$$\Sigma_{null}=\Big\{(\epsilon_i)_{i=1}^\infty\in\{1(0)^j(01)^\infty,0(1)^j(10)^\infty|1\leq j\leq k\}\Big|((\epsilon_i)_{i=1}^\infty)_{q^*}\text{ is a }q^*\text{ null infinite point }\Big\},$$

and

$$\Sigma_{bif}=\Big\{1(0)^j(01)^\infty,0(1)^j(10)^\infty|1\leq j\leq k\Big\}\setminus\Sigma_{null}.$$

For ease of exposition we let $\Sigma_{null}=\{(\epsilon_i^m)_{i=1}^\infty\}_{m=1}^M$ and $\Sigma_{bif}=\{(\epsilon_i^n)_{i=1}^\infty\}_{n=1}^N$. We will show that for each $(\epsilon_i^m)_{i=1}^\infty\in\Sigma_{null}$ there exists a finite sequence of transformations a and an open interval I_m such that, $q^*\in I_m$ and for each $q\in I_m\setminus\{q^*\}$ we have $T_{q,i}(a(((\epsilon_i^m)_{i=1}^\infty)_q))\notin U_q$, for $i\in\{0,1\}$. Similarly, we will show that for each $(\epsilon_i^n)_{i=1}^\infty\in\Sigma_{bif}$ there exists a finite sequence of transformations a and an open interval I_n such that, $q^*\in I_n$ and for all $q\in I_n$ we have $T_{q,i}(a(((\epsilon_i^n)_{i=1}^\infty)_q))\notin U_q$, for $i\in\{0,1\}$. Taking

$$I_{q^*}=I_1\cap(\bigcap_{m=1}^M I_m)\cap(\bigcap_{n=1}^N I_n),$$

it will follow from our construction that if $q\in I_{q^*}\setminus\{q^*\}$ then every element of P_q cannot be a q null infinite point, which by Proposition 4.4 implies $(I_{q^*}\setminus\{q^*\})\cap\mathcal{B}_{\aleph_0}=\emptyset$ and $\mathcal{B}_{\aleph_0}\cap([q_{\aleph_0},q_f)\setminus\{q_2\})$ is a discrete set.

To begin with let us consider $(\epsilon_i^m)_{i=1}^\infty\in\Sigma_{null}$, by an application of Lemma 3.1 there exists a finite sequence of transformations a such that $a(((\epsilon_i^m)_{i=1}^\infty)_{q^*})\in P_{q^*}$, $T_{q^*,i}(a(((\epsilon_i^m)_{i=1}^\infty)_{q^*}))\notin U_{q^*}$ and $T_{q^*,1-i}(a(((\epsilon_i^m)_{i=1}^\infty)_{q^*}))=((\delta_i)_{i=1}^\infty)_{q^*}\in U_{q^*}$, for some $i\in\{0,1\}$. By continuity we can assert that there exists an open interval I'_m satisfying: $q^*\in I'_m$, $a(((\epsilon_i^m)_{i=1}^\infty)_q\in S_q$ and $T_{q,i}(a(((\epsilon_i^m)_{i=1}^\infty)_q))\notin U_q$ for all $q\in I'_m$. Since $q^*\in[\alpha_k,\alpha_{k+1})$ we have

$$(\delta_i)_{i=1}^\infty\in\{(0)^j(01)^\infty,(1)^j(10)^\infty|1\leq j\leq k\},$$

from which it follows that satisfying $T_{q,1-i}(a(((\epsilon_i^m)_{i=1}^{\infty})_q)) = ((\delta_i)_{i=1}^{\infty})_q$ is equivalent to satisfying $f(q) = 0$ for some nontrivial polynomial $f(q) \in \mathbb{Z}[q]$. Clearly $f(q^*) = 0$, however, since $f(q) = 0$ has a finite number of solutions there exists an open interval I''_m satisfying: $q^* \in I''_m$, $a(((\epsilon_i^m)_{i=1}^{\infty})_q) \in S_q$ and $f(q) \neq 0$ for all $q \in I''_m \setminus \{q^*\}$. Moreover, by continuity we may assume that I''_m is sufficiently small such that $T_{q,1-i}(a(((\epsilon_i^m)_{i=1}^{\infty})_q)) \notin U_q \setminus \{((\delta_i)_{i=1}^{\infty})_q\}$, for all $q \in I''_m$. Taking $I_m = I'_m \cap I''_m$, we may conclude that for all $q \in I_m \setminus \{q^*\}$ we have $T_{q,i}(a(((\epsilon_i^m)_{i=1}^{\infty})_q)) \notin U_q$, for $i \in \{0,1\}$.

It remains to consider $(\epsilon_i^n)_{i=1}^{\infty} \in \Sigma_{bif}$, as $((\epsilon_i^n)_{i=1}^{\infty})_{q^*}$ is not a q^* null infinite point there exists a finite sequence of transformations a such that $a(((\epsilon_i^n)_{i=1}^{\infty})_{q^*}) \in S_{q^*}$ and $T_{q^*,i}(a(((\epsilon_i^n)_{i=1}^{\infty})_{q^*})) \notin U_{q^*}$, for $i \in \{0,1\}$. By continuity it follows that there exists an open interval I_n such that, $q^* \in I_n$, $a(((\epsilon_i^n)_{i=1}^{\infty})_q) \in S_q$ and $T_{q,i}(a(((\epsilon_i^n)_{i=1}^{\infty})_q)) \notin U_q$, for $i \in \{0,1\}$, for all $q \in I_n$. □

The discreteness of $\mathcal{B}_{\aleph_0} \cap ([\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\})$ leads to some interesting questions that we state in the next section.

5. Open questions

To conclude we shall pose some open questions.

- In [13] Sidorov constructs a sequence $(q_k)_{k=1}^{\infty}$ such that, $q_k \in \mathcal{B}_{\aleph_0}$ for all $k \geq 1$ and $q_k \searrow q_2$. As stated at the start of Section 4 $\alpha_k \nearrow q_f$, as such the following question seem natural. Suppose $q \in \mathcal{B}_m$ for some $m \geq 2$, is q a limit point of $\mathcal{B}_{\aleph_0}$? Moreover is the converse true, that is, if q is a limit point of $\mathcal{B}_{\aleph_0}$ does that imply $q \in \mathcal{B}_m$ for some $m \geq 2$? The discreteness of $\mathcal{B}_{\aleph_0} \cap ([\frac{1+\sqrt{5}}{2}, q_f) \setminus \{q_2\})$ guaranteed by Theorem 4.5 might seem to suggest so.
- Is $\mathcal{B}_{\aleph_0}$ closed?
- Is $q_2 \in \mathcal{B}_{\aleph_0}$? If $q_2 \in \mathcal{B}_{\aleph_0}$ then it would be a consequence of our above remarks, Theorem 4.5 and [3, Proposition 2.1] that $\mathcal{B}_{\aleph_0} \cap [\frac{1+\sqrt{5}}{2}, q_f]$ is a closed set.
- Given $q \in \mathcal{B}_{\aleph_0}$, what is the topology of the set of q null infinite points?

Acknowledgments. The author is grateful to Nikita Sidorov for his useful comments and insight.

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