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Abstract

Simple first-order closure remains an attractive way of formulating equations for complex canopy flows when the aim is to find analytic or simple numerical solutions to illustrate fundamental physical processes. Nevertheless, the limitations of such closures must be understood if the resulting models are to illuminate rather than mislead. We propose five conditions that first-order closures must satisfy then test two widely used closures against them. The first is the eddy diffusivity based on a mixing length. We discuss the origins of this approach, its use in simple canopy flows and extensions to more complex flows. We find that it satisfies most of the conditions and, because the reasons for its failures are well understood, it is a reliable methodology. The second is the velocity-squared closure that relates shear stress to the square of mean velocity. Again we discuss the origins of this closure and show that it is based on incorrect physical principles and fails to satisfy any of the five conditions in complex canopy flows; consequently its use can lead to actively misleading conclusions.

1. Introduction

At present over 400 FLUXNET ‘eddy-flux’ tower sites are operated on a long-term and continuous basis in order to infer net exchange of energy, carbon dioxide and other trace gases between the local ecosystem and the atmosphere (http://fluxnet.ornl.gov). In many cases such interpretation is confounded by the impacts of complex terrain on the atmospheric transport of matter, momentum and energy (Schimel et al., 2008; Finnigan 2008). Flows through fragmented forest canopies are also of great interest in the context of wind damage while urban canopies with changing building density define the conditions for urban pollution spread and the wind environment of cities. To understand these complex canopy flows at a fundamental level there is a need for simple (first-order) flow and transport models that allow analytic solutions or simple numerical computations that are not opaque 'black boxes' (e.g. Belcher et al., 2003; Finnigan and Belcher, 2004; Yi et al., 2005; Katul et al., 2006; Yi, 2008; Ross, 2012). It is important that the fluid mechanical principles underpinning such models be sound as the community increasingly relies on them for data interpretation. Although modelling is the primary motivation for this paper it is worth noting that several authors recently have used mixing lengths to interpret observed canopy flow statistics (eg., Poggi et al., 2004; Bai et al., 2012). The use of mixing lengths and eddy viscosities in canopy flows was common before the role of large eddy structure and the consequent absence of local equilibrium in the canopy turbulent kinetic energy and stress budgets was understood (Finnigan, 2000) but in fact the use of such concepts is very circumscribed as we shall show so the development below is also relevant to experimental studies.

Before going further, it is necessary to be more precise about what we mean by simple and complex canopy flows. All canopy flows are microscopically complex as the flow threads its way through the foliage airspace but at a macroscopic scale encompassing many plants, simple canopies are only heterogeneous in the vertical. So by a simple canopy flow we mean a statistically stationary flow in a horizontally homogeneous canopy on level ground. Such a flow has only one element in its mean rate-of-strain tensor: the vertical shear in the mean wind. In contrast, by complex canopy flows we mean flows in canopies on hills or with rapidly varying foliage area density such as gaps and clearings or flows that are unsteady on time scales comparable to the integral time scales of the turbulence. Such flows can exhibit mean strain rates along all three space axes. In this note we first discuss the fundamental requirements that simple first-order closures for models applied to complex canopy flows must satisfy. Then we review whether the closure schemes employed by two groups of models-mixing length closures eg. Finnigan and Belcher (2004) or Ross (2012) and the velocity-squared closure eg Yi et al. (2005, 2008)-satisfy these requirements and illustrate how failure to do so can lead to incorrect results or inferences.

We find that mixing-length-based eddy diffusivity closures satisfy the fundamental criteria most of the time. However, they fail to simulate canopy flows well in situations where foliage density changes significantly over length scales shorter than the integral length scale of the turbulence or where the turbulence is strained rapidly relative to its integral time scale. In the first case non-local effects degrade the relationship between local turbulent stress and local rate of mean straining and in the second, viscoelastic effects introduce 'memory' of earlier straining into the turbulent stress response. Nevertheless, in both cases, the nature of the closure ensures that the turbulent flow is modelled as a Stokesian fluid so that basic thermodynamic relationships are preserved.

The velocity-squared closure, in contrast, fails to give physically realistic results even in the simple canopy shear flows for which it was originally derived. In particular, it predicts that the canopy velocity profile is independent of canopy element drag coefficient. In more complex flows where both streamwise pressure gradients and shear stress gradients are combined, we find that the closure inevitably co-locates maxima in shear stress and maxima in mean velocity whereas in reality, shear stress maxima are found close to maxima in velocity gradient, independent of the velocity magnitude, which may be close to zero there. As a result, the closure fails to satisfy the fundamental criteria and contradicts basic thermodynamic requirements. We conclude that it is generally unsafe to use this closure in simple models.

2. Turbulence closure in complex canopy flows

The equations that describe flow in the canopy airspace are derived using a 'double averaging' technique with successive application of time and spatial averaging to the Navier Stokes equations (Raupach and Shaw, 1982; Finnigan, 1985; Brunet et al., 1994; Finnigan and Shaw, 2008). It is now usual to perform the spatial averaging over thin slabs confined between coordinate surfaces that follow the ground contour, such as surface-following or streamline coordinates. With the assumption that the canopy is laterally uniform on a scale much larger than the plants, choosing thin slabs as the averaging volume allows the averaged variables to reflect the characteristic vertical heterogeneity of the canopy but to smooth out smaller scale spatial fluctuations caused as air flows around leaves, stems and branches. The resulting time and space-averaged momentum and continuity equations are,

$$\frac{\partial\langle\bar{u}_i\rangle}{\partial t}+\langle\bar{u}_j\rangle\frac{\partial\langle\bar{u}_i\rangle}{\partial x_j}+\frac{\partial\langle\bar{u}_i''\bar{u}_j''\rangle}{\partial x_j}+\frac{\partial\langle\overline{u_i'u_j'}\rangle}{\partial x_j}=-\frac{\partial\langle\bar{p}\rangle}{\partial x_i}-\left\langle\frac{\partial\bar{p}''}{\partial x_i}\right\rangle+\nu\frac{\partial^2\langle\bar{u}_i\rangle}{\partial x_j\partial x_j}+\nu\left\langle\frac{\partial^2\bar{u}_i''}{\partial x_j\partial x_j}\right\rangle \quad a$$

$$\frac{\partial\langle\bar{u}_i\rangle}{\partial x_i}=\frac{\partial\bar{u}_i''}{\partial x_i}=\frac{\partial\bar{u}_i}{\partial x_i}=\frac{\partial u_i'}{\partial x_i}=0 \quad b \qquad (1)$$

where we use a right handed Cartesian coordinate system x_i with x_1 in the streamwise direction and x_3 normal to the ground surface; u_i is the corresponding velocity vector, ν is the kinematic viscosity and *p* the kinematic pressure departure from a hydrostatic reference state. In equation (1) and in the rest of this paper we ignore diabatic effects in the flow. The overbar denotes the time average with single primes the instantaneous departures from that average while angle brackets denote the spatial average with double primes the local departures from that average. In (1) we have ignored terms compensating for the volume fraction occupied by solids in the canopy space because these are negligible in natural vegetation although they must be included in urban canopies and some wind tunnel models (e.g. Böhm et al., 2013).

Just as in the conventional Reynolds equations for time or ensemble averaged velocity, in solving or modelling (1) we confront a closure problem because, even after using continuity (1)*b* to eliminate the pressure, equation (1)*a* contains terms that cannot be expressed as functions of $\langle\overline{u_i}\rangle$ without further assumptions. The two such terms on the right hand side of (1)*a* can be shown to correspond to F_{Di}, the aerodynamic drag exerted by the canopy on the air in the averaging volume. It is usual in the high Reynolds number conditions of natural canopies to parameterise the total aerodynamic drag as if it were all exerted by pressure forces and so proportional to the square of the windspeed past the canopy elements; hence we write,

$$F_{Di}=-\left\langle\frac{\partial\bar{p}''}{\partial x_i}\right\rangle+\nu\left\langle\frac{\partial^2\bar{u}_i''}{\partial x_j^2}\right\rangle=-c_d\,a\left\langle\overline{(u_i-v_i)|u_i-v_i|}\right\rangle \qquad (2)$$

where c_d is a dimensionless drag coefficient, *a* is the foliage area per unit volume of space and v_i is the canopy element velocity. It is apparent, however that equation (2) as it stands does not constitute a first order closure because the quadratic term,

$\left\langle \overline{(u_i - v_i)|u_i - v_i|} \right\rangle$ contains a mixture of mean and turbulent velocity fluctuations. As a result, the parameterization (2) is usually simplified to,

$$F_{Di} = -c_d\, a \left\langle \overline{(u_i - v_i)} \right\rangle \left| \left\langle \overline{(u_i - v_i)} \right\rangle \right| \tag{3}$$

which ensures that the drag force is a vector proportional to the square of the wind velocity relative to the foliage and always directed against the wind.

The inclusion of the element velocity, v_i in (3) serves two purposes. First, it allows us to deal with flexible canopies that wave in the wind. Although in most practical examples flexibility effects can be neglected, they can be important in the case of coherently waving canopies (de Langre, 2008; Dupont et al., 2010; Finnigan, 1985; Finnigan, 2010). Second, it satisfies the requirement that the parameterized equation be Galilean invariant, that is, that the equation remain physically correct if the axes are translated with constant velocity. For the case of axes fixed in space and rigid canopy elements, $v_i = 0$ and for simplicity, in the rest of this paper, we will assume this to be the case. Comparing equations (2) and (3) we see that parameterization (3) leaves some residual dependence on turbulent intensity and scale in the drag coefficient c_d as well as on canopy element Reynolds number because at lower wind speeds the viscous term, which varies as $u_i^{3/2}$, can make a significant contribution to the drag. These points are discussed at length in Brunet et al. (1994).

The terms on the left hand side of (1) that require closure assumptions are the first two terms in the expression for the total kinematic stress tensor,

$$\sigma_{ij} = -\left\langle \overline{u}_i'' \overline{u}_j'' \right\rangle - \left\langle \overline{u_i' u_j'} \right\rangle + \nu \left[\frac{\partial \langle \overline{u}_i \rangle}{\partial x_j} + \frac{\partial \langle \overline{u}_j \rangle}{\partial x_i} \right] - \langle \overline{p} \rangle \delta_{ij} \tag{4}$$

where δ_{ij} is the Kronecker delta and we have followed standard manipulations (eg. Hinze, 1975) and written the mean viscous stress as the product of the kinematic viscosity and the mean rate of strain tensor, to ensure invariance under exchange of indices. The dispersive stresses $\left\langle \overline{u}_i'' \overline{u}_j'' \right\rangle$ bear the same relationship to spatial averaging as the Reynolds stresses, $\overline{u_i' u_j'}$ do to time averaging but are almost always, when modelling, added to the Reynolds stresses to form the total ‘turbulent’ stress. For a fuller discussion see Finnigan (1985) and Finnigan and Shaw (2008). In what follows, for simplicity we combine Reynolds and dispersive stresses as the total turbulent stress,

$$\sigma_{ij}^{t} = -\left\langle \overline{u}_i'' \overline{u}_j'' \right\rangle - \left\langle \overline{u_i' u_j'} \right\rangle \qquad (5)$$

In atmospheric flows the mean viscous stress, $\nu\left[\partial\left\langle\overline{u_i}\right\rangle/\partial x_j + \partial\left\langle\overline{u_j}\right\rangle/\partial x_i\right]$ is two or three orders of orders of magnitude smaller than the other terms in (4) and so is usually neglected.

It is useful at this point to outline a set of requirements that any closure for the turbulent stress needs to satisfy to be applicable to canopy flows in complex terrain. These fundamental requirements are essentially the same as those that apply to higher order closures as set out in detail in Lumley (1978), namely that the closure must be:

I. coordinate invariant and material frame indifferent
II. unambiguous
III. complete
IV. ensure that the net rate of working of the turbulent stresses against the mean rate of strain over the region of turbulent flow is negative
V. not imply unphysical results.

The components of the kinematic stress, σ_{ij}, (4) form a second order tensor. Condition I is satisfied if the terms in (4) that contain the parameterized turbulent and dispersive stresses have the correct tensor form and exhibit Galilean invariance. These properties are critical as coordinate transformation is routinely used to simplify the equations of motion when dealing with flow over complex terrain. Condition II ensures that no additional information is required to define the parameterized terms other than what is contained in the first order moments. Completeness, III, is required because in complex terrain flows, all the components of the stress tensor need to be specified, even if some are later discarded should the equation be simplified using other criteria. Condition IV follows from the thermodynamics of irreversible processes (de Groot and Mazur, 1962) where the second law of thermodynamics demands that the net rate of working of the turbulent stresses against the mean rates of strain over the turbulent domain, V_t must constitute a sink of mean kinetic energy and a source of turbulent kinetic energy. Mathematically this implies that,

$$\iint_{V_t} \sigma^{t}_{ij} \frac{\partial\langle\overline{u_i}\rangle}{\partial x_j} dVdt < 0 \tag{6}$$

where it also assumed implicitly that the averaging time is much longer than any eddy timescale. Finally, condition V is included as an overall check on physical reality.

First order closure of (1) means that the second order terms, which appear after averaging, are parameterized in terms of the first order moments. Conventionally, though not exclusively, in first order closure the anisotropic part of the stress tensor is assumed to be proportional to S_{ij}, the mean rate-of-strain tensor,

$$\left(\sigma^{t}_{ij} - \frac{1}{3}\sigma^{t}_{kk}\delta_{ij}\right) = -2KS_{ij} = -K\left(\frac{\partial\langle\overline{u_i}\rangle}{\partial x_j} + \frac{\partial\langle\overline{u_j}\rangle}{\partial x_i}\right) \tag{7}$$

eg. Wyngaard (1982). The constant of proportionality, K, is known as the eddy diffusivity. As discussed in detail in Appendix A, this parameterization treats the time- and space-averaged turbulent flow as if it were a Stokesian fluid (Aris, 1962). First order closure was originally introduced in the context of parallel shear flows, where *a priori* only one component of the turbulent stress, σ^{t}_{13}, and one component of the deformation tensor, $\partial\langle\overline{u_1}\rangle/\partial x_3$, need be considered. The application of first order closure to complex two- and three-dimensional flows, such as we encounter over hills or around forest edges, reached a turning point at the AFOSR-IFP-Stanford Conference on Computation of Turbulent Boundary Layers in 1968 (Kline et al., 1969), where the observed limitations of such approaches switched attention to so-called second and even higher-order closures (Lumley, 1978). Despite this history, and the success of higher-order closure schemes in progressing the understanding of the mechanics of turbulence (Wyngaard, 2010), first-order closure of complex flows remains attractive primarily because it may allow analytical solutions of relatively complex problems with all the explanatory power such solutions confer or at least simple numerical solutions that do not become difficult-to-interpret ‘black boxes’.

Nevertheless, at least since the AFOSR-IFP-Stanford Conference it has been known that eddy diffusivity or ‘*K*-theory’ closures fail to predict the stress tensor reliably in complex flows. This weakness stems from three causes, all of which are important in complex canopy flows. The first follows from the fact that the eddy diffusivity is a scalar, implying that the principal axes of the stress tensor and rate of strain tensor are coincident. In many unsteady or spatially inhomogeneous flows this is demonstrably not so (see for example turbulent flows subjected to oscillating shear, Maxey and Hunt, 1978). In fact turbulent flows in general exhibit a viscoelastic response to straining as described by rapid distortion theory (Hunt and Caruthers, 1990) and the instantaneous stress tensor reflects the history of straining, not just the instantaneous rate of strain. Despite this, *K*-theory is now used to parameterize sub-filter scale stresses in the fundamentally unsteady and inhomogeneous ‘resolved flows’ in Large Eddy Simulations (LES) that use Smagorinsky-type closures (e.g. Sullivan and Patton, 2011). The second problem is that *K*-theory implies that the turbulent stress is determined by the local (in space) rate of strain. In reality, even in one-dimensional canopy flows, turbulent stresses and fluxes have a non-local character (e.g. Wilson and Shaw, 1977; Finnigan and Belcher, 2004). The third problem is that, if the flow field is complex, it is difficult or impossible to specify the eddy diffusivity in terms of first moments alone. Recognition of this has led researchers to develop so-called *one and a half* order closures such as the popular *k-epsilon* approach that overcome some of these problems but that preclude analytic solutions.

In the next section we discuss the application of *K*-theory to simple and complex canopy flows to illustrate the limitations these problems impose and in Appendix A, we illustrate the assumptions that are implicit when we do employ *K*-theory to model canopy flow.

3. Mixing length closures for simple and complex canopy flows

There are a variety of ways of specifying the eddy diffusivity but the best known is the concept of the mixing length. The Finnigan and Belcher (2004) group of models (Ross and Vosper 2005; Ross, 2011; Harman and Finnigan 2010, 2013) for canopy flow in complex terrain utilise this approach. While the concept of an eddy diffusivity dates back to Boussinesq (1877), a physical model for K awaited the introduction of mixing length closures by Prandtl (1925) (based on momentum) and Taylor (1932) (based on vorticity). Their models were direct analogues of molecular diffusion, with the mixing length replacing the mean free path of molecules. The concept was originally applied to simple (statistically steady, horizontally homogeneous) shear flows where only the turbulent shear stress, $-\left\langle \overline{u_1' u_3'} \right\rangle$ and the mean transverse shear, $\partial \left\langle \overline{u_1} \right\rangle / \partial x_3$ were related as,

$$-\left\langle \overline{u_1' u_3'} \right\rangle = -K \frac{\partial \left\langle \overline{u_1} \right\rangle}{\partial x_3} = -l_m^2 \left| \frac{\partial \left\langle \overline{u_1} \right\rangle}{\partial x_3} \right| \frac{\partial \left\langle \overline{u_1} \right\rangle}{\partial x_3} \qquad (8)$$

where l_m is the mixing length. So the eddy diffusivity K is the product of a length scale, l_m and a velocity scale, $l_m \left| \frac{\partial \left\langle \overline{u_1} \right\rangle}{\partial x_3} \right|$. The closure (8) applies to only one component of σ_{ij}^t (with dispersive stresses assumed negligible) and, as we shall below, cannot easily be generalized. However, it clearly satisfies the fundamental requirements II and III and also meets condition IV as the product of the shear stress and shear strain is always negative.

3.1 Mixing lengths in simple canopy flows

The analogy between mixing length and molecular mean free path implies that l_m represents the length scale of an actual process mixing momentum in the fluid. This interpretation can only hold if the mixing length is much smaller than the characteristic length over which the mean velocity gradient changes (Tennekes and Lumley, 1972;

Corrsin, 1974), i.e.

$$\left| \frac{\partial \langle \overline{u}_1 \rangle}{\partial x_3} \bigg/ \frac{\partial^2 \langle \overline{u}_1 \rangle}{\partial x_3^2} \right| << l_m \tag{9}$$

In the logarithmic surface layer it is well known that the apparent mixing length is $l_m = \kappa \left[x_3 - d \right]$ with κ, Von Karman's constant and d, the displacement height. This does not satisfy inequality (9). The reason a mixing length relationship appears to apply is a result of similarity scaling, as hypothesized by von Karman (1930) and later supported by formal asymptotic matching of wall and outer layers in shear flows (e.g. Tennekes and Lumley, 1972; Kader and Yaglom, 1978). Indeed, the derivation of the logarithmic law through asymptotic matching does not require any a priori assumption of a constant stress layer or that the mixing length be proportional to x_3. The logarithmic velocity profile is much more general, applying for example in a modified form to flows where $\partial \langle \overline{p} \rangle / \partial x_1 \neq 0$ and consequently, stress is not constant (Townsend, 1984). This illustrates that, while the physical analogy between the mixing length and the molecular mean free path does not hold even in simple turbulent shear flows (as the original authors of mixing length theory well knew; see Schlichting, 1975; pp 384 et seq.), there are some situations, such as the logarithmic layer, where similarity constraints lead to an apparent mixing length.

In canopy flow, it is known that the energy-containing canopy eddies are of large scale relative to individual canopy elements (Finnigan, 2000) and certainly do not satisfy (9). However, we can ask whether there exists any similarity-based reasoning, analogous to that producing the logarithmic law that yields a mixing length formula in canopies? Following arguments first set out by Inoue (1963), we can postulate that, if most of the streamwise mean momentum is absorbed as drag on the canopy and not at the ground surface, the relationship between velocity gradient and stress must depend on single length and velocity scales characteristic of the canopy flow. When the foliage area density a is constant, $L_c = \left(c_d \, a \right)^{-1}$ is the natural dynamic length scale for canopy flow. Physically L_c can be interpreted as the e-folding distance of streamwise velocity adjustment in a hypothetical one-dimensional canopy flow subject to a changing pressure gradient (Finnigan and Brunet, 1995). It appears as an essential parameter in models of flow in canopies on hills or fragmented canopies as well as in expressions for canopy kinetic energy dissipation (Belcher et al. 2012; Finnigan, 2000). As such it is more fundamental than canopy height, *h*, which only plays a dynamic role in sparser canopies where a significant amount of streamwise momentum is absorbed at the ground surface rather than by the foliage. The friction velocity $u*$ is the natural choice of velocity scale because $u*^2 = -\sigma_{13}(h)$ is the only source of streamwise momentum in simple canopy flows.

If L_c is constant, similarity arguments analogous to those used in deriving the logarithmic law lead to Inoue's (1963) well known exponential canopy velocity profile with the associated result that the mixing length is proportional to L_c. If L_c is not constant, but a function of x_3, then the straightforward similarity reasoning fails. Interestingly though, if a canopy's average value of L_c is used, then the Inoue (1963) formula yields an exponential velocity profile that can be a reasonable fit to measured profiles in the upper part of canopies even when L_c varies with height (Harman and Finnigan, 2007).

The second problematic feature of *K*-theory closure is its local nature; the components of the stress tensor are related only to the local rate-of-strain tensor. Thus far we have discussed the derivation of an eddy diffusivity by similarity arguments. A different derivation based on dynamical reasoning will clarify the issue of non-locality. Following Brunet et al. (1994) and applying simplifications discussed by Finnigan and Belcher (2004; Appendix A) and Ayotte et al. (1999) we write the equation for Reynolds shearing stress in a neutrally stratified simple canopy flow as,

$$\frac{D\left\langle \overline{u_1'u_3'} \right\rangle}{Dt} = 0 = -\left\langle \overline{u_3'u_3'} \right\rangle \frac{\partial \left\langle \overline{u_1} \right\rangle}{\partial x_3} - \frac{\partial}{\partial x_3}\left(\left\langle \overline{u_1'u_3'u_3'} \right\rangle + \left\langle \overline{p'u_1'} \right\rangle \right) + \left\langle \overline{p'\left(\frac{\partial u_3'}{\partial x_1} + \frac{\partial u_1'}{\partial x_3} \right)} \right\rangle \qquad (10)$$

D/Dt denotes the Eulerian derivative and the three terms on the right hand side of (10) are referred to as shear production, turbulent and pressure transport and pressure-strain interaction, respectively. Pressure-strain interaction is the main sink term for the covariance, $\left\langle \overline{u_1' u_3'} \right\rangle$. A standard set of parameterizations for the third moment expressions in terms of the first and second moments were proposed by Launder (1990). These expressions represent the third moments that appear in the transport term, $\left(\left\langle \overline{u_1' u_3' u_3'} \right\rangle + \left\langle \overline{p' u_1'} \right\rangle \right)$ by an effective diffusivity multiplied by the gradient of $\left\langle \overline{u_1' u_3'} \right\rangle$ while the pressure-strain terms are split into 'rapid' and 'return to isotropy' parts Launder (1990). Ayotte et al (1999) showed that these parameterizations worked satisfactorily in plant canopies without altering the values of Launder's coefficients so long as the expression for kinetic energy dissipation that appears in the parameterizations is adjusted to reflect canopy dynamics Finnigan (2000).

The parameterized version of equation (10) can be written,

$$A(x_3)\frac{\partial^2 \left\langle \overline{u_1' u_3'} \right\rangle}{\partial x_3^2} + A'(z)\frac{\partial \left\langle \overline{u_1' u_3'} \right\rangle}{\partial x_3} - \left\langle \overline{u_1' u_3'} \right\rangle = K(x_3)\frac{\partial \left\langle \overline{u_1} \right\rangle}{\partial x_3}, \qquad (11)$$

where, $A(x_3) = \frac{c_S c_T^2}{c_1} l^2$, $A'(x_3) = \partial A / \partial x_3$ and $K(x_3) = \frac{(1 - c_2) c_S}{c_1} lq$.

In boundary layer flows, l and q are height dependent length and velocity scales, respectively and c_S, c_T, c_1, c_2 are $O(1)$ constants. The values of these constants are given in Ayotte et al (1999). The first two terms on the left hand side of (11) result from the parameterization of the transport term in (10) so that, when this is negligible, shear stress may be related to the mean shear using the eddy viscosity $K(x_3)$[1]. In fact this analysis provides dynamical support for the use of eddy diffusivity-based parameterizations in the atmospheric surface layer where transport terms are small, at least in flows near neutral stratification. However, in canopy flows these terms are

[1] To the best of the current authors' knowledge, this derivation was first demonstrated

empirically large and cannot be discarded a priori (e.g. Wilson and Shaw, 1977; Raupach et al., 1996; Finnigan, 2000. In the following section we will show how this fact can be reconciled with the relative success of a canopy eddy diffusivity based on Inoue's scaling arguments.

The mixing layer analogy (Raupach et al, 1996, Finnigan et al., 2009) maintains that the production and character of turbulence in the canopy and overlying roughness sublayer is very similar to that in a plane mixing layer. A key result is that the dominant eddies in canopy flow are characterized by single length and velocity scales, which are invariant through the canopy-roughness sublayer. With *l* and *q* constant, equation (11) takes a simple form, whose solution can be written in terms of a Green's function $G\left(x_3, x_3'\right)$ (Finnigan and Belcher, 2004, Appendix A2),

$$\left\langle \overline{u_1' u_3'} \right\rangle \left(x_3\right) = \int_a^b G\left(x_3, x_3'\right) K \frac{\partial \left\langle u_1 \right\rangle}{\partial x_3'} dx_3' + G\left(x_3, a\right) \frac{\partial \left\langle \overline{u_1' u_3'} \right\rangle}{\partial x_3'} \left(a\right) - G\left(x_3, b\right) \frac{\partial \left\langle \overline{u_1' u_3'} \right\rangle}{\partial x_3'} \left(b\right) \quad (12)$$

Finnigan and Belcher (2004, Appendix A2) show that the Greens Function is symmetrical about and strongly peaked at x_3 and its width $\left(b - a\right)$ is determined by the strength and scale of the large eddies that effect the transport. Hence, the existence of turbulent transport means the shear stress at some level x_3 is determined by the balance of production and destruction terms (the first and third terms on the right hand side of (10)) weighted by the Greens function over a height interval $\left(b - a\right)$ centred on x_3.

In the case of horizontally homogeneous steady flow through a rigid canopy, equations (1) and (2) reduce to,

$$\frac{\partial \sigma_{13}^{t}}{\partial x_3} = - c_d a \left\langle \overline{u_1} \right\rangle^2 \quad (13)$$

so that the stress gradient cannot change sign. The spatial weighting implied by the

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by Wyngaard (1982) and applied to scalar transport in canopy flows by Finnigan (1985).

Greens function solution, however, can allow regions within the canopy where $\partial\langle\overline{u_1}\rangle/\partial x_3$ and σ^t_{13} are both negative, denoting transport of mean momentum towards the ground as required but implying a negative eddy diffusivity. This situation occurs because negative σ^t_{13}, produced at levels where $\partial\langle\overline{u_1}\rangle/\partial x_3 > 0$, is transported by the large canopy eddies into the region where the velocity gradient is negative. Note that the product of stress and rate of strain is then positive locally in apparent contradiction to condition IV. However, this is physically reasonable, which is why condition IV is framed as a global not a local requirement.

From the Greens function solution, Finnigan and Belcher (2004) inferred that, as long as $\partial\langle\overline{u_1}\rangle/\partial x_3$ is roughly constant over distances similar to the size of the energy-containing eddies then, an eddy diffusivity parameterization will not be greatly in error. In practice this requires that L_c not change significantly over the same distance. As a final comment on non-local effects, counter-gradient diffusion is more of a problem for scalar fluxes, where it is strongly linked to vertical heterogeneity in scalar sources (Raupach, 1989), than for momentum. In the authors' experience, despite the seminal analysis of Wilson and Shaw (1977), who first showed how turbulent transport terms can effect counter gradient diffusion of momentum in horizontally homogenous conditions, the existence of secondary wind speed maxima in canopy flows is more often the result of a mean hydrodynamic or hydrostatic pressure gradient than the non-local phenomenon described above.

3.2 Eddy Diffusivities in complex canopy flows

Clearly, mixing length closures for simple canopy flows can be constructed that satisfy the first four of our five requirements. Satisfying the final condition is more problematic because of the intrinsic limitations of eddy diffusivity closures. These are explored in more detail in Appendix A. However, the success of mixing length closures devolves from the existence of similarity scaling and is not a confirmation that the phenomenological analogy between the mixing length and the mean free path of molecules is valid. This makes the extension of the mixing length concept to complex flows, where similarity scaling is not generally available, problematic. Although Hinze (1975) has derived an extension to three-dimensional flow based on the phenomenological interpretation, it does not seem to have found application. Instead, practical eddy diffusivity models for complex flows have used formulae for *K* based upon physically relevant length and time scales, $K = BL^2/T$, where *B* is a dimensionless constant.

The first and most obvious generalization of shear as a time scale was the approach of Smagorinsky et al., (1965), which was adopted as a closure for sub-filter scale stresses in early LES models (eg. Deardorff, 1970). The Smagorinsky formula replaced $\partial\langle \overline{u_1}\rangle / \partial x_3$ as the time scale by the quadratic norm of the rate of strain tensor, S_{ij}. Hence the Smagorinsky time scale[2] is $T = \left(S_{ij}S_{ij}\right)^{-1/2}$. In LES applications the length scale *L* is related to the scale that separates resolved from sub-filter scale motions in the model solution. This scale is assumed to lie in the spectral inertial sub-range and an exact value for *B* can then be determined, if the Kolmogorov form for the energy spectrum in the inertial sub-range is assumed. In principle, the Smagorinsky time scale, $T = \left(S_{ij}S_{ij}\right)^{-1/2}$ could be combined with an empirically chosen mixing length *L*, representative of the dominant turbulent scale to provide an eddy diffusivity in a complex flow. In practice, more flexible ways of calculating *K* quickly replaced such approaches. In these 'two-equation' or 'one and half order' closure models, transport equations for length and time scales are solved to determine *K*. The so-called $k-\varepsilon$ model (or variants thereof) remains the most popular and widely applied. In this model, $k = 1/2\langle \overline{u'_i u'_i}\rangle$ is the turbulent kinetic energy and ε its dissipation rate so that $K = C_\mu k^2/\varepsilon$ with C_μ an empirically determined constant. Despite known limitations of such models going back to the AFOSR-IFP-Stanford Conference, they remain popular

[2] *T* is related to the first and second Cayley-Hamilton invariants of S_{ij} as follows: the first invariant is $\mathrm{Trace}\left(S_{ij}\right)$ and the second invariant is $S_{ij}S_{ij} - \left[\mathrm{Trace}\left(S_{ij}\right)\right]^2$.

and widely used in engineering applications and have been developed to a high level of sophistication for canopy flows; see especially Ross and Vosper (2005) and Sogachev et al, (2012). For recent reviews of the strengths and limitations of these models see Durbin (2004) and Hanjali´c and Kenjereš (2008). Interestingly in the light of our discussion of simple canopy flows, Katul et al. (2004) found that *K*-theory models with prescribed mixing lengths in the canopy performed as well as $k - \epsilon$ models in simple canopy flows.

Of course, two-equation models require full numerical solution and in this note we are focussed on first order closures that allow analytical insight. In practice this means that we are restricted to flows through canopies on gentle topography (eg. Finnigan and Belcher, 2004), with slowly varying foliage density (eg. Ross, 2012), across the edges of sparse canopies (eg. Belcher et al., 2003) or simple shear flows with applied hydrostatic pressure gradients (eg. Yi et al., 2005; Oldroyd et al., 2014). The first three situations lend themselves to analyses that treat the departure from horizontally homogeneous one-dimensional flow as small perturbations while the fourth remains one-dimensional but requires a physically realistic stress closure. In the first three situations, the perturbations to the background flow are caused by the pressure fields associated with the topography or the changed resistance to flow through the canopy. Unless separation occurs, it is reasonable to use a *K* that is derived from the undisturbed background value by a rational perturbation approach. Finnigan and Belcher (2004), for example, do this by transforming into potential flow streamline coordinates and using a constant canopy mixing length together with the cross-streamline shear as a time scale to form *K*. It is easy to show that this is equivalent to using the Smagorinsky time scale $T = \left(S_{ij}S_{ij}\right)^{-1/2}$ and retaining only the leading order term, $\left(2S_{13}^2\right)^{-1/2}$ after transforming into streamline coordinates.

Belcher et al. (2012), in their review of complex canopy flows, conclude that, “although canopy flows are highly turbulent, inviscid dynamics control many features of their adjustment to complex forcing, which suggests that simple turbulence closures are adequate in such applications”. Applying the *K*-theory closure will produce a flow pattern that is physically correct for a Stokesian fluid (see Appendix A and Aris, 1962).

To the extent that a turbulent flow is not Stokesian, the modelled flow will depart from the observed flow. These departures from observations will be largest where the stress is most affected by either the viscoelastic nature of the turbulence field's response to strain or by the departures from local-equilibrium. The first effect is likely to be important in regions where the flow is strained rapidly, in the sense that strain rate time scales, S_{ij}^{-1} are short compared to the integral time scales of the energy-containing turbulent eddies. An example would be where the flow encounters a dense forest edge or a steep hill. The second effect, non-locality, is likely to be important where the mean strain varies on space scales that are shorter than the energy containing eddies, for example where hydrodynamic or hydrostatic pressure gradients drive flow through a canopy with strong spatial variations in L_c. Of course this always occurs at the canopy top and even in simple canopy shear flows, the velocity profiles of the Inoue and logarithmic law similarity theories have to be augmented by a 'roughness sub layer' similarity theory that essentially recognizes the role played by large canopy eddies in altering the near-canopy velocity (and scalar) profiles (Harman and Finnigan, 2007, 2008).

4. Velocity-squared closure for canopy flows in complex terrain

The second closure considered here is that proposed by Yi et al. (2005) and then used in a series of papers primarily considering diabatically influenced flows in canopies on hills. The method is developed and applied in its complete form in Yi et al. (2005), Yi (2008) and Wang and Yi (2012). Assessing the efficacy of any turbulence closure applied to diabatically influenced complex canopy flows risks confusing the effects of hydrodynamic and hydrostatic pressure gradients (especially when the foliage is non-uniform) with issues of the validity of the closure. Consequently it is appropriate to test this closure first against the simplest case of uniform, neutrally stratified canopy flows on horizontally homogenous terrain.

The 'velocity-squared' closure relates the local shearing stress to the square of local mean velocity (Yi et al., 2005),

$$s^{t}_{13} = C\left\langle \overline{u}_1 \right\rangle^2 \tag{14}$$

with C a (positive definite) constant of proportionality that Yi identifies with a drag coefficient as explained below. This closure has the useful mathematical property that, when combined with (3), it removes one source of nonlinearity in (1). However, like the one-dimensional form of the mixing length (8), this closure fails condition I in that it is not a tensor and so is not invariant under transformations of the coordinate system. Unlike the mixing length, however, it is not obvious how (14) can be generalized. Referring to the discussion of drag parameterization in equations (2) and (3), it is clear that the closure also fails the test of Galilean invariance. Furthermore, the sign of the shearing stress does not change when the flow reverses, reflecting a fundamental ambiguity (condition II). Taken together these issues seem substantial enough to preclude the use of this closure for complex flows; nevertheless, given the several publications in which it has been employed, it is illuminating to examine the further issues that arise from the detailed specification of C.

Yi et al. (2005) derive the closure (14) through a confusion of two drag coefficients. The first is the c_d belonging to the local body force drag, which is only defined in the case of the volume averaged canopy flow. This c_d is defined by equation (3). The second drag coefficient, distinguished by capitals, C_D, is a function of height and is defined following Mahrt et al. (2000) using the integral of (3) between some reference height x_3, where the mean velocity equals $\langle \overline{u}_1 \rangle (x_3)$ and the ground, i.e,

$$C_D(x_3) = \frac{1}{\langle \overline{u}_1 \rangle^2 (x_3)} \int_0^{x_3} \left(c_d\, a \langle \overline{u}_1 \rangle^2 \right) dx_3' = \frac{\sigma^{\tau}_{13}(x_3) - \sigma^{\tau}_{13}(0)}{\langle \overline{u}_1 \rangle^2 (x_3)} \approx \frac{\sigma^{\tau}_{13}(x_3)}{\langle \overline{u}_1 \rangle^2 (x_3)}$$

(15)

where the exact equivalence between the penultimate and last term in (15) (as per the definition in Yi et al., 2005) is only valid if $\sigma^{\tau}_{13}(0) = 0$, that is when all the streamwise momentum is absorbed on the foliage and does not reach the ground[3]. The difference

[3] Ignoring $\sigma^{\tau}_{13}(0)$ prevents the use of further boundary conditions at the ground surface.

between c_d and C_D cannot be emphasized too strongly. c_d is the *local* coefficient of a non-linear body force, expressing the average aerodynamic drag of the canopy elements on the fluid in an averaging volume centred on a point in space. The second C_D is an *integral* coefficient, used to parameterize all the momentum absorbed in the foliage below some reference height in terms of a reference velocity at that height. Confusion of these two definitions of drag coefficient led to (Yi et al., 2005; equations 5 and 6),

$$\sigma^{t}_{13}\left(x_3\right) = C_D \left\langle \overline{u_1} \right\rangle^2 \left(x_3\right) \quad \text{and} \quad \frac{\partial \sigma^{t}_{13}}{\partial x_3} = C_D \, a \left\langle \overline{u_1} \right\rangle^2$$

which lead in turn to the full closure assumption,

$$C_D \left\langle \overline{u_1} \right\rangle^2 = \int_0^{x_3} C_D \, a \left\langle \overline{u_1} \right\rangle^2 \left(x_3'\right) dx_3' \tag{16}$$

For horizontally homogeneous flow within a canopy of height *h*, this implies,

$$\left\langle \overline{u_1} \right\rangle^2 \left(x_3\right) = \left\langle \overline{u_1} \right\rangle^2 \left(h\right) \exp\left[a \left(x_3 - h\right)\right] \tag{17}$$

This result is similar to the well-known exponential velocity profile solution of Inoue (1963) but with a critical difference. The Inoue (1963) result is,

$$\left\langle \overline{u_1} \right\rangle^2 \left(x_3\right) = \left\langle \overline{u_1} \right\rangle^2 \left(h\right) \exp\left[\frac{c_d \, a}{\beta^2} \left(x_3 - h\right)\right] \tag{18}$$

with $\beta = u_* \big/ \left\langle \overline{u_1} \right\rangle \left(h\right)$.[4] In contrast, Yi et al.'s exponential profile, equation (16), does not include the drag coefficient c_d. It therefore fails condition V since, according to this form of the velocity-squared closure, if the drag coefficient of the canopy elements is increased but the area density of the elements stays the same, the velocity profile will be unaffected. This is in clear contradiction to observations; see for example the compendium of canopy data collected in Raupach et al. (1996).

A more detailed justification of the velocity squared closure and of (15), in particular, is presented by Yi (2008) as depending on three hypotheses devolving from fundamental fluid dynamics. Unfortunately, for the first two of these hypotheses counter-examples

[4] Equation (18) follows from the assumption that the mixing length within the canopy is constant.

can easily be found while the third has unphysical consequences.

Hypothesis 1: within the canopy, the transport of horizontal momentum is continuous and downward. Meanwhile the horizontal momentum is continuously absorbed by canopy elements from the air.

This hypothesis indirectly addresses the issue of ambiguity of the closure identified earlier. The hypothesis is correct in the case of steady, horizontally homogeneous shear flows without applied pressure gradients but is easily violated in complex flows. An important and relevant example is flow over steep topography or flows over gentle topography covered by a tall canopy where the presence of a separation bubble with reversed flow means that momentum can be transferred upwards, away from the surface (e.g. Poggi and Katul, 2007a,b). More generally, many cases of secondary wind speed maxima in canopy flows result from the application of a terrain-generated streamwise pressure gradient to a canopy with a region of decreased $a\left(x_3\right)$ such as an open trunk space. In such cases, momentum will be transferred upwards from the velocity maximum, not downwards towards the surface.

Hypothesis 2: a local equilibrium exists between the rate of horizontal momentum transfer and its rate of loss . . . the local equilibrium relationship at level x_3 is,

$$s_{13}^{t} = C_D\left(x_3\right)\left\langle\overline{u_1}\right\rangle^2$$

Yi (2008) [P264] describes this hypothesis as the ‘velocity-squared law’ and attributes it to a list of investigators going back to Taylor (1916). However, local equilibrium implies that the turbulent transport terms can be neglected-in contradiction to observations (see the discussion on non-locality and counter-gradient diffusion of momentum in section 3.1 above). More fundamentally, proportionality between the shear stress and the square of the mean velocity is a consequence of the mechanism by which mean momentum is absorbed from the flow on solid surfaces and not a ubiquitous property of boundary layer flow. To illustrate this, consider that the structure of boundary layer turbulence in smooth and rough wall flows is essentially identical once one is far enough from the wall to be in the inertial sub-layer or logarithmic layer. That is, in smooth wall flows one needs to be well above the viscous sublayer and buffer layer, say $x_3^+ = x_3 u^*/n > 100$, and in rough wall flows above the roughness sublayer (Raupach et al., 1991). In aerodynamically fully rough wall flows such as the flow above a canopy, the wall stress or canopy drag is proportional to $\langle \overline{u_1} \rangle^2$ because the fluid momentum is absorbed almost entirely as pressure forces (form drag) on the roughness elements. Above a rough wall or canopy, where the wind speed profile assumes the familiar log law form, a velocity-squared relationship does emerge,

$$\langle \overline{u_1} \rangle^2 = \left[\frac{1}{k} \ln\left(\frac{x_3 - d}{z_0} \right) \right]^2 u^{*2} = - \left[\frac{1}{k} \ln\left(\frac{x_3 - d}{z_0} \right) \right]^2 s_{13}^{t}(h)$$

where z_0 is the friction velocity, with the proviso that the effective drag coefficient, $C_D = \left[1/k \ln\left([x_3 - d]/z_0 \right) \right]^2$, is a strong function of height. In contrast, in high Reynolds number turbulent flow over a smooth wall, the log law takes the form,

$$\overline{u_1}(x_3) = \frac{u^*}{k} \ln\left(\frac{x_3 u^*}{n} \right) + D \qquad (19)$$

where D is a constant. Equation (19) permits no simple relationship between the shear stress and the square of the wind speed. A linear relationship between the turbulent shear stress (momentum flux) and the square of mean velocity (kinematic momentum density) at all heights is therefore not a universal property of turbulent flows but depends entirely on how the momentum is absorbed on the surfaces bounding the flow.

Finally, paraphrasing slightly, the third hypothesis of Yi (2008) is,

Hypothesis 3: If their averaging operations are the same, the integral drag coefficient C_D in equation (10) of Yi (2008) (or defined properly in equation (15) above) is equal to the local drag coefficient c_d, defined in equation (3). As explained above, the two drag coefficients have qualitatively different meanings irrespective of their averaging operations. Quantitatively this hypothesis can only hold if the Yi (2008) exponential velocity profile formula (17) is valid and we have already shown this formula to be unphysical.

Other examples where the 'velocity-squared' closure clearly fails condition V can be found when considering flow in complex terrain-the application for which the closure was originally postulated-and closely related situations. First, consider the case of a pressure driven flow through a parallel two-dimensional duct fully occupied by vegetation (Figure 1a). This case, modelled in a wind tunnel by Seginer at al. (1976), is sufficiently closely related to real-world canopy flows that any successful canopy closure should provide at least qualitatively accurate results. Similar to Poiseuille flow, symmetry demands that the shear stress goes through zero at the duct centreline, where $\langle\overline{u_1}\rangle$ reaches a maximum, and has opposite signs on the two sides of the centreline so that momentum flows towards the walls from the velocity maximum. In contrast the velocity-squared closure results in a maximum in shear stress on the centreline , where it should be zero, and is zero on the walls where it should be maximal. Furthermore, the ambiguity in the sign of the closure means that the momentum flux is in one direction only rather than changing sign on the centreline. This ensures that, for this example, the parameterized flow also fails condition IV since the product of the rate of working of the shear stress, which, like $\langle\overline{u_1}\rangle$ has a single sign, and the rate of strain $\partial\langle\overline{u_1}\rangle/\partial x_3$, which is asymmetric across the duct, is zero.

Next consider flow over hills covered with canopies subject to diabatic and hydrodynamic pressure gradients. Two flow patterns are commonly observed in these situations, gravity currents with wind speed maxima within the canopy driven by hydrostatic pressure gradients (e.g. Goulden et al., 2006, Oldroyd et al., 2014) and reversed flow in the lee of the hill caused by the hydrodynamic pressure gradient in flows near neutral stratification (e.g.   Poggi and Katul 2007a,b). Gravity currents generated when radiative cooling produces a layer of higher-than-ambient density on a slope are ubiquitous at night. (Belcher et al., 2008, 2012). The gravity current/wall jet then is a case of great practical importance for flux tower studies but one where the phenomenon is unequivocally pressure driven (in contrast to turbulent transport driven). As in the duct case above, the velocity-squared closure produces a maximum in shear stress at the velocity peak, where physically the shear stress should be close to zero because it has to change sign around the windspeed maximum (Figure 1b). Just above the maximum in the gravity current windspeed, the shear stress is positive (upward momentum transfer), acting to remove momentum from the gravity current and balance the pressure gradient (e.g. van Gorsel et al. 2011). As shown in Figure 1b, the velocity-squared closure sees momentum being transferred towards the velocity peak, accelerating the current.

Our third example, flow separation, occurs when the combined effects of canopy drag (or surface friction) and an adverse pressure gradient reduce streamwise momentum faster than turbulent transfer of momentum from faster moving air aloft can redress the balance. Eventually a point is reached where the flow stops and reverses, creating a separation bubble. Modelling this requires capturing the balance between cross-streamline momentum transport and the pressure gradient. Since the velocity-squared closure predicts zero shear stress at the edges of the separation region, where it should be large or even maximal (Poggi and Katul 2007a,b), incorrect predictions of this balance occur. As a result, the size of the recirculation bubble cannot be accurately predicted (see Figure 1c).

These three important practical examples show that not only is the velocity-squared closure flawed at a basic level, in that it fails conditions I, II III and IV, but it also fails to meet condition V and regularly produces unphysical predictions. Consequently, we conclude that its use in modelling complex canopy flows is fundamentally wrong.

5. Summary and Conclusions

Although the deficiencies of first order closures for modelling complex turbulent flows and simple canopy flows are well known, they remain attractive when the main requirement is simplicity and if their shortcomings are well understood. Five requirements that such closures must satisfy, if they are to be used reliably in complex flow models, have been defined: they must be tensorially invariant, unambiguous, complete, globally satisfy the second law of thermodynamics and not lead to unphysical results. The most popular and well-tested first order turbulence closures are based on the eddy diffusivity concept. Such models treat the averaged turbulent flow as a linear Stokesian fluid (see Appendix) and yield physically plausible results to the extent that the turbulent flow is Stokesian. In reality, as discussed in section (3), canopy flows depart from this idealization in two ways: non-local dependence of stress on rate of strain and the viscoelastic response of the turbulent stresses to straining.

The Finnigan and Belcher (2004) group of models for canopy flow in complex terrain uses the mixing length approach to define an eddy diffusivity. This carries a set of implicit assumptions not all of which are automatically satisfied in all plant canopies, and so places conditions upon the closure's use. The methodology for deriving the diffusivity using mixing-lengths was introduced almost 90 years ago and has been developed and tested extensively since then. Despite the problems that are peculiar to canopies, mixing lengths can be defended if their formulation is linked to the dominant eddies responsible for turbulent transport in a robust way. Tensorially-invariant eddy diffusivity models based on mixing lengths satisfy four of the five conditions we have stipulated, and where they fail the fifth condition by giving unphysical results, it is because of deficiencies that are well understood so that their failure can be anticipated.

The velocity-squared first order closure scheme proposed by Yi et al. (2005) and Yi (2008) has a different form to the eddy diffusivity approach. This closure fails to satisfy any of the five conditions in both simple and complex canopy flows. In particular it is ambiguous (without further stipulations) and neither tensorially invariant nor material frame indifferent, rendering it immediately problematical for use in complex flows where simplified equations are typically derived using coordinate transformation. More fundamentally, the three hypotheses on which the closure is founded and which are proposed as principles of fluid mechanics (see Yi, 2008) can each be shown to be incorrect by straightforward examples.

All Reynolds stress closures are engineering approximations and those most appropriate to a particular problem need to be tailored to the circumstances. The success of two-equation models that employ sophisticated eddy diffusivities or of higher order closure models in engineering applications depends in part on the availability of more tunable constants as the degree of the closure increases (eg. Hanjali´c and Kenjereš, 2008). In this note we have concentrated on the simplest first-order closure schemes that are suitable for analytic modelling of complex canopy flows. In reality the desire for an analytic solution (or at least a transparent numerical solution) limits us to situations where the complexity can be treated as a small perturbation to a simple background shear flow. Nevertheless, the small perturbation equations must be derived by a rational simplification process from the general case and this inevitably requires coordinate transformations and turbulence closures that, at a minimum, do not violate the five principles we have set out. We have paid particular attention to mixing length-based closures but this does not imply that there are no other appropriate closure schemes for canopy flows in complex terrain. For example the approaches of Cowan (1968) or Massman (1997) could be generalized to be complete and invariant. In contrast, the fundamental and practical issues associated with the velocity-squared closure means that its use or any conclusion derived from it is unsound.

Appendix Derivation of a tensorially invariant first order closure

Following Aris (1962), we have referred to a fluid whose stress tensor is linearly dependent on its rate of strain tensor as Stokesian. Such a relationship requires the fluid to have certain properties. In the steps below we derive these properties in the course of moving from a completely general constitutive relationship to a scalar eddy diffusivity so as to clarify the assumptions we are making about the behaviour of the averaged turbulent field. The necessary steps are essentially those used to derive the form of the viscous stress in a Newtonian fluid (Batchelor, 1967) or that follow from the requirements of rational mechanics (Lumley, 1978) or the equivalent steps used to derive sub-filter scale closures in Large Eddy Simulations (Wyngaard, 2010). Note to begin with that the simplest conceptual first order closure relates σ_{ij} directly to $\langle \overline{u_i} \rangle$ but the need for Galilean invariance precludes this approach. Instead we wish to derive a linear relationship between σ_{ij} and $\partial \langle \overline{u_i} \rangle / \partial x_j$ while preserving tensor invariance.

We split the stress tensor into the sum of its isotropic and anisotropic parts

$$\sigma_{ij} = d_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{20}$$

The anisotropic part of the stress tensor, d_{ij}, contains the tangential turbulent stresses as well as 'deviatoric' normal stresses that sum to zero (Batchelor, 1967). Both d_{ij} and $\partial\langle \overline{u_i} \rangle / \partial x_j$ are second order tensors so linear dependence takes the general form,

$$d_{ij} = A_{ijkl}\, \partial\langle \overline{u_k} \rangle / \partial x_l \tag{21}$$

where the fourth order tensor A_{ijkl} is a property of the local state of the turbulent flow but does not depend directly on $\langle \overline{u_i} \rangle$ or its spatial derivatives. Since d_{ij} is symmetric in the indices *i* and *j*, so is A_{ijkl}.

Splitting the deformation tensor into the sum of its symmetric and antisymmetric parts we obtain,

$$\begin{aligned}\partial\langle \overline{u_i} \rangle / \partial x_j &= 1/2\left[\partial\langle \overline{u_i} \rangle / \partial x_j + \partial\langle \overline{u_j} \rangle / \partial x_i\right] + 1/2\left[\partial\langle \overline{u_i} \rangle / \partial x_j - \partial\langle \overline{u_j} \rangle / \partial x_i\right] \\ &= S_{ij} - 1/2\, e_{ijk} W_k\end{aligned} \tag{22}$$

where e_{ijk} is the alternating tensor and W_k the mean vorticity. The anisotropic component of the stress tensor is then,

$$d_{ij} = A_{ijkl} e_{kl} - 1/2\, A_{ijkl}\, e_{klm} W_m \tag{23}$$

Equation (23) can be simplified considerably if we assume that A_{ijkl}, as well as being symmetric in *i* and *j*, is isotropic in the sense that the deviatoric stress generated in an element of fluid by the deformation $\partial\langle \overline{u_i} \rangle / \partial x_j$ is independent of the orientation of the fluid element. This is another way of saying that the fluid itself has no preferred direction and in general this is not true of volume-averaged turbulent canopy flow, for two reasons. First, if the orientation of the solid canopy elements is predominantly in one direction then deformation along axes parallel or normal to the elements may produce different stresses. Second, the quadratic nature of canopy pressure drag (2) indicates that the simplest symmetry of turbulent canopy flow is not isotropy but axisymmetry with the axis aligned with the mean flow (Finnigan, 2000). However, if we assume that the turbulent flow *is* isotropic in the sense given above, the coefficient A_{ijkl} must then take the form of an isotropic tensor and so can be written as the sum and products of the basic isotropic second order tensor, the Kronecker delta,

$$A_{ijkl} = \mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk} + \mu''\delta_{ij}\delta_{kl} \tag{24}$$

where μ, μ', μ'' are scalar coefficients and, since we require A_{ijkl} to be symmetrical in *i* and *j*, then $\mu = \mu'$. A_{ijkl} is now symmetric in *k* and *l* also and the term containing $\varepsilon_{klm}\omega_m$ drops out of (23) because of the properties of the alternating tensor, leaving ,

$$d_{ij} = 2\mu e_{ij} + \mu'' e_{ii}\,\delta_{ij} \tag{25}$$

Finally, since from continuity $e_{ii} = 0$, a tensorially invariant closure can be written as,

$$\sigma_{ij} - \sigma_{kk}\,\delta_{ij}/3 = K\left(\partial\langle\overline{u_i}\rangle/\partial x_j + \partial\langle\overline{u_j}\rangle/\partial x_i\right) = 2KS_{ij} \tag{26}$$

The non-isotropic part of the turbulent stress tensor is thus linearly proportional to the mean rate of strain tensor, with the scalar eddy diffusivity *K*. As S_{ij} only involves velocity gradients and *K* is a scalar, the expression for σ_{ij} remains Galilean invariant. This derivation illustrates that the familiar tensorially invariant form of the first order stress closure required the assumption that the mean turbulent stresses in the 'fluid' defined by time and spatial averaging across the multiply connected canopy airspace had an isotropic response to straining and that $S_{ii} = 0$. If, in contrast, we had assumed that the response of a fluid element to straining was axisymmetric rather than isotropic, the simplest expression for σ_{ij} would involve two independent terms even when $S_{ii} = 0$ (Batchelor, 1953; p43; Finnigan, 2000).

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Figures

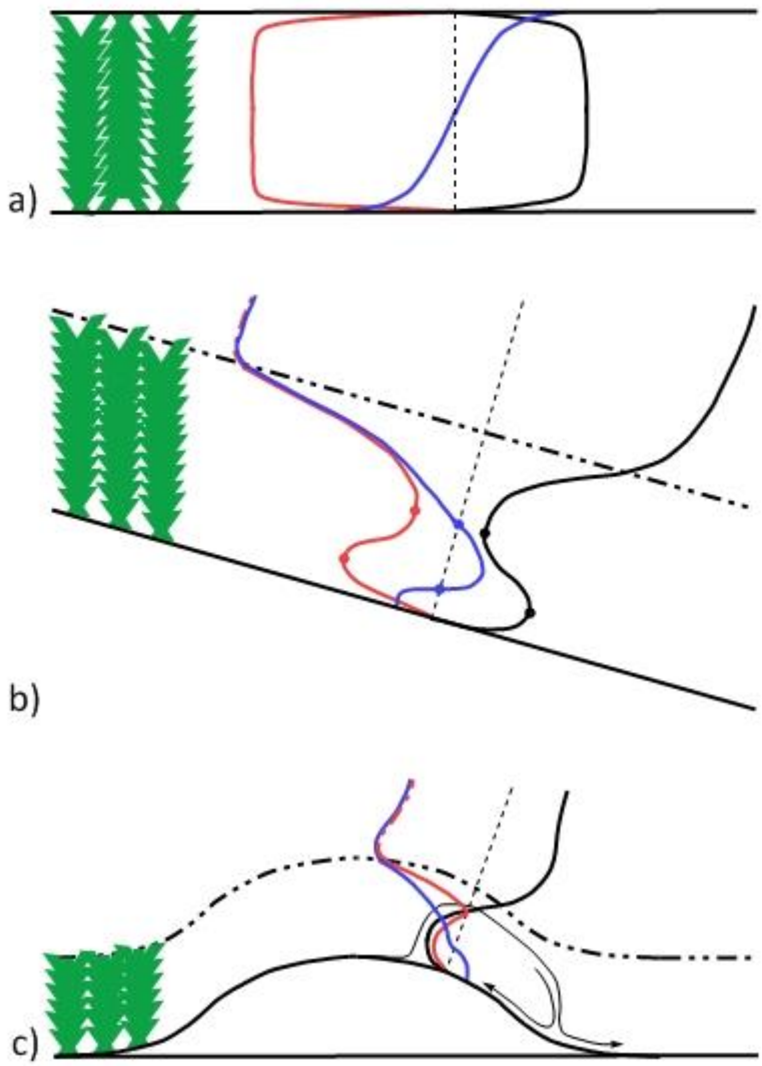

Figure 1: Schematic of three examples where the velocity-squared closure implies an unphysical shear stress. Black lines show the prescribed wind speed profile, red lines the shear stress according to the velocity-squared closure and blue lines the shear stress according to the mixing length closure. The zero line for all profiles is given by the dashed vertical line and velocity and stresses are referred to surface following coordinates.

a) Flow through a duct containing vegetation, as per Seginer et al. (1976).
b) Gravity current down slope beneath an ambient wind blowing from left to right. The dots denote heights where $\partial\langle\overline{u_1}\rangle/\partial x_3 = 0$.
c) Reversed flow within the canopy in the lee of a hill — thin lines with arrows denote two example streamlines, just above and just within the recirculation zone. Note the coincidence of $\langle\overline{u_1}\rangle = 0$ and $\sigma^{t}_{13} = 0$ at the top of the recirculation zone when using the velocity-squared closure. See main text for further discussion.