A high-wavenumber boundary-element method for an acoustic scattering problem

A High Wavenumber Boundary Element Method for an Acoustic Scattering Problem

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In this paper we show stability and convergence for a novel Galerkin boundary element method approach to the impedance boundary value problem for the Helmholtz equation in a half-plane with piecewise constant boundary data. This problem models, for example, outdoor sound propagation over inhomogeneous flat terrain. To achieve a good approximation with a relatively low number of degrees of freedom we employ a graded mesh with smaller elements adjacent to discontinuities in impedance, and a special set of basis functions for the Galerkin method so that, on each element, the approximation space consists of polynomials (of degree ν) multiplied by traces of plane waves on the boundary. In the case where the impedance is constant outside an interval $[a, b]$, which only requires the discretization of $[a, b]$, we show theoretically and experimentally that the L_2 error in computing the acoustic field on $[a, b]$ is $\mathcal{O}(\log^{\nu+3/2} |k(b-a)| M^{-(\nu+1)})$, where M is the number of degrees of freedom and k is the wavenumber. This indicates that the proposed method is especially commendable for large intervals or a high wavenumber. In a final section we sketch how the same methodology extends to more general scattering problems.

Keywords: high frequency scattering, Galerkin boundary element method

1. Introduction

In this paper we consider the numerical solution of the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{1.1}$$

in the upper half-plane $U := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$, with impedance boundary condition

$$\frac{\partial u}{\partial x_2} + \mathrm{i}k\beta u = f \tag{1.2}$$

on $\Gamma := \{(x_1, 0) : x_1 \in \mathbb{R}\}$, where $k > 0$ (the wavenumber) is some arbitrary positive constant and $\beta, f \in L_\infty(\mathbb{R})$, the set of complex-valued functions on $\mathbb{R}$ which are bounded and measurable.

This boundary value problem can arise when modelling the acoustic scattering of an incident wave by a planar surface with spatially varying acoustical surface impedance (Chandler-Wilde & Hothersall 1985; Habault 1985). In the case in which there is no variation in the acoustical properties of the surface or the incident

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field in some fixed direction parallel to the surface, the problem is effectively two-dimensional. Adopting Cartesian coordinates $0x_1x_2x_3$, let this direction be that of the x_3-axis and the surface be the plane $x_2 = 0$. Under the further assumption that the incident wave and scattered fields are time harmonic, the acoustic pressure at time t, position (x_1, x_2, x_3) is given by $\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega t} u^t(x))$, where $x = (x_1, x_2) \in \overline{U}$, $\omega = 2\pi\mu$ and μ is the frequency of the incident wave.

The total acoustic field $u^t \in C(\overline{U}) \cap C^2(U)$ satisfies (1.1) and (1.2), with $f \equiv 0$. The wavenumber $k = \omega/c$, with c being the speed of sound in U. In the case of an incident plane wave, the incident field u^i is given by

$$u^i(x) = \mathrm{e}^{\mathrm{i}kx\cdot d} = \mathrm{e}^{\mathrm{i}k(x_1 \sin\theta - x_2 \cos\theta)}, \tag{1.3}$$

where $d = (d_1, d_2) = (\sin\theta, -\cos\theta)$ and $\theta \in (-\pi/2, \pi/2)$ is the angle of incidence. The reflected or scattered part of the wave field is $u \in C(\overline{U}) \cap C^2(U)$, defined by $u = u^t - u^i$. The scattered field also satisfies (1.1) and (1.2) with $f := -\partial u^i/\partial x_2 - \mathrm{i}k\beta u^i$.

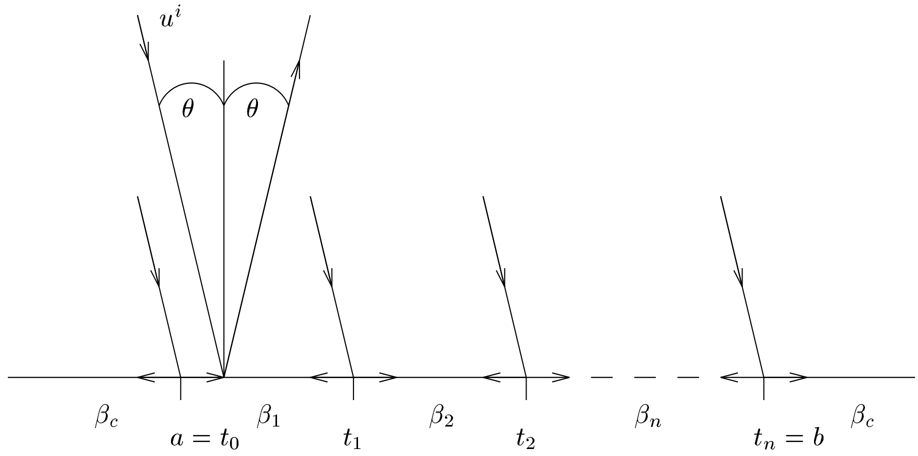

Figure 1. Acoustic scattering by an impedance boundary. Typical incident and reflected rays are shown as well as some of the rays arising from diffraction at impedance discontinuities.

The function β in (1.2) is the relative admittance of the surface and, in outdoor sound propagation, depends on the frequency and the ground properties. Usually the case when β is piecewise constant is of interest (Habault 1985; Hothersall & Chandler-Wilde 1987) with β taking a different value for each ground surface type (grassland, forest floor, road pavement, etc. (Attenborough 1985)). If the ground surface is to absorb rather than emit energy, the condition $\mathrm{Re}(\beta) \geq 0$ must be satisfied. We assume throughout that $\mathrm{Re}(\beta) \geq \epsilon$, for some $\epsilon > 0$, which in physical terms ensures that the boundary is everywhere energy absorbing.

The impedance boundary value problem in a half-plane is also of interest as a model of the scattering of an incident acoustic or electromagnetic wave by an infinite rough surface, in which case (1.1) holds in a region $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > g(x_1)\}$, for some bounded and at least Lipschitz continuous function g; see e.g. Boulanger et al. (1998), Poirier et al. (2000).

In this paper we are concerned with solving (1.1)–(1.2) numerically with particular emphasis on the case in which k may be large. This corresponds to the high frequency/low wavelength case (the wavelength is $\lambda = c/\mu = 2\pi/k$), and presents a number of numerical difficulties.

Standard numerical schemes for solving scattering problems become prohibitively expensive as $k \to \infty$. For standard boundary element schemes, where the approximation space consists of piecewise polynomials, the rule of thumb in the engineering literature (see e.g. Perrey-Debain et al. 2003a, b) seems to be that between five to ten elements are required per wavelength, in order to achieve reasonable accuracy. If the wavelength is small compared to the size of the obstacle then large dense systems of equations have to be solved. Much successful effort has been devoted to solving these large systems quickly, using preconditioned iterative methods (e.g. Amini & Maines 1998; Christiansen & Nedelec 2000) combined with fast multipole or fast Fourier transform (FFT) based methods (e.g. Darve 2000; Bruno & Kunyansky 2001a, b; Donepudi et al. 2003) to carry out the matrix-vector multiplications efficiently. These methods considerably decrease computing costs allowing the solution of higher frequency problems than would be possible using more standard methods. However they still become impractical when k becomes very large, since the size of the linear systems to be solved grows at least linearly with respect to k.

For the specific problem (1.1)–(1.2) a standard boundary element method using piecewise constant collocation is discussed in Chandler-Wilde et al. (2002). An approximate two-grid iterative solver is proposed with the matrix-vector multiplications required carried out by the FFT if a uniform grid is used. A rigorous numerical analysis shows that, if β is constant outside $[a, b]$, as in figure 1, then the solution u on $[a, b]$ can be computed with error $\mathcal{O}(kh|\log kh|\|f\|_\infty)$, for kh sufficiently small, in $\mathcal{O}(M \log M)$ operations, where h is the grid spacing and $M = (b-a)/h$ is the number of degrees of freedom. This appears to be the first paper to present a method for a scattering problem in more than one dimension in which the dependance of the error estimates on k is established. The convergence rate is modest however, and the method is still hindered by the constraint that the linear system to be solved grows linearly in size with increasing k.

A recent approach in the literature for higher frequencies is to use either a finite element or a boundary element method in which the approximation space is designed specifically to take advantage of the behaviour of the solution of (1.1) for large k. Rather than using piecewise polynomials, one can enrich the approximation space with plane wave or Bessel function solutions of (1.1), in order to represent efficiently the highly oscillatory solution. This idea is applied in Monk & Wang (1999) and Giladi & Keller (2001) in a finite element framework, and Perrey-Debain et al. (2003a, b) in a boundary element framework. Other related methods are the microlocal discretisation approach (de La Bourdonnaye 1994; Darrigrand 2002; Abboud et al. 1994) the ultra weak variational formulation (Cessenat & Després 1998) and the partition of unity method (Melenk & Babuška 1996). Although promising numerical results are reported for all these methods, the only method for which an error estimate exists specifying the dependence on the wavenumber k is a microlocal discretisation approach for plane wave scattering by smooth convex obstacles in which a standard Galerkin boundary element method is applied to the ratio of the scattered field to the incident field (Abboud et al. 1994). The error estimate in this case is that the relative error in the best approximation from a boundary element space of piecewise polynomials of degree $\leq \nu$ is $\mathcal{O}(h^\nu) + \mathcal{O}((hk^{1/3})^{\nu+1})$. However, the analysis does not guarantee that the Galerkin method solution is close to this best approximation.

In this paper we present a new high frequency boundary element method for (1.1)–(1.2). We consider the case in which β is piecewise constant, and constant outside a finite interval $[a, b]$. To achieve good approximations with a relatively low number of degrees of freedom, we obtain representations for the solution on the boundary in the spirit of the geometrical theory of diffraction (Keller 1962). These representations can be viewed as explicitly summing the reflected and diffracted ray path contributions to the field on the boundary shown in figure 1. Precisely, we show that after subtracting off the leading order behaviour as $k \to \infty$ on each interval the remaining scattered wave can be expressed as the product of the oscillatory functions $\mathrm{e}^{\pm \mathrm{i}kx_1}$ and non-oscillatory functions which we denote as $f_j^{\pm}$. Rigorous bounds are established on the derivatives of the non-oscillatory functions $f_j^{\pm}$ both adjacent to and away from discontinuities in impedance.

In §3 we present our Galerkin method for solving the integral equation. A graded mesh is employed with elements very large compared to the wavelength away from discontinuities in β, in order to take advantage of the smooth behaviour of $f_j^{\pm}$ away from impedance discontinuities as deduced in §2. We use a special set of basis functions so that on each element the approximation space consists of polynomials (of degree ν) multiplied by $\mathrm{e}^{\pm \mathrm{i}kx_1}$, so that we obtain a piecewise polynomial representation of the non-oscillatory functions $f_j^{\pm}$.

In §4 we present an error analysis for the method. In our main result (theorem 4.3) we show that the error in computing an approximation to $u^t|_\Gamma$ on $[a, b]$ in the L_2 norm is $\mathcal{O}(\log^{\nu+3/2}|k(b-a)|M^{-(\nu+1)})$ where M is the number of degrees of freedom. This estimate implies that, to maintain accuracy, M has to increase only approximately in proportion to $\log k(b-a)$ as $k(b-a)$ increases, as compared to in proportion to $k(b-a)$ for standard boundary element methods, e.g. that analysed in Chandler-Wilde et al. (2002). In §5 we discuss the practical implementation of our approach, and we present some numerical results demonstrating that the theoretically predicted behaviour is achieved.

Finally in §6 we summarise our results and discuss the extent to which the numerical schemes and analysis presented in this paper are applicable to more general scattering problems. As an initial step in this direction we briefly consider the problem of 2D scattering by a convex polygon, using on each side of the polygon the same basis functions and boundary element mesh as proposed in §3. This problem will be considered in detail in Chandler-Wilde & Langdon (2003, in preparation).

2. Integral equation formulation and regularity

We begin by stating the problem we wish to solve precisely and reformulating it as an integral equation. For simplicity of exposition, we restrict our attention to the case of plane wave incidence, so that u^i is given by (1.3). The scattered field satisfies the impedance boundary condition (1.2) with f at the point $(x_1, 0)$ on Γ explicitly given by

$$f(x_1) = \mathrm{i}k\mathrm{e}^{\mathrm{i}kx_1 \sin\theta}(\cos\theta - \beta(x_1)), \quad x_1 \in \mathbb{R}. \tag{2.1}$$

We also assume throughout that β is piecewise constant, and constant outside some finite interval $[a, b]$. Thus, for some real numbers $a = t_0 < t_1 < \cdots < t_n = b$, the relative surface admittance at $(x_1, 0)$ on Γ is given by

$$\beta(x_1) = \begin{cases} \beta_j, & x_1 \in (t_{j-1}, t_j], \quad j = 1, \ldots, n, \\ \beta_c, & x_1 \in \mathbb{R} \backslash (t_0, t_n], \end{cases} \tag{2.2}$$

(see figure 1) where we assume throughout that

$$\mathrm{Re}\,\beta_c > 0, \quad \mathrm{Re}\,\beta_j > 0, \quad j = 1, \ldots, n. \tag{2.3}$$

For $H \geq 0$, let $U_H := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 > H\}$ and $\Gamma_H := \{(x_1, H) : x_1 \in \mathbb{R}\}$. To determine u uniquely we impose the radiation condition proposed in Chandler-Wilde (1997) that, for some $H > 0$, u can be written in the half-plane U_H as the double layer potential

$$u(x) = \int_{\Gamma_H} \frac{\partial H_0^{(1)}(k|x-y|)}{\partial y_2} \phi(y)\, \mathrm{d}s(y), \quad x \in U_H, \tag{2.4}$$

for some density $\phi \in L_\infty(\Gamma_H)$, where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. The boundary value problem for the scattered field u that we wish to solve is thus as follows:

Boundary Value Problem. *Given $k > 0$ (the wavenumber), $\theta \in (-\pi/2, \pi/2)$ (the angle of incidence) and β given by (2.2), find $u \in \mathcal{C}(\overline{U}) \cap \mathcal{C}^2(U)$ such that:*

(i) u is bounded in the horizontal strip $U \backslash U_H$ for every $H > 0$;

(ii) u satisfies the Helmholtz equation (1.1) in U;

(iii) u satisfies the impedance boundary condition (1.2) on Γ (in the weak sense explained in Chandler-Wilde (1997)), with $f \in L_\infty(\Gamma)$ given by (2.1);

(iv) u satisfies the radiation condition (2.4), for some $H > 0$ and $\phi \in L_\infty(\Gamma_H)$.

For $\beta^* \in \mathbb{C}$ with $\mathrm{Re}\,\beta^* > 0$ let $G_{\beta^*}(x, y)$ denote the Green's function for the above problem which satisfies (1.2), with $\beta \equiv \beta^*$ and $f \equiv 0$, and the standard Sommerfeld radiation and boundedness conditions. I.e. $G_{\beta^*}(x, y)$ is the Green's function for constant relative surface admittance β^*. Then explicitly (Chandler-Wilde & Hothersall 1995*a*),

$$G_{\beta^*}(x, y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) + \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y'|) + P_{\beta^*}(k(x-y')), \quad x, y \in \overline{U}, x \neq y, \tag{2.5}$$

where $y = (y_1, y_2)$, $y' = (y_1, -y_2)$ and, for $z = (z_1, z_2)$, $z_2 \geq 0$,

$$P_{\beta^*}(z) := -\frac{\mathrm{i}\beta^*}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}(z_2(1-s^2)^{1/2} - z_1 s))}{(1-s^2)^{1/2}((1-s^2)^{1/2} + \beta^*)}\, \mathrm{d}s, \quad 0 \leq \arg((1-s^2)^{1/2}) \leq \pi/2. \tag{2.6}$$

In Chandler-Wilde & Hothersall (1995*a*) an alternative representation for P_{β^*} is also given, that $P_{\beta^*}(z) := P_{\beta^*}^I(z) + P_{\beta^*}^P(z)$, where

$$P_{\beta^*}^I(z) := \frac{\beta^* \mathrm{e}^{\mathrm{i}|z|}}{\pi} \int_0^\infty \frac{t^{-1/2} \mathrm{e}^{-|z|t} (\beta^* + \gamma(1 + \mathrm{i}t))}{(t - 2\mathrm{i})^{1/2} (t^2 - 2\mathrm{i}(1 + \beta^*\gamma)t - (\beta^* + \gamma)^2)}\, \mathrm{d}t, \tag{2.7}$$

and $P^P_{\beta^*}(z) = C_{\beta^*} e^{\mathrm{i}|z|(1-a_+)}$, with

$$C_{\beta^*} := \begin{cases} \frac{\beta^*}{(1-\beta^{*2})^{1/2}}, & \operatorname{Im}\beta^* < 0, \operatorname{Re}(a_+) < 0, \\ \frac{\beta^*}{2(1-\beta^{*2})^{1/2}}, & \operatorname{Im}\beta^* < 0, \operatorname{Re}(a_+) = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.8}$$

$a_\pm := 1+\beta^*\gamma \mp (1-\beta^{*2})^{\frac{1}{2}}(1-\gamma^2)^{\frac{1}{2}}$, $\operatorname{Re}\{(1-\beta^{*2})^{1/2}\} \geq 0$, and $\gamma = z_2/|z|$. Note that (Chandler-Wilde & Hothersall 1995*a*) $P_{\beta^*} \in C(\overline{U}) \cap C^\infty(\overline{U}\setminus\{0\})$ and P_{β^*} satisfies the Sommerfeld radiation and boundedness conditions in U (for wavenumber $k = 1$).

Suppose that, for some $\epsilon > 0$, $\operatorname{Re}\beta^* \geq \epsilon$, $\quad |\beta^*| \leq \epsilon^{-1}$. From (2.6) we see that $|P_{\beta^*}(z)| \leq C_\epsilon$ for $|z| \leq 1$, where the constant C_ϵ depends only on ϵ. Thus, and from properties of the Hankel function for small argument, we deduce that

$$|G_{\beta^*}(x,y)| \leq C_\epsilon(1 - \log(k|x-y|)), \quad x \in \overline{U},\ y \in \Gamma,\ 0 < k|x-y| \leq 1, \tag{2.9}$$

where the constant C_ϵ depends only on ϵ. From (2.9), the asymptotic behaviour of the Hankel function for large argument and the uniform asymptotic expansions for P_{β^*} for large argument in Chandler-Wilde & Hothersall (1995*b*), it follows that

$$|G_{\beta^*}(x,y)| \leq \frac{C_\epsilon(1+kx_2)}{(k|x-y|)^{\frac{3}{2}}}, \quad x \in \overline{U},\ y \in \Gamma,\ x \neq y, \tag{2.10}$$

where, again, the value of $C_\epsilon > 0$ depends only on ϵ.

The following result, a reformulation of the above boundary value problem as a boundary integral equation, is shown for $\beta^* = 1$ in Chandler-Wilde (1997). The extension to arbitrary β^* with $\operatorname{Re}\beta^* > 0$ is straightforward, using the properties of P_{β^*} and G_{β^*} we have just stated.

Theorem 2.1. *If u satisfies the above boundary value problem and* $\operatorname{Re}\beta^* > 0$ *then*

$$u(x) = \int_\Gamma G_{\beta^*}(x,y)(\mathrm{i}k(\beta(y)-\beta^*)u(y) - f(y))\,\mathrm{d}s(y), \quad x \in \overline{U}. \tag{2.11}$$

Conversely, if $u|_\Gamma \in BC(\Gamma)$ (the space of bounded and continuous functions on Γ) and u satisfies (2.11), for some β^* with $\operatorname{Re}\beta^* > 0$*, then u satisfies the above boundary value problem.*

Note that the estimates (2.9) and (2.10) guarantee that the integral in (2.11) is well-defined. The following result holds regarding the solvability of (2.11).

Theorem 2.2. *If* $\operatorname{Re}\beta^* > 0$*, (2.11) has exactly one solution with $u|_\Gamma \in BC(\Gamma)$.*

Proof. This result is shown for $\beta^* = 1$ in Chandler-Wilde (1997, theorem 4.17). That this result holds for all β^* with $\operatorname{Re}\beta^* > 0$ follows from theorem 2.1. □

In view of theorem 2.1, theorem 2.2 has the following corollary.

Corollary 2.3. *The boundary value problem has exactly one solution.*

Now, denote the solution of the above boundary value problem in the special case $\beta \equiv \beta^*$ by u_{β^*}. Then, by theorem 2.1,

$$u_{\beta^*}(x) = -\int_\Gamma G_{\beta^*}(x,y) f^*(y)\,\mathrm{d}s(y), \quad x \in \overline{U}, \tag{2.12}$$

where f^* is given by (2.1) with $\beta(s) \equiv \beta^*$. Using the identity $f^* - f = -\mathrm{i}k(\beta^* - \beta)u^i$, subtracting (2.12) from (2.11), and adding the incident field u^i to both sides,

$$u^t(x) = u^t_{\beta^*}(x) + \mathrm{i}k \int_\Gamma G_{\beta^*}(x,y)(\beta(y) - \beta^*)u^t(y)\,\mathrm{d}s(y), \quad x \in \overline{U}, \tag{2.13}$$

where $u^t_{\beta^*} := u^i + u_{\beta^*}$ is the corresponding total field when $\beta(s) \equiv \beta^*$.

In fact u_{β^*}, given by (2.12), is just the reflected plane wave

$$u_{\beta^*}(x) = R_{\beta^*}(\theta)\mathrm{e}^{\mathrm{i}kx\cdot d'}, \quad x \in \overline{U}, \tag{2.14}$$

where $d' = (d_1, -d_2) = (\sin\theta, \cos\theta)$ and $R_{\beta^*}(\theta)$ is the reflection coefficient $R_{\beta^*}(\theta) = (\cos\theta - \beta^*)/(\cos\theta + \beta^*)$. To see this we just check that (2.14) satisfies all the conditions of the boundary value problem, which it does (the upward propagating radiation condition holds as discussed in Chandler-Wilde (1995)). Thus the total field for constant relative surface admittance β^* is

$$u^t_{\beta^*}(x) = \mathrm{e}^{\mathrm{i}kx\cdot d} + R_{\beta^*}(\theta)\mathrm{e}^{\mathrm{i}kx\cdot d'}, \quad x \in \overline{U}. \tag{2.15}$$

Equation (2.13), restricted to Γ, is a boundary integral equation for $u^t|_\Gamma$ and it is the main concern in the remainder of the paper to solve this equation numerically in the case when $\beta^* = \beta_c$. To make explicit the dependence on the wavenumber k in the results we obtain it is useful to introduce new, dimensionless variables. Thus, define $\phi(s) := u^t((s/k,0))$, $\psi_{\beta^*}(s) := u^t_{\beta^*}((s/k,0))$, and $\kappa_{\beta^*}(s) := G_{\beta^*}((s/k,0),(0,0))$, $s \in \mathbb{R}$. Then (2.13) restricted to Γ is the following second kind integral equation for ϕ:

$$\phi(s) = \psi_{\beta^*}(s) + \mathrm{i}\int_{-\infty}^{\infty} \kappa_{\beta^*}(s-t)(\beta(t/k) - \beta^*)\phi(t)\,\mathrm{d}t, \quad s \in \mathbb{R}. \tag{2.16}$$

From (2.15) and (2.5),

$$\begin{aligned}
\psi_{\beta^*}(s) &= (1 + R_{\beta^*}(\theta))\mathrm{e}^{\mathrm{i}s\sin\theta}, && (2.17)\\
\kappa_{\beta^*}(s) &= \frac{\mathrm{i}}{2}H_0^{(1)}(|s|) + P_{\beta^*}((s,0)), \quad s \in \mathbb{R}, && (2.18)
\end{aligned}$$

so the only dependence on k in the known terms in (2.16) is in the impedance function $\beta(t/k)$.

Using (2.5)–(2.8) (noting that in this case $\gamma = 0$) and the identity (Oberhettinger & Badii 1973, [12.31])

$$H_0^{(1)}(s) = \frac{-2\mathrm{i}}{\pi}\int_0^\infty \frac{\mathrm{e}^{(\mathrm{i}-t)s}}{t^{\frac{1}{2}}(t-2i)^{\frac{1}{2}}}\,\mathrm{d}t, \quad s > 0,$$

we see that an explicit formula for $\kappa_{\beta^*}(s)$ is

$$\begin{aligned}
\kappa_{\beta^*}(s) &= \frac{\mathrm{i}}{2}H_0^{(1)}(|s|) + \frac{{\beta^*}^2\mathrm{e}^{\mathrm{i}|s|}}{\pi}\int_0^\infty \frac{t^{-\frac{1}{2}}(t-2\mathrm{i})^{-\frac{1}{2}}\mathrm{e}^{-|s|t}}{(t^2 - 2\mathrm{i}t - {\beta^*}^2)}\,\mathrm{d}t + P^P_{\beta^*}((s,0))\\
&= \mathrm{e}^{\mathrm{i}|s|}\check{\kappa}_{\beta^*}(s), \quad s \in \mathbb{R}\backslash\{0\}, && (2.19)
\end{aligned}$$

where

$$\check{\kappa}_{\beta^*}(s) := \frac{1}{\pi}\int_0^\infty \frac{r^{\frac{1}{2}}(r-2\mathrm{i})^{\frac{1}{2}}}{r^2-2\mathrm{i}r-{\beta^*}^2}\mathrm{e}^{-r|s|}\,\mathrm{d}r + \check{P}^P_{\beta^*}(s), \quad s\in\mathbb{R}\backslash\{0\}, \tag{2.20}$$

with $\check{P}^P_{\beta^*}(s) := C_{\beta_*}\mathrm{e}^{-\mathrm{i}|s|\hat{a}_+}$, C_{β^*} given by (2.8) and $\hat{a}_\pm = 1\mp(1-{\beta^*}^2)^{\frac{1}{2}}$. We shall see shortly that the oscillating part of $\kappa_{\beta^*}(s)$ is contained in the factor $\mathrm{e}^{\mathrm{i}|s|}$ in (2.19), $\check{\kappa}_{\beta^*}(s)$ becoming increasingly smooth as $s\to\pm\infty$.

In view of (2.2), if we set $\beta^*=\beta_c$ in (2.16) the interval of integration reduces to the finite interval $[\tilde{a},\tilde{b}]$, where $\tilde{a} := ka = kt_0$, $\tilde{b} := kb = kt_n$. Explicitly, (2.16) becomes

$$\phi(s) = \psi_{\beta_c}(s) + \mathrm{i}\int_{\tilde{a}}^{\tilde{b}} \kappa_{\beta_c}(s-t)(\beta(t/k)-\beta_c)\phi(t)\,\mathrm{d}t, \quad s\in\mathbb{R}, \tag{2.21}$$

with ψ_{β_c} and κ_{β_c} given by (2.17) and (2.18) respectively with $\beta^*=\beta_c$. We write (2.21) in operator form as

$$\phi = \psi_{\beta_c} + K^{\beta_c}_{\beta}\phi, \tag{2.22}$$

where

$$K^{\beta_c}_{\beta}\chi(s) := \mathrm{i}\int_{\tilde{a}}^{\tilde{b}} \kappa_{\beta_c}(s-t)(\beta(t/k)-\beta_c)\chi(t)\,\mathrm{d}t.$$

As discussed in the introduction, our numerical scheme for solving (2.21) is based on a consideration of the contribution of the reflected and diffracted ray paths in the spirit of the geometrical theory of diffraction and as predicted by an exact solution of the canonical problem of a plane wave incident on a boundary with a single impedance discontinuity (Heins & Feshbach 1954). The relevant ray paths incident on the boundary Γ are depicted in figure 1. In particular, to leading order as $k\to\infty$, on the interval (t_{j-1},t_j) the geometrical theory of diffraction predicts that the total field $\phi\approx\psi_{\beta_j}$, the total field there would be if the whole boundary had the admittance β_j of the interval (t_{j-1},t_j), given explicitly by (2.17) with $\beta^*=\beta_j$. Thus, for $s\neq\tilde{t}_j := kt_j$, $j=0,\ldots,n$, the geometrical theory of diffraction predicts that $\phi(s)\to\Psi(s)$ as $k\to\infty$, where

$$\Psi(s) := \begin{cases} \psi_{\beta_j}(s), & s\in(\tilde{t}_{j-1},\tilde{t}_j], \quad j=1,\ldots,n, \\ \psi_{\beta_c}(s), & s\in\mathbb{R}\backslash(\tilde{t}_0,\tilde{t}_n]. \end{cases}$$

In our numerical scheme we compute the difference between ϕ and Ψ, i.e.

$$\Phi(s) := \phi(s) - \Psi(s), \quad s\in\mathbb{R},$$

which can be viewed as the correction to the leading order field due to scattering from impedance discontinuities. From (2.22),

$$\Phi = \Psi^{\beta_c}_{\beta} + K^{\beta_c}_{\beta}\Phi, \tag{2.23}$$

where $\Psi^{\beta_c}_{\beta}\in L_\infty(\mathbb{R})$ is given by

$$\Psi^{\beta_c}_{\beta} := \psi_{\beta_c} - \Psi + K^{\beta_c}_{\beta}\Psi.$$

Equation (2.23) will be the integral equation that we solve numerically.

By setting $\beta^* = \beta_j$ in (2.16) we obtain explicit expressions for Φ on each subinterval, namely

$$\Phi(s) = \mathrm{e}^{\mathrm{i}s} f_j^+(s - \tilde{t}_{j-1}) + \mathrm{e}^{-\mathrm{i}s} f_j^-(\tilde{t}_j - s), \quad s \in (\tilde{t}_{j-1}, \tilde{t}_j],\ j = 1, \ldots, n, \tag{2.24}$$

where, for $j = 1, \ldots, n$, $f_j^+, f_j^- \in C[0, \infty)$ are defined by

$$\begin{aligned} f_j^+(r) &:= \int_{-\infty}^{\tilde{t}_{j-1}} \breve{\kappa}_{\beta_j}(r + \tilde{t}_{j-1} - t)\mathrm{e}^{-\mathrm{i}t}\mathrm{i}(\beta(t/k) - \beta_j)\phi(t)\,\mathrm{d}t, \\ f_j^-(r) &:= \int_{\tilde{t}_j}^{\infty} \breve{\kappa}_{\beta_j}(t - \tilde{t}_j + r)\mathrm{e}^{\mathrm{i}t}\mathrm{i}(\beta(t/k) - \beta_j)\phi(t)\,\mathrm{d}t, \end{aligned}$$

for $r \geq 0$, with $\breve{\kappa}_{\beta_j}$ given by (2.20) with $\beta^* = \beta_j$.

In geometrical theory of diffraction terms, our interpretation of the first term in (2.24) is that it is an explicit summation of all the diffracted rays scattered at the discontinuity in impedance at t_{j-1} which travel from left to right along (t_{j-1}, t_j). Similarly, the other term in (2.24) is the contribution to the diffracted field diffracted by the discontinuity at t_j. In the remainder of this section, so as to design an efficient discretisation for Φ, we investigate in detail the behaviour of the integrals $f_j^\pm$. As a tool in this investigation we need first the following result which follows from Chandler-Wilde et al. (2002, theorem 17). We note that our earlier assumption (2.3) ensures that (2.25) below holds for some $\epsilon > 0$.

Theorem 2.4. *For every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$, dependent only on ϵ, such that, provided*

$$\mathrm{Re}\,\beta_c \geq \epsilon, \quad \mathrm{Re}\,\beta_j \geq \epsilon, \quad |\beta_c| \leq \epsilon^{-1}, \quad |\beta_j| \leq \epsilon^{-1}, \quad j = 1, \ldots, n, \tag{2.25}$$

the unique solution of the boundary value problem satisfies

$$|u(x)| \leq C_\epsilon(1 + kx_2)^{1/2}, \quad x \in \overline{U}.$$

We will also require the following bounds on $|\breve{\kappa}_{\beta^*}^{(m)}(s)|$; see Langdon & Chandler-Wilde (2003) for the proof, and note that for $m = 0$ these bounds follow from (2.9) and (2.10).

Lemma 2.5. *Suppose that $\mathrm{Re}\,\beta^* \geq \epsilon$, $|\beta^*| \leq \epsilon^{-1}$ hold for some $\epsilon > 0$. Then, for $m = 0, 1, \ldots$, there exist constants c_m, dependent only on m and ϵ, such that, for $0 \leq s \leq 1$,*

$$|\breve{\kappa}_{\beta^*}^{(m)}(s)| \leq \begin{cases} c_0(1 + |\log s|), & m = 0, \\ c_m s^{-m}, & m \geq 1, \end{cases}$$

and, for $s > 1$,

$$|\breve{\kappa}_{\beta^*}^{(m)}(s)| \leq c_m s^{-\frac{3}{2} - m}.$$

We are now ready to establish the final key result of this section which quantifies the smoothness of the functions $f_j^\pm \in C[0, \infty) \cap C^\infty(0, \infty)$.

Theorem 2.6. *Suppose that (2.25) holds for some $\epsilon > 0$. Then, for $r > 0$, $m = 0, 1, \ldots$, it holds that*

$$\left| f_j^{\pm(m)}(r) \right| \leq c_m F_m(r),$$

where, for $0 < r \leq 1$,

$$F_m(r) := \begin{cases} 1, & m = 0, \\ 1 - \log r, & m = 1, \\ r^{1-m}, & m \geq 2, \end{cases}$$

and, for $r > 1$,

$$F_m(r) := r^{-\frac{1}{2}-m}.$$

The constants c_m depend only on m and ϵ.

Proof. We prove this result for $f_j^+(r)$, the proof for $f_j^-(r)$ follows analogously. Recalling that $\phi(s) = u^t((s/k, 0))$, and using theorem 2.4 to get that $\|\phi\|_\infty \leq 1 + C_\epsilon$, it is straightforward to show that

$$\begin{aligned} |f_j^{+(m)}(r)| &\leq \frac{2}{\epsilon}\|\phi\|_\infty \int_{-\infty}^{\tilde{t}_{j-1}} |\check{\kappa}_{\beta_j}^{(m)}(r + \tilde{t}_{j-1} - t)| \, \mathrm{d}t \\ &\leq C \int_{-\infty}^{\tilde{t}_{j-1}} |\check{\kappa}_{\beta_j}^{(m)}(r + \tilde{t}_{j-1} - t)| \, \mathrm{d}t = C \int_r^\infty |\check{\kappa}_{\beta_j}^{(m)}(s)| \, \mathrm{d}s, \end{aligned}$$

where $C = 2(1 + C_\epsilon)/\epsilon$. Applying Lemma 2.5 the result follows. □

3. The Galerkin method

Our aim now is to design an optimal method to solve (2.23) numerically, supported by a full error analysis. To achieve this we will work in $L_2(\mathbb{R})$, and to that end we begin by introducing the operator $Q : L_\infty(\mathbb{R}) \to L_2(\mathbb{R})$ defined by

$$Q\chi(s) := \begin{cases} \chi(s), & s \in [\tilde{a}, \tilde{b}] = [\tilde{t}_0, \tilde{t}_n], \\ 0, & s \in \mathbb{R} \backslash [\tilde{a}, \tilde{b}]. \end{cases}$$

Writing $\Phi^* := Q\Phi$, and noting that $K_\beta^{\beta_c}\Phi = K_\beta^{\beta_c}\Phi^*$, it follows from (2.23) that

$$\Phi^* = Q\Psi_\beta^{\beta_c} + QK_\beta^{\beta_c}\Phi^*, \tag{3.1}$$

where Φ^* and $Q\Psi_\beta^{\beta_c}$ are both in $L_2(\mathbb{R})$.

Before considering our approximation scheme, we will first establish existence and boundedness for $(I - QK_\beta^{\beta_c})^{-1}$, and so bound Φ^* in terms of $Q\Psi_\beta^{\beta_c}$. It is well known that the convolution operator $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, defined by $K\chi = \kappa * \chi$ for some $\kappa \in L_1(\mathbb{R})$, satisfies $\|K\|_2 \leq \|\hat{\kappa}\|_\infty$, where $\hat{\kappa}$ denotes the Fourier transform of κ. In the case of our integral equation (3.1), we have $QK_\beta^{\beta_c}\Phi^* = Q(\kappa_{\beta_c} * (\mathrm{i}(\beta - \beta_c)\Phi^*))$, and then noting that $\|Q\|_2 = 1$,

$$\|QK_\beta^{\beta_c}\|_2 \leq \|K_\beta^{\beta_c}\|_2 \leq \|\hat{\kappa}_{\beta_c}\|_\infty \|\beta - \beta_c\|_\infty \leq \frac{\|\beta - \beta_c\|_\infty}{\mathrm{Re}\,\beta_c}. \tag{3.2}$$

This last inequality follows by recalling (2.18), using the standard Fourier transform of the Hankel function (e.g. Chandler-Wilde & Hothersall 1995a, equation (12)) and noting that (2.6) takes the form of an inverse Fourier transform, to get $\hat{\kappa}_{\beta_c}(s) = i/((1-s^2)^{1/2}+\beta_c)$, with $0 \leq \arg(\sqrt{1-s^2}) \leq \pi/2$, and hence $\|\hat{\kappa}_{\beta_c}\|_\infty \leq 1/\operatorname{Re}\beta_c$.

Thus $\|QK_\beta^{\beta_c}\|_2 < 1$ if

$$|\beta_j - \beta_c| < \operatorname{Re}\beta_c, \quad j = 1, \ldots, n. \tag{3.3}$$

The existence and boundedness of $(I - QK_\beta^{\beta_c})^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ then follows from standard operator perturbation results (e.g. Kress 1989, theorem 2.8), from which we deduce that (3.1) has the unique solution $\Phi^* = (I - QK_\beta^{\beta_c})^{-1} Q\Psi_\beta^{\beta_c}$, and

$$\|\Phi^*\|_2 \leq \frac{1}{1 - \|QK_\beta^{\beta_c}\|_2} \|Q\Psi_\beta^{\beta_c}\|_2 \leq \frac{\operatorname{Re}\beta_c}{\operatorname{Re}\beta_c - \|\beta - \beta_c\|_\infty} \|Q\Psi_\beta^{\beta_c}\|_2,$$

provided (3.3) holds. If (3.3) does not hold, this stability result still follows, but without an explicit expression for the stability constant. For Arens et al. (2001, corollary 3.12) combined with theorem 2.2 imply that $\|(I - K_\beta^{\beta_c})^{-1}\|_2 \leq C_\epsilon$, provided (2.25) holds, where the constant C_ϵ depends only on ϵ and β_c, and thus

$$\|(I - QK_\beta^{\beta_c})^{-1}\|_2 = \|QK_\beta^{\beta_c}(I - K_\beta^{\beta_c})^{-1} + I\|_2 \leq C'_\epsilon,$$

where C'_ϵ again depends only on ϵ and β_c.

To approximate the solution $\Phi^* = Q\Phi$ of (3.1) we will use a Galerkin method. The novelty of the scheme we propose is that the approximation space chosen is such that, on each interval $(\tilde{t}_{j-1}, \tilde{t}_j)$, we approximate $f_j^+(s - \tilde{t}_{j-1})$ and $f_j^-(\tilde{t}_j - s)$ in (2.24) by conventional piecewise polynomial approximations, rather than approximating Φ itself. This makes sense since, as quantified by theorem 2.6, the functions $f_j^+(s - \tilde{t}_{j-1})$ and $f_j^-(\tilde{t}_j - s)$ are smooth (their higher order derivatives are small) away from $\tilde{t}_{j-1}$ and $\tilde{t}_j$, respectively. To approximate $f_j^+(s - \tilde{t}_{j-1})$ and $f_j^-(\tilde{t}_j - s)$ we use piecewise polynomials of a fixed degree $\nu \geq 0$ on a graded mesh, the mesh grading adapted in an optimal way to the bounds on $f_j^{\pm(m)}$ in theorem 2.6.

To begin, we define a graded mesh on a general interval $[0, A]$, with more mesh points near 0 and less near A. Then, to mesh the interval $[\tilde{t}_{j-1}, \tilde{t}_j]$, we generate a mesh on $[0, A]$ with $A = A_j := (\tilde{t}_j - \tilde{t}_{j-1})/2$, shift the mesh to $[\tilde{t}_{j-1}, (\tilde{t}_{j-1} + \tilde{t}_j)/2]$, and reflect it around $(\tilde{t}_{j-1} + \tilde{t}_j)/2$, thereby creating a symmetric mesh on the whole interval. To mesh the interval $[0, A]$ we pick a positive integer N (the size of N determining the density of the mesh on $[0, A]$) and mesh $[0, A]$ with the mesh $\Lambda_{N,A}$ defined below. As we are primarily concerned with solving the high frequency problem we will assume for simplicity that $A_j := (\tilde{t}_j - \tilde{t}_{j-1})/2 > 1$ for $j = 1, \ldots, n$. We remark however that, in the case $A_j \leq 1$ for some value of j, we would use an appropriate subset of the points y_i given by (3.4) below as our mesh, and this would give us similar approximation properties to those we achieve with $\Lambda_{N,A}$ in the case $A > 1$.

Definition 3.1. *For $A > 1$, $N = 2, 3, \ldots$, the mesh $\Lambda_{N,A} = \{y_0, y_1, \ldots, y_{N+N_A}\}$ consists of the points*

$$y_i = \left(\frac{i}{N}\right)^q, \quad i = 0, \ldots, N, \tag{3.4}$$

where $q = 1 + \frac{2\nu}{3}$, *together with the points*

$$y_{N+j} = A^{j/N_A}, \quad j = 1, \ldots, N_A, \tag{3.5}$$

where $N_A = \lceil N^* \rceil$, *the smallest integer* $\geq N^*$, *and*

$$N^* := -\frac{\log A}{q \log(1 - \frac{1}{N})}. \tag{3.6}$$

The mesh $\Lambda_{N,A}$ is a composite mesh with a polynomial grading on $[0, 1]$ and a geometric grading on $[1, A]$. The definition of N_A ensures a smooth transition between the two parts of the mesh. Precisely, the definition of N^* is such that, in the case $N_A = N^*$, it holds that $y_{N+1}/y_N = y_N/y_{N-1}$, so that y_{N-1} and y_N are points in both the polynomial and the geometric parts of the mesh.

Clearly, there are N subintervals of the mesh on $[0, 1]$ and N_A on $[1, A]$. By the mean value theorem, $-\log(1 - 1/N) = \log 1 - \log(1 - 1/N) = 1/\xi N$, for some $\xi \in (1 - 1/N, 1)$. Since $N \geq 2$ it then holds that $N \log A/2q < N_A < N \log A/q + 1$, so that

$$N + N_A < \left(1 + \frac{1}{N} + \frac{\log A}{q}\right) N < \left(\frac{3}{2} + \frac{\log A}{q}\right) N. \tag{3.7}$$

In earlier work (Ritter 1999) a less optimal mesh, essentially the polynomial mesh developed in Mendes (1988) for a class of Wiener-Hopf integral equations on the half-line, was used for the same impedance boundary value problem. On $[0, 1]$ our mesh (3.4) is identical to that in Mendes (1988), except that here we establish the optimal value for q (the error estimate in Ritter (1999) is proved under the assumption that $q > \nu + 1$). The type of graded mesh we use on $[0, 1]$ has a long history, used first in Rice (1969), and first in an integral equation context in Schneider (1981). In Rice (1969) it is shown, for $0 < \alpha < 1$, that x^α for $x \in [0, 1]$ is optimally approximated in L_2 norm, using piecewise polynomials of degree $\leq \nu$ on the graded mesh (3.4), by taking $q = (3 + 2\nu)/(1 + 2\alpha)$. Setting $\alpha = 1$ recovers the value of q that we propose in definition 3.1. Recalling theorem 2.6, we observe that the mth derivatives of our functions $f_j^\pm$ behave like the limit as $\alpha \to 1$ of the mth derivative of x^α, so that our mesh on $[0, 1]$ appears consistent with that proposed in Rice (1969).

Let

$$\Pi_{A,N,\nu} := \{\sigma : \sigma|_{[y_{j-1}, y_j]} \text{ is a polynomial of degree } \leq \nu, \text{ for } j = 1, \ldots, N + N_A\},$$

and let P_N^* be the orthogonal projection operator from $L_2(0, A)$ to $\Pi_{A,N,\nu}$, so that setting $p = P_N^* f$ minimises $\|f - p\|_{2,(0,A)}$ over all $p \in \Pi_{A,N,\nu}$. The mesh $\Lambda_{N,A}$ is designed to minimise, to a good approximation, the error $\|f - P_N^* f\|_{2,(0,A)}$ in the case when $f \in C^\infty(0, \infty)$ with $|f^{(\nu+1)}(s)| = F_{\nu+1}(s)$, $s > 0$, where $F_{\nu+1}$ is defined as in Theorem 2.6. It achieves this by ensuring that $\|f - P_N^* f\|_{2,(y_{j-1}, y_j)}$ is approximately constant for $j = 1, \ldots, N + N_A$, i.e. by equidistributing the approximation error over the intervals of the mesh. That $\|f - P_N^* f\|_{2,(y_{j-1}, y_j)}$ is approximately constant for the mesh $\Lambda_{N,A}$ in the case $|f^{(\nu+1)}(s)| = F_{\nu+1}(s)$ can be seen by inspecting the proof of the following key error estimate.

Theorem 3.2. *Suppose that $f \in \mathcal{C}^\infty(0,\infty)$ and $|f'(s)| \le F_1(s)$, $|f^{(\nu+1)}(s)| \le F_{\nu+1}(s)$, $s > 0$. Then*

$$\|f - P_N^* f\|_{2,(0,A)} \le C_\nu \frac{1 + \log^{1/2} A}{N^{\nu+1}},$$

where the constant C_ν depends only on ν.

Proof. Throughout the proof let $C_\nu > 0$ denote a constant whose value depends only on ν, not necessarily the same at each occurence. For $A < B$ and $f \in L_2(A,B)$ let $p_{A,B,\nu}$ denote the polynomial of degree $\le \nu$ which is the best approximation to f in the L_2 norm. Clearly

$$\|f - p_{A,B,\nu}\|_{2,(A,B)} \le \|f - f(A)\|_{2,(A,B)}.$$

If $f \in \mathcal{C}^{\nu+1}[A,B]$ we have further that

$$\|f - p_{A,B,\nu}\|_{2,(A,B)} \le C_\nu \|f^{\nu+1}\|_{\infty,[A,B]} (B-A)^{\nu+3/2}$$

(see e.g. Prössdorf & Silbermann 1991, lemma 5.21). Now $\|f - P_N^* f\|^2_{2,(0,A)} = \sum_{j=1}^{N+N_A} e_j$, where $e_j := \|f - p_{y_{j-1},y_j,\nu}\|^2_{2,(y_{j-1},y_j)}$. To bound e_1, as $|f'(s)| \le F_1(s)$ we have

$$e_1 \le \int_0^{y_1} |f(s) - f(0)|^2 \,\mathrm{d}s = \int_0^{y_1} \left| \int_0^s f'(t)\,\mathrm{d}t \right|^2 \mathrm{d}s \le C y_1^3 (1 - \log^2 y_1),$$

where $C > 0$ is an absolute constant. Thus, and since $y_1 = N^{-q}$ and $q = 1 + 2\nu/3$,

$$e_1 \le C N^{-3-2\nu}(1 + q^2 \log^2 N) \le C_\nu N^{-2-2\nu}.$$

For $j = 2, \ldots, N + N_A$, $e_j \le C_\nu (y_j - y_{j-1})^{2\nu+3} \|F_{\nu+1}\|^2_{\infty,(y_{j-1},y_j)}$. For $j = 2, \ldots, N$,

$$y_j - y_{j-1} = \left(\frac{j}{N}\right)^q - \left(\frac{j-1}{N}\right)^q \le \frac{q}{N}\left(\frac{j}{N}\right)^{2\nu/3}, \tag{3.8}$$

by the mean value theorem. For $\nu \ge 1$, $\|F_{\nu+1}\|^2_{\infty,(y_{j-1},y_j)} = y_{j-1}^{-2\nu} \le (2N/j)^{2\nu q}$, while, for $\nu = 0$, $\|F_{\nu+1}\|^2_{\infty,(y_{j-1},y_j)} = (1 - q\log(j - 1/N))^2$. Thus, for $j = 2, \ldots, N$,

$$e_j \le \begin{cases} C_\nu N^{-3-2\nu}, & \nu \ge 1, \\ C_\nu N^{-3}\left(1 - q\log\left(\frac{j-1}{N}\right)\right)^2, & \nu = 0. \end{cases}$$

For $j = N+1, \ldots, N + N_A$, $\|F_{\nu+1}\|^2_{\infty,(y_{j-1},y_j)} = y_{j-1}^{-2\nu-3}$, so that $e_j \le C_\nu((y_j - y_{j-1})/y_{j-1})^{2\nu+3}$. But, for $j = N+1, \ldots, N+N_A$, it follows from the definition of the mesh (in particular (3.5) and (3.6)) and (3.8) that

$$\frac{y_j - y_{j-1}}{y_{j-1}} = \frac{y_{N+1} - y_N}{y_N} \le \frac{y_N - y_{N-1}}{y_{N-1}} \le \left(\frac{N}{N-1}\right)^q \frac{q}{N} \le \frac{2^q q}{N},$$

so that $e_j \le C_\nu N^{-3-2\nu}$. Thus

$$\|f - P_N^* f\|^2_{2,(0,A)} = \sum_{j=1}^{N+N_A} e_j \le C_\nu \left(\frac{1}{N^{2+2\nu}} + \frac{N+N_A}{N^{3+2\nu}}\right) \tag{3.9}$$

in the case $\nu \geq 1$. In fact (3.9) holds also in the case $\nu = 0$ since then

$$\sum_{j=2}^{N} e_j \leq \frac{C_\nu}{N^3} \sum_{j=1}^{N} (1 - q \log \frac{j}{N})^2 < \frac{C_\nu}{N^3} \int_0^N (1 - q \log \frac{s}{N})^2 \, \mathrm{d}s = \frac{C_\nu}{N^2} \int_0^1 (1 - q \log t)^2 \, \mathrm{d}t.$$

Using (3.7) the result follows from (3.9). □

To form a mesh on the whole interval $[\tilde{a}, \tilde{b}] = [\tilde{t}_0, \tilde{t}_n]$ we place $n_j + 1$ mesh points, $s_{j,i}$, $i = 1, ..., n_j + 1$, on the interval $[\tilde{t}_{j-1}, \tilde{t}_j]$, for $j = 1, ..., n$, with $\tilde{t}_{j-1} = s_{j,1} < s_{j,2} < \ldots < s_{j,n_j+1} = \tilde{t}_j$. Under the assumption that $A_j := (\tilde{t}_j - \tilde{t}_{j-1})/2 > 1$, $j = 1, ..., n$, we set $n_j := 2(N + N_{A_j})$ and define a symmetric mesh on $[\tilde{t}_{j-1}, \tilde{t}_j]$ by

$$s_{j,i} := \begin{cases} \tilde{t}_{j-1} + y_{i-1}, & i = 1, \ldots, N + N_{A_j} + 1, \\ \tilde{t}_j - y_{2(N+N_{A_j})+1-i}, & i = N + N_{A_j} + 2, \ldots, 2(N + N_{A_j}) + 1, \end{cases}$$

where the points y_i are given by (3.4) and (3.5) with $A = A_j = (\tilde{t}_j - \tilde{t}_{j-1})/2$. The total mesh on $[\tilde{t}_0, \tilde{t}_n]$ is then denoted by

$$\Omega := \{s_{j,l} : j = 1, \ldots, n, \, l = 1, \ldots, n_j + 1\}.$$

Let $e_\pm(s) := \mathrm{e}^{\pm \mathrm{i} s}$, $s \in \mathbb{R}$. Then, in the Galerkin method we propose we shall seek an approximation to $\Phi^*, \Phi_N \in V_{\Omega,\nu}$, where

$$V_{\Omega,\nu} := \{\sigma e_\pm : \sigma \in \Pi_{\Omega,\nu}\},$$

and

$$\begin{aligned} \Pi_{\Omega,\nu} \quad := \quad & \{\sigma \in L_2(\mathbb{R}) : \sigma|_{[s_{j,m}, s_{j,m+1}]} \text{ is a polynomial of degree } \leq \nu, \\ & \text{for } j = 1, \ldots, n, \, m = 1, \ldots n_j, \text{ and } \sigma|_{\mathbb{R} \setminus [\tilde{t}_0, \tilde{t}_n]} = 0\}. \end{aligned}$$

Let $(\cdot, \cdot)$ denote the usual inner product on $L_2(\mathbb{R})$. Then our Galerkin method approximation, $\Phi_N \in V_{\Omega,\nu}$, is defined by

$$(\Phi_N, \rho) = (\Psi_\beta^{\beta_c}, \rho) + (K_\beta^{\beta_c} \Phi_N, \rho), \qquad \text{for all } \rho \in V_{\Omega,\nu}; \tag{3.10}$$

equivalently,

$$\Phi_N = P_N Q \Psi_\beta^{\beta_c} + P_N K_\beta^{\beta_c} \Phi_N, \tag{3.11}$$

where $P_N : L_2(\mathbb{R}) \to V_{\Omega,\nu}$ is the operator of orthogonal projection onto $V_{\Omega,\nu}$. Equation (3.10) can be written explicitly as a system of M_N linear algebraic equations, where M_N is the dimension of $V_{\Omega,\nu}$, i.e. the number of degrees of freedom, given by

$$M_N = 2(\nu + 1) \sum_{j=1}^{n} n_j = 4(\nu + 1) \sum_{j=1}^{n} (N + N_{A_j}). \tag{3.12}$$

By (3.7), $M_N < 4(\nu + 1)nN(1 + N^{-1} + q^{-1} \log \bar{A})$, where $\bar{A} = (A_1 \ldots A_n)^{1/n}$ is the geometric mean of $A_1, \ldots, A_n$. Since $\bar{A} \leq (A_1 + \ldots A_n)/n = (\tilde{b} - \tilde{a})/2n$, we have

$$M_N < 4(\nu + 1)nN \left(1 + \frac{1}{N} + \frac{1}{q} \log \frac{k(b - a)}{2n}\right). \tag{3.13}$$

4. Error analysis

We begin by rewriting (3.1) and (3.11) as

$$\Phi^* - QK_\beta^{\beta_c}\Phi^* = Q\Psi_\beta^{\beta_c} \tag{4.1}$$

and

$$\Phi_N - P_N K_\beta^{\beta_c}\Phi_N = P_N Q\Psi_\beta^{\beta_c} \tag{4.2}$$

respectively, where both equations hold in $L_2(\mathbb{R})$. Our goal now is to show that (4.2) has a unique solution Φ_N, and to establish an error bound on $\|\Phi^* - \Phi_N\|_2$.

From (3.2) and that $\|P_N\|_2 = 1$, $\|P_N K_\beta^{\beta_c}\|_2 \leq \|K_\beta^{\beta^c}\|_2 \leq \|\beta - \beta_c\|_\infty / \operatorname{Re}\beta_c$. Thus, arguing as we did to establish the existence and boundedness of $(I - QK_\beta^{\beta_c})^{-1}$ in §3, we see that $(I - P_N K_\beta^{\beta_c})^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ exists, so that (4.2) is uniquely solvable, and

$$\|(I - P_N K_\beta^{\beta_c})^{-1}\| \leq \frac{\operatorname{Re}\beta_c}{\operatorname{Re}\beta_c - \|\beta - \beta_c\|_\infty}, \tag{4.3}$$

provided (3.3) holds. Further, operating on (4.1) with P_N, subtracting (4.2), and rearranging, we see that $\Phi^* - \Phi_N = (I - P_N K_\beta^{\beta_c})^{-1}(\Phi^* - P_N\Phi^*)$ so that

$$\|\Phi^* - \Phi_N\|_2 \leq \frac{\operatorname{Re}\beta_c}{\operatorname{Re}\beta_c - \|\beta - \beta_c\|_\infty}\|\Phi^* - P_N\Phi^*\|_2, \tag{4.4}$$

provided (3.3) holds.

If necessary, one can force (3.3) to hold at the cost of changing the surface impedance outside some finite interval, replacing β by $\check{\beta}$, where

$$\check{\beta}(s) := \begin{cases} \beta(s), & s \in (\check{t}_0, \check{t}_n], \\ \check{\beta}_c, & s \in \mathbb{R}\backslash(\check{t}_0, \check{t}_n], \end{cases}$$

with $\check{t}_0 < t_0$, $\check{t}_n > t_n$, and $\check{\beta}_c$ chosen to ensure that $|\beta_j - \check{\beta}_c| < \operatorname{Re}\check{\beta}_c$, $j = 1, \ldots, n$. The error estimates derived in Chandler-Wilde et al. (2002), Rahman (2000) quantify the change in u^t caused by replacing β by $\check{\beta}$ and show convergence of the perturbed total field, resulting from replacing β by $\check{\beta}$, to the true total field at every $x \in U$ as $\check{t}_0 \to -\infty$ and $\check{t}_n \to \infty$, with $\check{\beta}_c$ fixed. In fact this convergence is uniform on bounded sets, so that $\check{\beta}_c$, $\check{t}_0$, and $\check{t}_n$ can be chosen so that the original problem is replaced by one for which (3.3) holds and whose solution, in any given bounded region of interest, is arbitrarily close to the solution of the original problem. For this reason and in order to focus on other difficulties (namely efficient solution for large k), we will assume that (3.3) holds in our analysis. In any case, in §5 we will present numerical results suggesting that the Galerkin scheme we propose is stable and convergent even when (3.3) does not hold. In this case the bound (4.3) does not apply however.

It finally just remains to bound $\|\Phi^* - P_N\Phi^*\|_2$, showing that our mesh and approximation space are well-adapted to approximate Φ^*. We introduce a further projection operator, P'_N, the operator of orthogonal projection from $L_2(\mathbb{R})$ onto $\Pi_{\Omega,\nu}$. A consequence of theorem 3.2 is the following error estimate.

Theorem 4.1. *Suppose that $f \in C^\infty((\tilde a,\tilde b)\backslash\{\tilde t_1,\ldots,\tilde t_{n-1}\})$, and that, for some $c>0$,*

$$\begin{aligned}|f'(s)| &\leq c\max(F_1(s-\tilde t_{j-1}),F_1(\tilde t_j - s)),\\ |f^{(\nu+1)}(s)| &\leq c\max(F_{\nu+1}(s-\tilde t_{j-1}),F_{\nu+1}(\tilde t_j - s)),\end{aligned}$$

for $s\in(\tilde t_{j-1},\tilde t_j)$, $j=1,\ldots,n$. Then

$$\|f-P_N'f\|_{2,(\tilde a,\tilde b)} \leq c\frac{C_\nu' n^{1/2}}{N^{\nu+1}}(1+\log^{1/2}\bar A),$$

where $\bar A = (A_1\ldots A_n)^{1/n}$, and the constant C_ν' depends only on ν.

Proof. Setting $\bar t_j = (\tilde t_j + \tilde t_{j-1})/2$, and recalling that $\bar t_j$ is one of the mesh points, for $j=1,\ldots,n$, it holds that $\|f-P_N'f\|^2_{2,(\tilde a,\tilde b)} = \sum_{j=1}^n (l_j^2+r_j^2)$, where $l_j := \|f - p_{\tilde t_{j-1},\bar t_j,\nu}\|_{2,(\tilde t_{j-1},\bar t_j)}$, $r_j := \|f-p_{\bar t_j,\tilde t_j,\nu}\|_{2,(\bar t_j,\tilde t_j)}$, and where we have used the notation $p_{A,B,\nu}$ from the proof of theorem 3.2 to denote the best polynomial approximation to f of degree $\leq \nu$ on the interval (A,B). By theorem 3.2, l_j and r_j are both $\leq cC_\nu N^{-\nu-1}(1+\log^{1/2}A_j)$. Thus, and since $(1+\log^{1/2}A_j)^2\leq 2(1+\log A_j)$, the result follows. □

To use the above error estimate, we note from (2.24) that $\Phi^* = e_+f_+ + e_-f_-$, where

$$f_+(s) := \begin{cases} f_j^+(s-\tilde t_{j-1}), & s\in(\tilde t_{j-1},\tilde t_j],\ j=1,\ldots,n,\\ 0, & s\in\mathbb{R}\backslash(\tilde t_0,\tilde t_n],\end{cases}$$

and

$$f_-(s) := \begin{cases} f_j^-(\tilde t_j - s), & s\in(\tilde t_{j-1},\tilde t_j],\ j=1,\ldots,n,\\ 0, & s\in\mathbb{R}\backslash(\tilde t_0,\tilde t_n].\end{cases}$$

Further, provided (2.25) holds for some $\epsilon>0$, it follows from theorem 2.6 that f_+ and f_- both satisfy the conditions of theorem 4.1, with a constant $c>0$ which depends only on ν and ϵ. Thus, applying theorem 4.1, we have that

$$\|f_+ - P_N'f_+\|_2 = \|f_+ - P_N'f_+\|_{2,(\tilde a,\tilde b)} \leq C\frac{n^{1/2}}{N^{\nu+1}}(1+\log^{1/2}\bar A),$$

if (2.25) holds, where the constant $C>0$ depends only on ν and ϵ. Further, the same bound holds on $\|f_- - P_N'f_-\|_2$. But $e_+P_N'f_+ + e_-P_N'f_- \in V_{\Omega,\nu}$. Thus, and since $P_N\Phi^*$ is the best approximation to Φ^* in $V_{\Omega,\nu}$,

$$\begin{aligned}&\|\Phi^* - P_N\Phi^*\|_2\\ &\quad\leq \|\Phi^* - (e_+P_N'f_+ + e_-P_N'f_-)\|_2 = \|e_+(f_+ - P_N'f_+) + e_-(f_- - P_N'f_-)\|_2\\ &\quad\leq \|e_+\|_\infty\|f_+ - P_N'f_+\|_2 + \|e_-\|_\infty\|f_- - P_N'f_-\|_2 \leq 2\frac{Cn^{1/2}}{N^{\nu+1}}(1+\log^{1/2}\bar A).\end{aligned}$$

We have shown the following result.

Theorem 4.2. *Suppose that (2.25) holds, for some $\epsilon>0$. Then it holds that*

$$\|\Phi^* - P_N\Phi^*\|_2 \leq \frac{Cn^{1/2}}{N^{\nu+1}}(1+\log^{1/2}\bar A),$$

where the constant $C>0$ depends only on ν and ϵ.

Combining this result with the stability bound (4.4) and the bound (3.13) on the number of degrees of freedom M_N, we obtain our final error estimate for the approximation of Φ by Φ_N.

Theorem 4.3. *Provided (2.25) holds, for some $\epsilon > 0$, and (3.3) is satisfied, we have that*

$$\begin{aligned}\|\Phi - \Phi_N\|_{2,(\tilde{a},\tilde{b})} = \|\Phi^* - \Phi_N\|_2 &\leq \frac{Cn^{1/2}(1+\log^{1/2}\bar{A})}{(\mathrm{Re}\,\beta_c - \|\beta-\beta_c\|_\infty)N^{\nu+1}}, \\ &\leq \frac{C[n(1+\log(k(b-a)/n))]^{\nu+3/2}}{(\mathrm{Re}\,\beta_c - \|\beta-\beta_c\|_\infty){M_N}^{\nu+1}},\end{aligned}$$

where the constant $C>0$ depends only on ν and ϵ.

5. Implementation and numerical results

Throughout this section, we will restrict our attention to the case $\nu = 0$. For higher values of ν the implementation of the scheme is similar and the coefficients can be evaluated in an analogous way. However, the formulae will be considerably more complicated. Recalling (3.10), the equation we wish to solve is then

$$(\Phi_N,\rho) - (K_\beta^{\beta_c}\Phi_N,\rho) = (\Psi_\beta^{\beta_c},\rho), \qquad \text{for all } \rho \in V_{\Omega,0}. \tag{5.1}$$

Writing Φ_N as a linear combination of the basis functions of $V_{\Omega,0}$, we have $\Phi_N(s) = \sum_{j=1}^{M_N} v_j\rho_j(s)$, where M_N is given by (3.12) and ρ_j is the jth basis function,

$$\rho_j(s) := \begin{cases} \mathrm{e}^{\mathrm{i}s}\chi_{[s_{k,l-1},s_{k,l})}(s), & j = 2l-1, \\ \mathrm{e}^{-\mathrm{i}s}\chi_{[s_{k,l-1},s_{k,l})}(s), & j = 2l, \end{cases}$$

for $l = \sum_{m=1}^{k-1} n_m + 1, \ldots, \sum_{m=1}^{k} n_m$, $k = 1,\ldots,n$, where $\chi_{[y_1,y_2)}$ denotes the characteristic function of the interval $[y_1,y_2)$. Equation (5.1) then becomes

$$\sum_{j=1}^{M_N} v_j((\rho_j,\rho_k) - (K_\beta^{\beta_c}\rho_j,\rho_k)) = (\Psi_\beta^{\beta_c},\rho_k), \quad k = 1,\ldots,M_N,$$

and thus we need to determine (ρ_j,ρ_k), $(K_\beta^{\beta_c}\rho_j,\rho_k))$ and $(\Psi_\beta^{\beta_c},\rho_k)$. We can evaluate (ρ_j,ρ_k) analytically. To evaluate $(K_\beta^{\beta_c}\rho_j,\rho_k)$ and $(\Psi_\beta^{\beta_c},\rho_k)$ most of the integrations can be carried out analytically, but we need to compute some integrals numerically. The most difficult of these take the forms

$$\int_0^\infty \frac{(i-r)J(r)}{r(r-2\mathrm{i})}\,\mathrm{d}r, \quad \int_0^\infty \frac{(1-\mathrm{e}^{rs})J(r)}{r^2}\,\mathrm{d}r, \quad \int_0^\infty \frac{(1-\mathrm{e}^{rs})J(r)}{r(r-2\mathrm{i})}\,\mathrm{d}r,$$

where $s<0$ and $J(r) = r^{1/2}(r-2\mathrm{i})^{1/2}/(r-\mathrm{i}\hat{a}_+)(r-\mathrm{i}\hat{a}_-)$, with $\hat{a}_\pm = 1 \mp \sqrt{1-\beta_c^2}$ as in §2. These integrals are similar in difficulty to the integral representation for the Green's function, equation (2.7), for which very efficient numerical schemes are proposed in Chandler-Wilde & Hothersall (1995*a*). In particular, we remark that the integrands are not oscillatory; the oscillating part of the integrands is removed by the integrations which are carried out analytically.

As a first numerical example we take $\theta = \pi/4$, $n = 1$, and

$$\beta(s) = \begin{cases} 0.505 - 0.3i, & s \in [-m\lambda, m\lambda], \\ 1 & s \notin [-m\lambda, m\lambda], \end{cases}$$

for $m = 5, 10, 20, 40, 80, 160, 320, 640, 1280, 2560$ and 5120, where $k = 1$ and $\lambda = 2\pi$ is the wavelength. Assumption (3.3) is satisfied so that theorem 4.3 holds. For each value of m, we compute Φ_N with $\nu = 0$ and $N = 2, 4, 8, 16, 64$, taking the solution with $N = 64$ to be the "exact" solution, for the purpose of computing errors.

In figures 2 and 3 we plot $|\Phi_N|$ for $N = 4$ and $N = 64$ (the "exact" solution) for $m = 10$ and $m = 5120$ respectively. Note the logarithmic scale. We also plot the grid points of Ω. As m increases, with N fixed, the density of grid points near the discontinuities in impedance does not vary, and only a few more points are added near the centre of the interval. The value of $|\Phi|$ is highly peaked at the discontinuities in impedance. Recalling that Φ is a correction term, namely the difference between the true solution and the solution that there would be if the impedance was constant everywhere, the reason for this is clear.

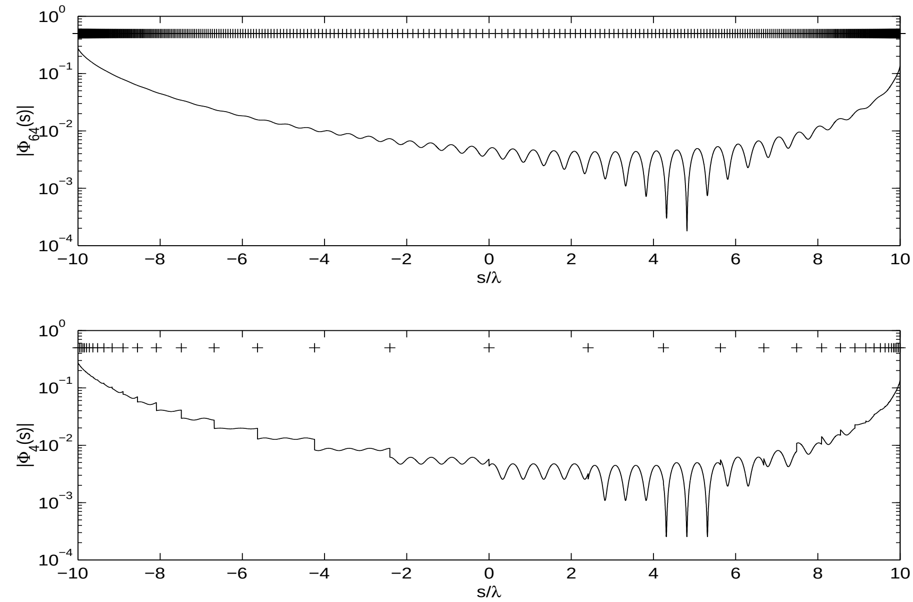

Figure 2. Plot of $|\Phi_N|$, $N = 4$ and $N = 64$ for $m = 10$, so that $b - a = 20\lambda$

The relative L_2 errors $\|\Phi_{64} - \Phi_N\|_2/\|\Phi_{64}\|_2$ are shown in table 1. (All L_2 norms are computed by approximating by discrete L_2 norms, sampling at 100000 evenly spaced points in the relevant interval for the function whose norm is to be evaluated.) The estimated order of convergence is given by

$$EOC := \log_2 \left(\frac{\|\Phi_{64} - \Phi_N\|_2}{\|\Phi_{64} - \Phi_{2N}\|_2} \right).$$

As $\nu = 0$ we would expect, from theorem 4.3, that $EOC \approx 1$. In fact the convergence rate is better than this. The number of degrees of freedom for each example is

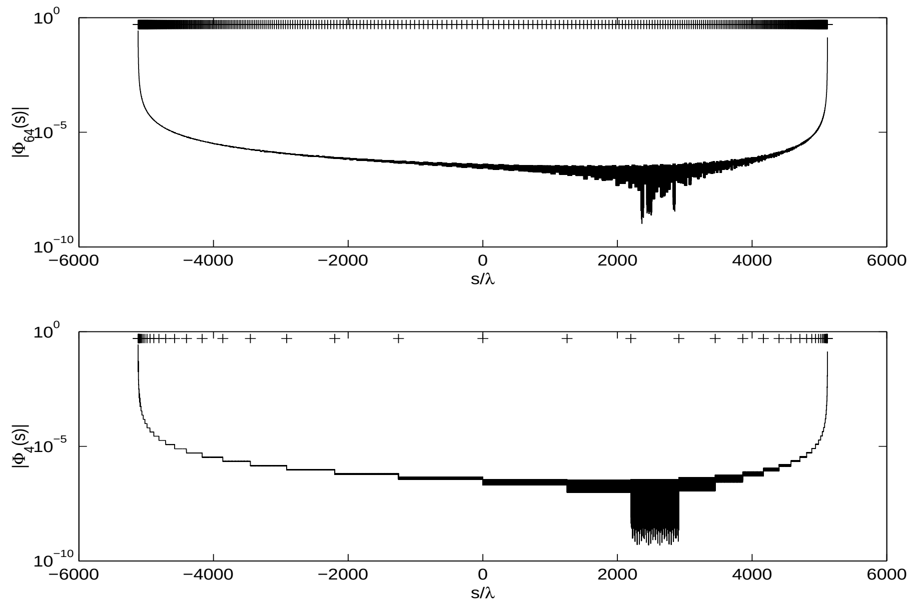

Figure 3. Plot of $|\Phi_N|$, $N = 4$ and $N = 64$ for $m = 5120$, so that $b - a = 10240\lambda$

	$b-a=20\lambda$,			$b-a=10240\lambda$		
N	M_N	$\|\Phi_N-\Phi_{64}\|_2/\|\Phi_{64}\|_2$	EOC	M_N	$\|\Phi_N-\Phi_{64}\|_2/\|\Phi_{64}\|_2$	EOC
2	32	1.392×10^{-1}	1.7	68	1.404×10^{-1}	1.7
4	76	4.191×10^{-2}	1.7	164	4.324×10^{-2}	1.7
8	160	1.285×10^{-2}	1.8	344	1.331×10^{-2}	1.7
16	324	3.619×10^{-3}		708	4.006×10^{-3}	
64	1308			2896		

Table 1. $\|\Phi_N - \Phi_{64}\|_2/\|\Phi_{64}\|_2$ *for* $m = 10$ *and* $m = 5120$*, and increasing* N*.*

denoted by M_N. Note that the relative L_2 error is almost the same for the two cases $b - a = 20\lambda$ and $b - a = 10240\lambda$.

In table 2 we fix the value of N at $N = 8$ and show $\|\Phi_N - \Phi_{64}\|_2/\|\Phi_{64}\|_2$ and also $\|\Phi_N - \Phi_{64}\|_2$ for increasing values of $m = (b - a)/2\lambda$. As m increases, the number of degrees of freedom increases only logarithmically, while both the relative and actual error remain roughly constant. For $m = 5120$ the interval is of length greater than 10000 wavelengths, and yet we achieve roughly one per cent relative error with only 400 degrees of freedom.

As a second numerical example we take $\theta = \pi/4$, $n = 3$, and

$$\beta(s) = \begin{cases} 0.5 - 2i, & s \in [-10\lambda, 0], \\ 1 - i, & s \in [0, 10\lambda], \\ 2 - 0.5i & s \in [10\lambda, 50\lambda], \\ 0.505 - 0.3i & s \notin [-10\lambda, 50\lambda], \end{cases}$$

where $k = 1$ and $\lambda = 2\pi$ is the wavelength. In this case assumption (3.3) is not

$(b-a)/\lambda$	M_N	$\|\Phi_8 - \Phi_{64}\|_2 / \|\Phi_{64}\|_2$	$\|\Phi_8 - \Phi_{64}\|_2$
10	136	1.228×10^{-2}	5.579×10^{-3}
20	160	1.285×10^{-2}	5.843×10^{-3}
40	180	1.325×10^{-2}	6.027×10^{-3}
80	200	1.345×10^{-2}	6.118×10^{-3}
160	220	1.356×10^{-2}	6.167×10^{-3}
320	240	1.364×10^{-2}	6.202×10^{-3}
640	260	1.366×10^{-2}	6.211×10^{-3}
1280	284	1.364×10^{-2}	6.204×10^{-3}
2560	304	1.355×10^{-2}	6.162×10^{-3}
5120	324	1.342×10^{-2}	6.097×10^{-3}
10240	344	1.331×10^{-2}	6.028×10^{-3}

Table 2. $\|\Phi_8 - \Phi_{64}\|_2 / \|\Phi_{64}\|_2$ *for increasing interval length.*

satisfied. We compute Φ_N for $\nu = 0$ and $N = 2, 4, 8, 16, 32$, taking the solution with $N = 32$ to be the "exact" solution, for the purpose of computing errors.

In figure 4 we plot $|\Phi_N|$ for $N = 4$ and $N = 32$ (the "exact" solution). We also plot the grid points of Ω. The way in which the grid on each interval $[t_{j-1}, t_j]$ is dependent on $t_j - t_{j-1}$ can be seen. Specifically, we point out that the density of grid points is the same monotonic decreasing function of distance to the nearest impedance discontinuity on each interval.

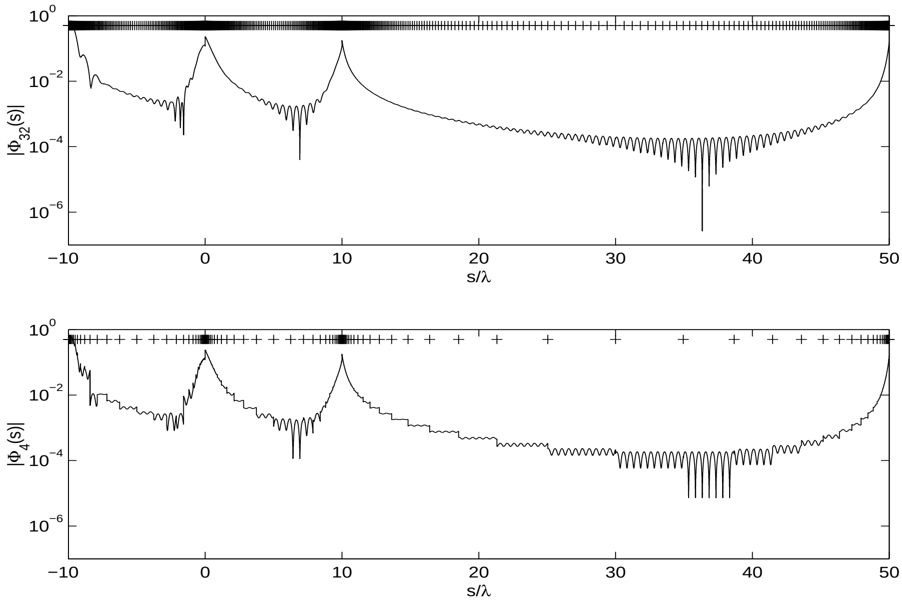

Figure 4. Plot of $|\Phi_N|$, $N = 4$ and $N = 32$, second example

The relative L_2 errors, $\|\Phi_{32} - \Phi_N\|_2 / \|\Phi_{32}\|_2$, are shown in table 3. The errors and convergence rate achieved are close to those in table 1 for the same values of N. This suggests that the numerical scheme is stable and convergent in this case,

N	M_N	$\|\Phi_N - \Phi_{32}\|_2 / \|\Phi_{32}\|_2$	EOC
2	92	2.886×10^{-1}	1.8
4	212	8.450×10^{-2}	2.2
8	452	1.837×10^{-2}	2.0
16	924	4.513×10^{-3}	
32	1866		

Table 3. $\|\Phi_N(s) - \Phi_{32}(s)\|_2 / \|\Phi_{32}\|_2$ *as* N *increases, second example*

even though assumption (3.3), needed for our proof of stability of the numerical scheme and of the error estimate of theorem 4.3, does not hold.

6. Extension to more general scattering problems

In this paper we have presented a Galerkin boundary element method for a problem of acoustic scattering by an unbounded surface with piecewise constant surface impedance, and we have demonstrated via both theoretical analysis and numerical examples that the number of degrees of freedom required for an accurate solution depends only logarithmically on the wavenumber. Although the method and analysis described here is for a very specific scattering problem, similar ideas can be applied to solve problems of scattering by more general objects. In Chandler-Wilde and Langdon (2003, in preparation), the authors consider the problem of scattering in 2D by convex polygons. To give a flavour of how the ideas presented here extend to this case, we briefly discuss this extension now.

Consider the problem of scattering of a given incoming acoustic plane wave u^i by a sound-soft bounded convex polygon Ω. In the spirit of the geometrical theory of diffraction, one expects, on a typical side PQ, incident, reflected and diffracted ray contributions, as indicated in figure 5.

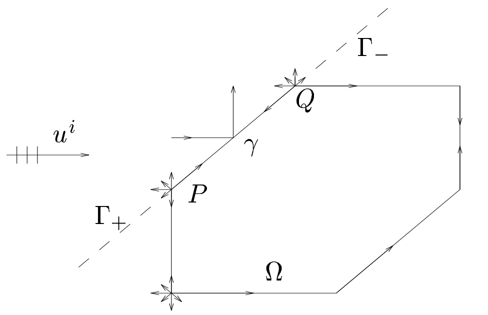

Figure 5. Acoustic scattering by a convex polygon. A typical incident and reflected ray are shown as well as some of the rays arising from diffraction at the corners.

The total acoustic field u^t satisfies the Helmholtz equation $\Delta u^t + k^2 u^t = 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}$, with boundary data $u^t = 0$ on $\Gamma := \partial\Omega$, supplemented with an appropriate Sommerfeld radiation condition to ensure uniqueness of solution. A direct integral equation formulation leads to the following second kind boundary integral equation for the unknown complementary boundary data $\partial u^t/\partial n$:

$$\frac{1}{2}\frac{\partial u^t}{\partial n}(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y)\right)\frac{\partial u^t}{\partial n}(y)\,\mathrm{d}s(y) = f(x), \quad x \in \Gamma\backslash\{\text{corners}\}, \tag{6.1}$$

with $f := \partial u^i/\partial n + \mathrm{i}\eta u^i$, n the normal direction directed out of Ω, η a coupling parameter, and $\Phi(x,y) := (\mathrm{i}/4)H_0^{(1)}(k|x-y|)$ the standard fundamental solution for the Helmholtz equation.

To solve (6.1) for $\partial u^t/\partial n$ a similar Galerkin boundary element method approach to that described in this paper can be used. On each side of the polygon, we begin by separating off the leading order behaviour as $k \to \infty$, namely the known incident and reflected waves. (On sides in the shadow zone, there are no incident and reflected waves to subtract, and this step is omitted.) The remaining scattered wave, which consists of the contributions from the diffracted waves at the corners, can then be represented as a product of oscillatory plane waves and non-oscillatory functions. For example, on the side γ between corners P and Q in figure 5, by applying Green's theorem in the half-plane to the left of the line $\Gamma_+ \cup \gamma \cup \Gamma_-$, which is the extension of γ, one finds that (cf. equation(2.13))

$$\frac{\partial u^t}{\partial n}(x) = \text{known leading order behaviour } + 2\int_{\Gamma_+\cup\Gamma_-} \frac{\partial^2 \Phi(x,y)}{\partial n(x)\partial n(y)} u^t(y)\,\mathrm{d}s(y), \; x \in \gamma, \tag{6.2}$$

where Γ_+ and Γ_- are the half-lines below P and above Q, respectively, in figure 5. If we represent the side γ parametrically as $P + s(Q-P)/|Q-P|$, $s \in [0, |P-Q|]$, then on the side γ it follows that

$$\frac{1}{k}\frac{\partial u^t}{\partial n}(s) = \text{known leading order behaviour } + \mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s),$$

where the terms $\mathrm{e}^{\mathrm{i}ks}v_+(s)$ and $\mathrm{e}^{-\mathrm{i}ks}v_-(s)$ are the integrals over Γ_+ and Γ_-, respectively, in equation (6.2), and can be thought of as the contribution to $\partial u^t/\partial n$ on γ due to the diffracted rays travelling from P to Q and from Q to P, respectively, including all multiply diffracted ray components. Explicitly, from (6.2) it follows that

$$v_+(s) = \frac{\mathrm{i}k}{2}\int_{-\infty}^0 \mu(k(s-t))e^{-\mathrm{i}kt}\phi(t)\,\mathrm{d}t,$$

where $\phi(t) = u^t(P + t(Q-P)/|Q-P|)$, $t < 0$, denotes the total field on Γ_+ at distance $|t|$ from P, and $\mu(s) := e^{-\mathrm{i}|s|}H_1^{(1)}(|s|)/|s|$. A similar explicit expression holds for v_-.

To achieve good approximation with a low number of degrees of freedom, one can then approximate v_+ and v_- by piecewise polynomials, using a similar graded mesh on each side of Γ to that used on each constant impedance interval $[t_{j-1}, t_j]$ in §3, with larger elements away from the corners of the polygon and the mesh grading near the corners depending on the internal angles. The construction of such a mesh relies on the derivation of rigorous regularity results for $v_\pm$, similar to those for $f_j^\pm$ achieved in theorem 2.6 above. These will appear in detail in Chandler-Wilde and Langdon (2003, in preparation), but we point out that, for $s \geq 1$, $\mu(s)$ satisfies exactly the same bound as that on $\check{\kappa}_\beta(s)$ in lemma 2.5, from which it follows that, arguing exactly as in the proof of theorem 2.6,

$$|v^{+(j)}(s)| \leq c_j(ks)^{-1/2-j}, \quad ks \geq 1,$$

for $j = 0, 1, \ldots$. From this bound and the argument of the proof of theorem 3.2 it follows that, except in a neighbourhood of the corners which is of the order of the wavelength, the appropriate mesh grading on γ is exactly that used on each constant impedance interval $[t_{j-1}, t_j]$ in the method of section 3.

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