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Abstract

In this paper we study the problem of maximizing a quadratic form $\langle Ax, x\rangle$ subject to $\|x\|_q = 1$, where A has matrix entries $f(\frac{[i,j]}{(i,j)})$ with $i, j|k$ and $q \geq 1$. We investigate when the optimal is achieved at a 'multiplicative' point; i.e. where $x_1 x_{mn} = x_m x_n$. This turns out to depend on both f and q, with a marked difference appearing as q varies between 1 and 2. We prove some partial results and conjecture that for f is multiplicative such that $0 < f(p) < 1$, the solution is at a multiplicative point for all $q \geq 1$.

2010 AMS Mathematics Subject Classification: 11A05, 11C20, 11N99, 15A36
Keywords and phrases: Optimization problem, Multiplicative functions

§1. Introduction

In optimization problems involving multiplicative structure, there is a tendency for multiplicative functions to play a crucial role. This can appear in various ways; the optimal may itself be multiplicative, or the point where the optimal occurs may be multiplicative.

For instance in [3], Codecá and Nair considered (amongst others) the problem of minimizing a quadratic form $\langle Bx, x\rangle$ subject to $\|x\|_2 = 1$ where B is the $d(k) \times d(k)$ matrix with entries $\frac{h((i,j))}{ij}$ where $i, j|k$, (i, j) is the gcd of i and j, and k is squarefree. They proved that any real multiplicative function f with $0 < f(p) < 1$ (for primes $p|k$) can be realised as such as minimum. Further, they explicitly determined this minimum when h is multiplicative and of the form $h = 1 * g$, with $g \geq 0$.

Another example comes from [7], where Perelli and Zannier considered the problem of minimizing $\langle Ax, x\rangle$ subject to $\|x\|_2 = 1$ where A is the $d(k) \times d(k)$ matrix (again with k squarefree) with entries $f(\frac{[i,j]}{(i,j)})$ (here $[i, j]$ is the lcm of i and j) in the special case that $f(n) = \frac{1}{4} + \frac{1}{12n}$. They show that the minimum is $\frac{\varphi(k)}{12k}$ and that this is achieved at the point $x_d = \frac{\mu(d)}{\sqrt{d(k)}}$.

In [6], it was noted that the operation $c \circ d = \frac{[c,d]}{(c,d)}$ is a group operation on $D(k) = \{d : d|k\}$ if k is squarefree and, as an application of this algebraic structure, the problem of maximizing $\langle A_f x, x\rangle$ was considered, where $A_f = (f(c \circ d))_{c,d|k}$ but now subject to $\|x\|_q = 1$ with $q \geq 2$. It was found that for any $f : D(k) \to (0, \infty)$, the optimal is

$$d(k)^{1-\frac{2}{q}} \sum_{d|k} f(d),$$

and that it occurs at x_d constant. Notice that in both of the above examples, $\frac{x_d}{x_1}$ is multiplicative at the optimal, even if f is not. In the latter, the optimal itself is also multiplicative precisely when f is.

In this paper we consider the above optimization problem for the range $1 < q < 2$, which turns out to be highly non-trivial. This has its origin in a problem concerning gcd sums. Briefly, one wishes to maximize the sum

$$F_\alpha(S) = \sum_{m,n \in S} \frac{1}{(m \circ n)^\alpha}$$

over all sets S of size N (see [5] for the case $\alpha = 1$ and [4] and [1] for other values of $\alpha > 0$). For $\alpha \geq \frac{1}{2}$ good bounds for this maximum have been established (sharp for $\alpha = 1$ [5] and close to best possible for $\frac{1}{2} \le \alpha < 1$ see [1], [2]), but for $0 < \alpha < \frac{1}{2}$ little is as yet known, except for rather crude upper and lower bounds. Thus it is known that in this range

$$N^{2-2\alpha} \ll \max_{|S|=N} F_\alpha(S) \ll N^{2-2\alpha} \exp\left\{ c\alpha \sqrt{\frac{\log N \log\log\log N}{\log\log N}} \right\}$$

for some absolute constant c (see [2]), but the true order is far from settled. In work in progress, a new lower bound $N^{2-2\alpha}(\log\log N)^{2\alpha}$ can be established which may also turn out to be the correct order of magnitude. This hinges (in part) on maximizing $\langle A_f x, x\rangle$ with $f(n) = n^{-\alpha}$ over $\|x\|_q = 1$, where $q = \frac{1}{1-\alpha} \in (1, 2)$. This motivates studying the following

Optimization problem: *Let $f : D(k) \to (0, \infty)$ where k is squarefree. Find the supremum of*

$$\langle A_f x, x\rangle = \sum_{c,d|k} f(c \circ d) x_c x_d \quad \textit{subject to} \quad \|x\|_q = 1.$$

Throughout the article, k is squarefree, $q \ge 1$ and $\|x\|_q$ is the usual q-norm: with $x = (x_d)_{d|k}$, $\|x\|_q = (\sum_{d|k} |x_d|^q)^{1/q}$. Also let $F(k) = \sum_{d|k} f(d)$.

Remarks 1

(a) Note the following symmetry: *let $x' = (x'_d)$ where $x'_d = x_{c\circ d}$ for some $c|k$ (for all $d|k$); then $\langle A_f x', x'\rangle = \langle A_f x, x\rangle$, and $\|x'\|_q = \|x\|_q$.* Thus if x is optimal, then so is x'. Also, as $f > 0$, the maximum occurs for $x \ge 0$. Hence, without loss of generality, by permuting the x_d, we may always assume that at the optimal, $x_1 \ge x_d \ge 0$ for every $d|k$.

(b) For A_f positive definite, $A_f = B^*B$ for some B, so that $\langle A_f x, x\rangle = \|Bx\|^2$ and the problem becomes one of evaluating the norm $\|B\|_{q,2}$. We discuss the details in §5.

For $q = 2$ the problem is standard: optimizing a (Hermitian) quadratic form. The optimal is just the largest eigenvalue of A_f, which is $F(k) = \sum_{d|k} f(d)$. As mentioned earlier, for $q > 2$ the answer is also relatively straightforward as shown in [6], and we briefly outline the proof. Our main interest shall be the range $1 < q < 2$.

Let Λ (or Λ_q if we wish to emphasize the dependence on q) denote the optimum, indeed maximum. Also let

$$M_q = \max\left\{ \langle A_f x, x\rangle \ : \ \|x\|_q = 1 \text{ and } \tfrac{x_d}{x_1} \text{ is multiplicative} \right\}$$

denote the maximum over 'multiplicative' x; i.e. when $x_1 x_{mn} = x_m x_n$ for $(m, n) = 1$.

Our main results are the following:

Theorem 1
Let $f : D(k) \to (0, \infty)$. Then there exists $c > 0$, depending on f and k, such that for $q \ge 2 - c$, the optimal solution occurs at x_d constant and $\Lambda_q = d(k)^{1-2/q} F(k)$.

Theorem 2
Let f be multiplicative on $D(k)$ such that $0 < f(p) < 1$ for all $p|k$. Then there exists $c > 0$, depending on f and k, such that for $q \in [1, 1 + c)$, the optimal solution occurs at a multiplicative point; i.e. where $x_1 x_{mn} = x_m x_n$ whenever $(m, n) = 1$.

Combining these, we see that for f multiplicative, $M_q = \Lambda_q$ for $q \in [1, 1 + c_1) \cup (2 - c_2, \infty)$ for some $c_1, c_2 > 0$, depending on f and k. However, we believe that the result is true throughout $[1, \infty)$. In other words, we make the following

Conjecture: Let f be multiplicative on $D(k)$ such that $0 < f(p) < 1$ for all $p|k$. Then the optimal solution occurs at a multiplicative point and so $M_q = \Lambda_q$ for all $q \ge 1$.

Briefly we outline the rest of the paper. In §2, we indicate how the method of Lagrange multipliers deals with the $q \geq 2$ case and what it tells us about the range $1 < q < 2$. We take a particular look at the first non-trivial case $k = 6$.

In §3, we evaluate M_q explicitly, while in §4 we give the proofs of our main results. In §5, we show how we can view the problem as a problem of determining a norm, giving an equivalent form of the above conjecture.

§2. The method of Lagrange multipliers

To find the optimal, we use the method of Lagrange multipliers. We observe that, for $q > 1$, the maximum must occur at an interior point; i.e. where each $x_d > 0$. For suppose $x_a = 0$ for some $a|k$ at a local maximum. There exists b such that $x_b > 0$. Let

$$G(x) = \langle A_f x, x\rangle = \sum_{c,d|k} f(c \circ d) x_c x_d$$

and consider $G(x+h) - G(x)$ with $h = (h_d) = (\ldots, \varepsilon, \ldots, -\varepsilon', \ldots)$ where there is an $\varepsilon > 0$ in the a^{th} place and $-\varepsilon'$ in the b^{th} place and zeros elsewhere, with ε' chosen so that $\|x+h\|_q = 1$. As such

$$\varepsilon' = x_b - (x_b^q - \varepsilon^q)^{\frac{1}{q}} \sim \frac{\varepsilon^q}{q x_b^{q-1}} = o(\varepsilon),$$

as $\varepsilon \to 0$. Now

$$\begin{aligned} G(x+h) - G(x) &= \sum_{c,d|k} f(c \circ d)\Big\{(x_c + h_c)(x_d + h_d) - x_c x_d\Big\} \\ &= 2\sum_{c,d|k} f(c \circ d) x_c h_d + \sum_{c,d|k} f(c \circ d) h_c h_d \\ &= 2\varepsilon \sum_{c|k} f(c \circ a) x_c + o(\varepsilon) \geq 2\varepsilon f(a \circ b) x_b + o(\varepsilon) > 0, \end{aligned}$$

for ε sufficiently small and positive. Thus $G(x)$ cannot be maximal.

For $x = (x_d)_{d|k} \in \mathbb{R}_{\geq 0}^{d(k)}$, let $H(x) = G(x) - 2A(\sum_{d|k} x_d^q - 1)$, where A is to be determined. Then at the optimal solution, we must have $\frac{\partial H}{\partial x_d} = 0$ for every $d|k$; i.e.

$$A x_d^{q-1} = \sum_{c|k} f(c \circ d) x_c \quad (\forall d|k).$$

Multiplying through by x_d and summing over d shows that we must take $A = \Lambda$. Thus, *at the optimal,*

$$\Lambda x_d^{q-1} = \sum_{c|k} f(c \circ d) x_c \quad \textit{for every } d|k. \tag{2.1}$$

2.1 The case $q \geq 2$

Using equations (2.1), the case $q \geq 2$ can be easily dealt with.

Theorem A (see [6])
Let k be squarefree, $f : D(k) \to (0, \infty)$ and $q \geq 2$. Then $\Lambda = d(k)^{1-\frac{2}{q}} F(k)$, where the optimal occurs for x_d constant; i.e. $x_d = \frac{1}{\sqrt[q]{d(k)}}$.

Proof. Let $x = (x_d)$ denote the optimal and $\underline{x}$ and $\overline{x}$ the minimum and maximum of x_d respectively. By (2.1), for some $d|k$,

$$\Lambda \underline{x}^{q-1} = \sum_{c|k} f(c \circ d) x_c \geq \underline{x} \sum_{c|k} f(c \circ d) = \underline{x} F(k)$$

since $(D(k), \circ)$ is a group. On the other hand, for some $d'|k$,

$$\Lambda \overline{x}^{q-1} = \sum_{c|k} f(c \circ d') x_c \leq \overline{x} \sum_{c|k} f(c \circ d') = \overline{x} F(k).$$

Combining these gives $\Lambda \underline{x}^{q-2} \geq F(k) \geq \Lambda \overline{x}^{q-2}$. For $q = 2$ this forces $\Lambda = \sum_{d|k} f(d)$. For $q > 2$, we must have $\overline{x} \leq \underline{x}$; i.e. x_d must be constant. As $\sum_{d|k} x_d^q = 1$, this forces $x_d = 1/\sqrt[q]{d(k)}$. This must give the maximum value of G as it exists and it lies in the interior of the region. Hence $\Lambda = d(k)^{1-\frac{2}{q}} F(k)$ follows.

□

2.2 The case $1 < q < 2$

If $q \in (1, 2)$, the above analysis using Lagrange Multipliers leading to (2.1) is still valid, but the conclusion that x_d is constant at the optimum no longer holds in general. However, as we shall prove in Theorem 1, this constant solution continues to hold in an interval $q \in (2 - c, 2)$ for some $c > 0$, depending on both f and k.

For smaller q though, the optimal changes. Indeed, looking at the behaviour of the optimal solution when q is close to 1, shows precisely what is required for multiplicativity. Indeed, for $q = 1$, one can construct examples with $f > 1$ where the optimal is not multiplicative, even if f is (see Remarks 2). By continuity, this shows it also fails for some $q > 1$. However, if $f(n) \leq f(1) = 1$ for all n, then the optimal when $q = 1$ occurs at $x = (1, 0, \ldots, 0)$. For q close to 1, we shall see that in this case (taking $x_1 \geq x_d$)

$$x_d^{q-1} \sim f(d) \quad \text{as } q \to 1+, \text{ for every } d|k.$$

Thus for x_d/x_1 to be multiplicative, we need f to be multiplicative.

However, there are indications that it is also sufficient. Note that for f multiplicative, the eigenvalues of A_f are $\prod_{p|k}(1 \pm f(p))$ (where any combination of $\pm$ is possible – see [6]) and A_f is positive definite precisely when $-1 < f(p) < 1$ for all prime divisors p of k. The condition that f is at most 1 in Theorem 2 is therefore quite natural.

2.3 The simplest non-trivial case; $k = 6$

The reason why we expect multiplicativity at the optimum may not be clear at this stage. That it is true in a fairly trivial way for $q \geq 2$ is not sufficient reason. Also it is vacuously true when k is prime. A look at the first non-trivial case gives some indication why multiplicativity is expected.

Writing $f(2) = a$ and $f(3) = b$ (so that $f(6) = ab$), the problem for the $k = 6$ case now becomes: *maximize*

$$x_1^2 + x_2^2 + x_3^2 + x_6^2 + 2a(x_1x_2 + x_3x_6) + 2b(x_1x_3 + x_2x_6) + 2ab(x_1x_6 + x_2x_3)$$

subject to $x_1, x_2, x_3, x_6 \geq 0$ *and* $x_1^q + x_2^q + x_3^q + x_6^q = 1$.

The Conjecture says that, if $0 < a, b < 1$ then, at the maximum, $x_1x_6 = x_2x_3$.
Let us see why this is plausible. Equations (2.1) give

$$\begin{aligned}
\Lambda x_1^{q-1} &= x_1 + ax_2 + bx_3 + abx_6 \\
\Lambda x_2^{q-1} &= ax_1 + x_2 + abx_3 + bx_6 \\
\Lambda x_3^{q-1} &= bx_1 + abx_2 + x_3 + ax_6 \\
\Lambda x_6^{q-1} &= abx_1 + bx_2 + ax_3 + x_6.
\end{aligned}$$

Multiplying the cases $d = 1$ and $d = 6$ together and subtracting the product of $d = 2$ and $d = 3$ gives (after some cancellation)

$$\Lambda^2 \Big((x_1x_6)^{q-1} - (x_2x_3)^{q-1} \Big) = (1 - a^2)(1 - b^2)(x_1x_6 - x_2x_3).$$

This indicates the special role played by the quantity $x_1x_6 - x_2x_3$.

If $x_1x_6 \neq x_2x_3$, then we may divide through:

$$\Lambda^2 = (1-a^2)(1-b^2)\frac{x_1x_6 - x_2x_3}{(x_1x_6)^{q-1} - (x_2x_3)^{q-1}} < \frac{x_1x_6 - x_2x_3}{(x_1x_6)^{q-1} - (x_2x_3)^{q-1}},$$

It is not difficult to show that the RHS has its supremum (over all x such that $\|x\|_q = 1$ and $x_1x_6 \neq x_2x_3$) when x_d is constant, interpreted in the limit as $x_1x_6 \to x_2x_3$. (We omit the details.) As a result,

$$\Lambda^2 \le \frac{(1/4)^{\frac{2(2-q)}{q}}}{q-1}.$$

But $\Lambda \ge 1$ (by taking $x_1 = 1$ and $x_d = 0$ for $d > 1$). Thus $x_1x_6 \neq x_2x_3$ implies

$$(q-1)4^{\frac{2(2-q)}{q}} < 1.$$

But this is (fairly easily) shown to be false for $q \in (1.1076, 2]$. Thus the conjecture holds when $k = 6$ for $q \in (1.1076, 2]$ at least. By Theorem 2, it also holds for q in an interval $[1, 1+c)$ but, unfortunately, c is not an absolute constant, depending as it does on a and b. So the case $k = 6$ is still open.

§3. The maximum over multiplicative x for f multiplicative

Now we calculate the maximum over 'multiplicative' x (i.e. evaluate M_q) when f is multiplicative. We shall require some preliminaries. For $1 \le q < 2$, $a \in (0,1)$ and $x \ge 0$, define the functions

$$h_q(a,x) = ax^q + x^{q-1} - a - x$$
$$L_q(a,x) = \frac{1 + 2ax + x^2}{(1+x^q)^{2/q}}.$$

Note that $h_q(a,1) = 0$, and for $x > 0$, $h_q(a, \frac{1}{x}) = -x^{-q}h_q(a,x)$ and $L_q(a, \frac{1}{x}) = L_q(a,x)$.

Lemma 3.1

Fix $q \in (1,2)$ and $a \in (0,1)$ and let $\gamma = \frac{2}{q} - 1$, so that $\gamma \in (0,1)$. Then

(a) if $a \ge \gamma$, then $h_q(a,x) < 0$ in $[0,1)$;

(b) if $a < \gamma$, then $h_q(a,x)$ has precisely one root in $[0,1)$.

Proof. We have $h_q(a,0) = -a < 0$, $h_q(a,1) = 0$ and $h_q'(a,1) = q(a-\gamma)$. Thus we have a zero at 1 in any case, while if $a < \gamma$ we must have (at least) one more in $(0,1)$. But also

$$h_q''(a,x) = q(q-1)x^{q-3}(ax - \gamma).$$

If $a < \gamma$, then h is concave in $[0,1]$ and so there is precisely one zero in $(0,1)$. If $a \ge \gamma$, then h' is decreasing on $[0, \frac{\gamma}{a}]$ and increasing on $[\frac{\gamma}{a}, 1]$. Thus

$$\min_{0 \le x \le 1} h_q'(a,x) = h_q'\Big(a, \frac{\gamma}{a}\Big) = \Big(\frac{a}{\gamma}\Big)^{2-q} - 1 \ge 0$$

and so $h_q(a,x)$ is (strictly) increasing in $[0,1]$.

□

Now let $r_q(a)$ denote the unique root of $h_q(a,x)$ in $(0,1)$ for $a < \gamma$. Thus

$$r_q(a)^{q-1} = \frac{a + r_q(a)}{1 + ar_q(a)}. \tag{3.1}$$

Also extend to $(0,1)$ by defining $r_q(a) = 1$ for $\gamma \le a < 1$. Let

$$Q_q(a) = \sup_{x \ge 0} L_q(a,x) = \max_{0 \le x \le 1} L_q(a,x).$$

Since $L_q'(a,x) = -\frac{2h_q(a,x)}{(1+x^q)^{2/q}}$, it is quickly seen that for $q > 1$, $Q_q(a) = L_q(a, r_q(a))$ while

$$Q_1(a) = \begin{cases} 1 & \text{if } a \le 1 \\ \frac{1+a}{2} & \text{if } a > 1 \end{cases}.$$

Lemma 3.2
Fix $a \in (0,1)$. Then, as $q \to 1+$, $r_q(a) \to 0$. More precisely, for $a < \gamma = \frac{2}{q} - 1$,

$$r_q(a) \le \frac{q-1}{1-aq}.$$

Hence (3.1) *implies $r_q(a)^{q-1} \to a$ as $q \to 1+$.*

Proof. For $a < \gamma$, $h_q(a,x)$ has one turning point in $(0,1)$, say at $s(a)$. This is necessarily a maximum and $r(a) < s(a)$. We have $aqs(a)^{q-1} + (q-1)s(a)^{q-2} = 1$. In particular, $1 \le (q-1)s(a)^{q-2} + aq$. Thus

$$r(a) \le r(a)^{2-q} \le s(a)^{2-q} \le \frac{q-1}{1-aq}.$$

□

Proposition 3.3
Let f be multiplicative and positive on $D(k)$. Then, with $\gamma = \frac{2}{q} - 1$

$$M_q = \prod_{p|k} Q_q(f(p)) = \prod_{f(p)<\gamma} Q_q(f(p)) \prod_{f(p)\ge\gamma} \frac{1+f(p)}{2^\gamma}.$$

In particular for $q = 1$,

$$M_1 = \prod_{f(p)>1} \frac{1+f(p)}{2}.$$

Proof. For $x = (x_d)$ such that $\|x\|_q = 1$, we may write

$$x_d = \frac{g(d)}{G(k)}, \qquad \text{where } g \ge 0 \text{ is multiplicative and } G(k) = (\textstyle\sum_{d|k} g(d)^q)^{1/q}.$$

We recall from [6] that with $F \otimes G$ defined on $D(k)$ by $(F \otimes G)(n) = \sum_{d|k} F(d)G(n \circ d)$, then $(F \tilde{\otimes} G)(n) := \frac{(F \otimes G)(n)}{(F \otimes G)(1)}$ is multiplicative whenever F and G are, provided that $(F \otimes G)(1) \ne 0$. Further, $(F \tilde{\otimes} G)(p) = \frac{F(p)+G(p)}{1+F(p)G(p)}$ for a prime p. As such,

$$\begin{aligned}
\langle A_f x, x\rangle &= \frac{1}{G(k)^2} \sum_{c,d|k} f(c \circ d) g(c) g(d) = \frac{1}{G(k)^2} \sum_{d|k} g(d)(f \otimes g)(d) \\
&= \frac{1}{G(k)^2} \sum_{c|k} f(c)g(c) \sum_{d|k} g(d)(f \tilde{\otimes} g)(d) \\
&= \prod_{p|k} \left\{ \frac{1+f(p)g(p)}{(1+g(p)^q)^{2/q}} \cdot (1 + g(p)(f \tilde{\otimes} g)(p)) \right\} \qquad \text{(by multiplicativity)} \\
&= \prod_{p|k} \left\{ \frac{1+2f(p)g(p)+g(p)^2}{(1+g(p)^q)^{2/q}} \right\} = \prod_{p|k} L_q(f(p), g(p)).
\end{aligned}$$

In order to maximize this, we maximize each factor independently of the others. Since there is no restriction on $g(p)$, we need to maximize $L_q(f(p),t)$ over t in (0,1). Thus we take $g(p) = r_q(f(p))$ giving the maximum $Q_q(f(p))$, and so

$$M_q = \prod_{p|k} Q_q(f(p)).$$

The second formula follows on using $Q_q(f(p)) = \frac{1+f(p)}{2^\gamma}$ whenever $f(p) \geq \gamma$.

□

Remarks 2

(a) Note that if $f(p) < 1$ for each $p|k$ then, for q close to 1, $g(p)^{q-1} = f(p) + O(q-1)$ by Lemma 3.2 and, by multiplicativity, $g(d)^{q-1} = f(d) + O(q-1)$.

(b) From the formula for M_1 we can show that the maximum need *not* necessarily occur at a 'multiplicative' point, even if f is multiplicative. As an example, take $k = 6$ and let f be multiplicative with $f(2), f(3) > 1$. Then

$$M_1 = \frac{(1+f(2))(1+f(3))}{4}.$$

But at $x = (\frac{1}{2}, 0, 0, \frac{1}{2})$, $\langle A_f x, x\rangle = \frac{1+f(2)f(3)}{2}$, which is larger. (Indeed this can be shown to be the maximum.) By continuity, for this f, $M_q < \Lambda_q$ if q is a little larger than 1.

§4. Proof of Theorems 1 and 2

Proof of Theorem 1. We need only consider $q < 2$. Since Λ_q varies continuously with q and $\Lambda_2 = F(k)$, we must have

$$\Lambda_q = F(k) + o(1) \qquad \text{as } q \to 2-.$$

Let x be such that $\|x\|_q = 1$ and $x_1 \geq x_d$ without loss of generality. Then we have

$$1 = \sum_{d|k} x_d^q \leq x_1^q d(k),$$

so that $x_1 \geq d(k)^{-1/q} = \frac{1}{\sqrt{d(k)}} + o(1)$. Now put $d = 1$ in (2.1). Thus

$$\sum_{c|k} f(c)x_c = \Lambda_q x_1^{q-1} \sim F(k)x_1.$$

It follows that, for every $d|k$,

$$0 \leq f(d)(x_1 - x_d) \leq \sum_{c|k} f(c)(x_1 - x_c) = F(k)x_1 - \Lambda_q x_1^{q-1} \to 0$$

as $q \to 2-$. Thus $x_d = x_1 + o(1)$ for every $d|k$. We may therefore write

$$x_d = x_1 e^{-\eta_d}, \qquad \text{where } 0 \leq \eta_d \to 0 \text{ as } q \to 2-.$$

Let $\eta = \max_{d|k} \eta_d$ and $H = \frac{1}{d(k)} \sum_{d|k} \eta_d$. Note that $H \leq \eta$, and $\eta \to 0$ as $q \to 2-$. Then

$$1 = \sum_{d|k} x_d^q = x_1^q \sum_{d|k} e^{-q\eta_d} = x_1^q \sum_{d|k} (1 - q\eta_d + O(\eta^2)) = x_1^q d(k)(1 - qH + O(\eta^2)).$$

Thus

$$x_1 = \frac{1 + H + O(\eta^2)}{d(k)^{1/q}}. \tag{4.1}$$

Next,

$$\begin{aligned}\Lambda_q &= x_1^2 \sum_{c,d|k} f(c\circ d)e^{-\eta_c-\eta_d} = x_1^2 \sum_{c,d|k} f(c\circ d)(1-\eta_c-\eta_d+O(\eta^2))\\ &= x_1^2\Big(\sum_{c|k}\sum_{d|k} f(c\circ d) - 2\sum_{c|k}\eta_c\sum_{d|k} f(c\circ d) + O(\eta^2)\Big)\\ &= x_1^2 F(k)d(k)(1-2H+O(\eta^2)).\end{aligned}$$

Inserting (4.1) gives

$$\Lambda_q = F(k)d(k)^{1-2/q}(1+O(\eta^2)). \tag{4.2}$$

Now, with $d=1$ in (2.1), and dividing through by x_1,

$$\Lambda_q x_1^{q-2} = \sum_{c|k} f(c)e^{-\eta_c} = \sum_{c|k} f(c)(1-\eta_c+O(\eta^2)) = F(k) - \sum_{c|k} f(c)\eta_c + O(\eta^2).$$

Rearranging and inserting (4.1) and (4.2),

$$\begin{aligned}\sum_{c|k} f(c)\eta_c &= F(k) - \Lambda_q x_1^{q-2} + O(\eta^2) = F(k) - F(k)(1+(q-2)H) + O(\eta^2)\\ &= (2-q)HF(k) + O(\eta^2) \le (2-q)\eta F(k) + O(\eta^2).\end{aligned}$$

But the left-hand side is at least $f(d)\eta$ for some d. If $\eta > 0$, we may divide through to get

$$f(d) \le (2-q)F(k) + O(\eta).$$

This is a contradiction for all q sufficiently close to 2. Thus $\eta = 0$ and x_d is constant.

□

For the proof of Theorem 2, we first determine the asymptotic behaviour of the solution and Λ_q as $q \to 1$. For the following result we do not require f to be multiplicative, only to be bounded by 1.

Proposition 4.1
Let $f : D(k) \to (0,1]$ such that $f(d) = 1$ at $d = 1$ only. Then, at the optimal, as $q \to 1+$

$$\Lambda_q = 1 + O(q-1) \quad \text{and} \quad x_d^{q-1} = f(d) + O(q-1).$$

Proof. Since $f \le 1$, we have for $\|x\|_q = 1$,

$$1 \le \Lambda_q \le \Big(\sum_{d|k} x_d\Big)^2 \le \Big(\sum_{d|k} x_d^q\Big)^{\frac{2}{q}}\Big(\sum_{d|k} 1\Big)^{2(1-\frac{1}{q})} = d(k)^{\frac{2(q-1)}{q}} = 1 + O(q-1).$$

Also $1 = \sum_{d|k} x_d^q \le d(k)x_1^q \le d(k)x_1$, so that $\frac{1}{d(k)} \le x_1 \le 1$ and hence $x_1^{q-1} = 1 + O(q-1)$. Now (2.1) with $d = 1$ implies

$$\sum_{c|k} f(c)x_c = \Lambda_q x_1^{q-1} = 1 + O(q-1).$$

But $\sum_{c|k} x_c = 1 + O(q-1)$ also, and subtracting gives

$$\sum_{c|k}(1-f(c))x_c = O(q-1).$$

As $f(c) < 1$ whenever $c > 1$, we see that $x_d = O(q-1)$ for each $d > 1$, and hence $x_1 = 1+O(q-1)$. This implies

$$\Lambda_q x_d^{q-1} = \sum_{c|k} f(c\circ d)x_c = f(d) + O(q-1),$$

with $c = 1$ giving the main term. Thus $x_d^{q-1} = f(d) + O(q-1)$ as required.

□

Proof of Theorem 2. Again we may assume that at the optimal solution $x_1 \geq x_d > 0$ for all $d|k$. We shall also assume that $q > 1$, the $q = 1$ case being trivial, so that the method of Lagrange multipliers is valid and equations (2.1) hold.

These may be rewritten by letting $h(d) = \frac{x_d}{x_1}$ as follows. Then dividing (2.1) through by the $d = 1$ case gives

$$h(d)^{q-1} \sum_{c|k} f(c)h(c) = \sum_{c|k} f(c \circ d)h(c) \quad \text{or} \quad h(d)^{q-1} = (f \tilde{\otimes} h)(d). \tag{4.3}$$

The aim is now to show that $h(d) = g(d)$, where $g(d)$ is the optimal chosen in the multiplicative case in Proposition 3.3. There we found that

$$g(p)^{q-1} = \frac{f(p) + g(p)}{1 + f(p)g(p)} = (f \tilde{\otimes} g)(p).$$

Since f and g are multiplicative, it follows that

$$g(d)^{q-1} = (f \tilde{\otimes} g)(d).$$

Thus $g(d)$ also satisfies (4.3).

Furthermore, both $g(d)^{q-1} = f(d) + O(q-1)$ and $h(d)^{q-1} = f(d) + O(q-1)$ as $q \to 1+$ (from Remarks 2(a) and Proposition 4.1 respectively). Thus $h(d) \asymp g(d) \asymp f(d)^{\frac{1}{q-1}}$ and we may write

$$h(d) = g(d)e^{\eta_d},$$

where $\eta_d = O(1)$. As such, (4.3) becomes

$$\sum_{c|k} \Big(f(c \circ d) - f(c)g(d)^{q-1}e^{\eta_d(q-1)} \Big) h(c) = 0.$$

Splitting $e^{\eta_d(q-1)}$ into $1 + (e^{\eta_d(q-1)} - 1)$ and using (4.3) for g leads to

$$\sum_{c|k} \Big(f(c \circ d) - f(c)g(d)^{q-1} \Big) g(c)(e^{\eta_c} - 1) = g(d)^{q-1}(e^{\eta_d(q-1)} - 1) \sum_{c|k} f(c)h(c). \tag{4.4}$$

Choose d such that $|\eta_d| \geq |\eta_c|$ for all $c|k$ and suppose for a contradiction that $|\eta_d| > 0$. Then the RHS in (4.4) is, in modulus, at least

$$g(d)^{q-1}|e^{\eta_d(q-1)} - 1| \sim f(d)|\eta_d|(q-1).$$

But on the left of (4.4), the $c = 1$ term is zero, while for $c > 1$, $g(c)$ is *exponentially* small, as $g(c)^{q-1} \to f(c) < 1$. Thus the LHS of (4.4) is, in modulus,

$$\ll |\eta_d| \sum_{c>1} g(c) \ll |\eta_d| (\max_{c>1} f(c))^{\frac{1}{q-1}} = o(|\eta_d|(q-1)).$$

We have our desired contradiction, and so $h = g$, making h multiplicative.

□

§5. Problem transposed into one of norms

If A_f is positive definite, which is our main interest, then $A_f = B^*B$ for some B, so that $\langle A_f x, x \rangle = \|Bx\|^2$ and the problem becomes one of evaluating the norm

$$\|B\|_{q,2} = \sup_{x \neq 0} \frac{\|Bx\|_2}{\|x\|_q}.$$

Such norms are generally difficult to find, there being no general formulae. Indeed, for bounded linear operators $\varphi : l^p \to l^q$, a general formula (in terms of the associated matrix entries) is only known for the cases $p = 1$ or $q = \infty$ (see for example [8], Chapter 4).

Now if f is multiplicative, then A_f is positive definite precisely when $f(p) \in (-1, 1)$ for all $p|k$. We can give a precise form for B in this case. We require some concepts from [6].

Every $f : D(k) \to \mathbb{C}$ has a Fourier series

$$f(n) = \frac{1}{d(k)} \sum_{\chi \in \hat{D(k)}} \widehat{f}(\chi)\chi(n),$$

where χ ranges over the characters of $D(k)$ and $\widehat{f}(\chi)$ are the Fourier coefficients of f, given by

$$\widehat{f}(\chi) = \sum_{d|k} \chi(d)f(d) \quad \Big(= \prod_{p|k}(1 + \chi(p)f(p)) \text{ if } f \text{ is multiplicative}\Big).$$

If $\widehat{f}(\chi) \geq 0$ for all χ, we may define for $\alpha > 0$,

$$f^{\otimes\alpha}(n) = \frac{1}{d(k)} \sum_{\chi \in \hat{D(k)}} \widehat{f}(\chi)^\alpha \chi(n). \tag{5.1}$$

Equivalently, we may write $A_f = U^* D U$ where U is the unitary matrix with entries $(\chi(d))_{d|k, \chi \in \widehat{D(k)}}$ and $D = \operatorname{diag}(\hat{f}(\chi))_{\chi \in \widehat{D(k)}}$, in which case $A_f^\alpha = A_{f^{\otimes\alpha}}$.

Also let $f^{\tilde{\otimes}\alpha}(n) = \frac{f^{\otimes\alpha}(n)}{f^{\otimes\alpha}(1)}$ whenever the denominator is non-zero.

Proposition 5.1
Let f be multiplicative on $D(k)$ such that $0 < f(p) < 1$ for all primes $p|k$. Then $f^{\tilde{\otimes}\alpha}$ is multiplicative for every $\alpha > 0$, and furthermore for each $n|k$,

$$f^{\tilde{\otimes}\alpha}(n) = \prod_{p|n} \frac{(1 + f(p))^\alpha - (1 - f(p))^\alpha}{(1 + f(p))^\alpha + (1 - f(p))^\alpha}.$$

Proof. Denote the $d(k)$ characters of $\widehat{D(k)}$ by $\chi_d(\cdot) = \mu((\cdot, d))$ where $d|k$ and $\mu(\cdot)$ is the Möbius function. We prove by induction on $w(k)$ (the number of prime factors of k) that

$$f^{\otimes\alpha}(n) = \frac{1}{d(k)} \prod_{p|k} \Big\{(1 + f(p))^\alpha + \chi_p(n)(1 - f(p))^\alpha\Big\}. \tag{5.2}$$

For if (5.2) holds, then dividing through by the $n = 1$ case and using $\chi_p(n) = -1$ if $p|n$ and 1 otherwise, gives the result.

Now if $w(k) = 2$, then k is prime and $\widehat{D(k)}$ consists of two characters 1 and μ. Thus by (5.1)

$$f^{\otimes\alpha}(n) = \frac{1}{2}(\widehat{f}(1)^\alpha 1(n) + \widehat{f}(\mu)^\alpha \mu(n)) = \frac{1}{2}\Big((1 + f(k))^\alpha + \mu(n)(1 - f(k))^\alpha\Big)$$

which is the RHS of (5.2).

For the inductive step, suppose (5.2) holds for some k squarefree and all $n|k$. Let q be prime and such that $q \not| k$, and consider (5.2) for qk.

Observe that (i) $D(qk) = D(k) \cup qD(k)$ (since every divisor $d|qk$ satisfies either $d|k$ or $d = qd'$, $d'|k$), and (ii) $\chi \in \widehat{D(qk)} \Leftrightarrow \chi = \chi_d$ or $\chi = \chi_{qd} = \chi_q \chi_d$ for $d|k$ since $(q, d) = 1$.

Thus for $\chi \in \widehat{D(qk)}$, we have

$$\begin{aligned}
\widehat{f}(\chi) &= \prod_{p|qk}(1 + \chi(p)f(p)) = (1 + \chi(q)f(q)) \prod_{p|k}(1 + \chi(p)f(p)) \\
&= \begin{cases} (1 + f(q)) \prod_{p|k}(1 + \chi(p)f(p)) & \text{if } \chi = \chi_d \\ (1 - f(q)) \prod_{p|k}(1 + \chi(p)f(p)) & \text{if } \chi = \chi_{qd} \end{cases} \quad (d|k),
\end{aligned}$$

using the fact that $\chi_q(p) = 1$ if $p|k$ and -1 otherwise. Thus

$$\begin{aligned}
\sum_{\chi \in \widehat{D(qk)}} \chi(n)\widehat{f}(\chi)^\alpha &= \sum_{\chi \in \widehat{D(k)}} \chi(n)(1+f(q))^\alpha \prod_{p|k}(1+\chi(p)f(p))^\alpha \\
&+ \sum_{\chi \in \widehat{D(k)}} \chi_q(n)\chi(n)(1-f(q))^\alpha \prod_{p|k}(1+\chi(p)f(p))^\alpha \\
&= \Big((1+f(q))^\alpha + \chi_q(n)(1-f(q))^\alpha\Big) \sum_{\chi \in \widehat{D(k)}} \chi(n) \prod_{p|k}(1+\chi(p)f(p))^\alpha \\
&= \Big((1+f(q))^\alpha + \chi_q(n)(1-f(q))^\alpha\Big) \prod_{p|k}\Big\{(1+f(p))^\alpha + \chi_p(n)(1-f(p))^\alpha\Big\} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by assumption)} \\
&= \prod_{p|qk}\Big\{(1+f(p))^\alpha + \chi_p(n)(1-f(p))^\alpha\Big\}.
\end{aligned}$$

□

Note also that $0 < f^{\tilde{\otimes}\alpha}(p) < 1$ for all $p|k$.

It follows from Proposition 5.1 that for f multiplicative on $D(k)$ satisfying $0 < f(p) < 1$ for $p|k$, we have

$$A_f = A_g^2 = g(1)^2 A_h^2,$$

where $g = f^{\otimes\frac{1}{2}}$ and h is the multiplicative function $f^{\tilde{\otimes}\frac{1}{2}}$. Thus

$$\Lambda_q = f^{\otimes\frac{1}{2}}(1)^2 \|A_h\|_{q,2}^2,$$

and an equivalent problem is therefore to evaluate $\|A_h\|_{q,2}$ for a general multiplicative function h.

As such, let $h_p : D(k) \to (0,\infty)$ denote the function restricted to $D(p)$; i.e.

$$h_p(n) = \begin{cases} h(n) & \text{if } n = 1, p \\ 0 & \text{otherwise} \end{cases}.$$

Using the above relation to Λ_q, it is readily seen[1] that $\|A_{h_p}\|_{q,2} = \sqrt{1+h(p)^2}\sqrt{Q_q(f(p))}$ with Q_q as in section 3. But also Proposition 3.3 gives

$$\max\Big\{\frac{\|A_h x\|_2}{\|x\|_q} : x \text{ is multiplicative}\Big\} = \frac{\sqrt{M_q}}{f^{\otimes\frac{1}{2}}(1)} = \prod_{p|k} \|A_{h_p}\|_{q,2},$$

by using (5.2). On replacing h by f, the conjecture (made after the statement of Theorem 2) is therefore equivalent to

Conjecture: Let f be multiplicative on $D(k)$ such that $0 < f(p) < 1$ for all $p|k$. Then

$$\|A_f\|_{q,2} = \prod_{p|k} \|A_{f_p}\|_{q,2}, \tag{5.3}$$

and the norm is achieved at a multiplicative point.

Note that since $A_f = \prod_{p|k} A_{f_p}$ (see Theorem 3.3, [6]), (5.3) may equally be written as

$$\Big\|\prod_{p|k} A_{f_p}\Big\|_{q,2} = \prod_{p|k} \|A_{f_p}\|_{q,2}.$$

[1]Use the formula $\sqrt{1+f(p)} + \sqrt{1-f(p)} = \frac{2}{1+h(p)^2}$.

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