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The Signature of a Rough Path: Uniqueness

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Abstract

In the context of controlled differential equations, the signature is the exponential function on paths. B. Hambly and T. Lyons proved that the signature of a bounded variation path is trivial if and only if the path is tree-like. We extend Hambly-Lyons' result and their notion of tree-like paths to the setting of weakly geometric rough paths in a Banach space. At the heart of our approach is a new definition for reduced path and a lemma identifying the reduced path group with the space of signatures.

1 Introduction

In K.T. Chen's work [8] on the cohomology of the loop space, he defined and systematically studied the formal series of iterated integrals

$$S(x) = 1 + \sum_{i_1} \int_0^T \mathrm{d}x_{t_1}^{i_1} X_{i_1} + \sum_{i_1,i_2} \int_0^T \int_0^{t_2} \mathrm{d}x_{t_1}^{i_1} \mathrm{d}x_{t_2}^{i_2} X_{i_1} \cdot X_{i_2} + \ldots \tag{1.1}$$

where $x : [0,T] \to \mathbb{R}^d$ is a path with bounded variation and $X_1, \ldots, X_d$ are formal non-commutative indeterminates. After proving a homomorphism property of the map S ([8], see (2.1) below), he gave an argument [10] that the map S restricted to appropriate classes of paths is, up to translation and reparametrisation, injective. Hambly and Lyons [16], motivated by the application of the map S in rough path theory, posed the following problem:

How to characterise the kernel of the map S?

Hambly and Lyons [16] proved that for a bounded variation path x, $S(x) = 1$ if and only if x is *tree-like*. They conjectured that the result extends to weakly geometric rough paths, a fundamental class of control paths for which controlled differential equations can be defined. Their result directly implies that the space

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of bounded variation paths, quotiented by the space of tree-like paths, forms a group with respect to the concatenation operation. They called this quotient space the *reduced path group*.

In [18], LeJan and Qian answered a special case of Hambly-Lyons conjecture. They proved that, when restricted on the complement of a Wiener measure zero set, the map S (defined using Stratonovich integration) is injective. There has been a number of other partial results for particular cases of weakly geometric rough paths ([17, 3, 4, 2]). A key observation in the proof of Hambly-Lyons and further refined by LeJan and Qian is that the iterated integrals of 1-forms along a path is a linear functional of the signature of the path. It turns out that a subtle variant of this idea, by considering the 1-form along the iterated integrals of the path work to prove Hambly-Lyons' conjecture.

To formulate the extension of Hambly-Lyons result, we must find the correct notion of tree-like weakly geometric rough paths. Hambly-Lyons' definition of tree-like path is inappropriate in the setting of weakly geometric rough paths. In fact, it is easy to prove that if x is an injective path with finite p-variation ($p > 1$), but not finite 1-variation, then $S(x \star \overleftarrow{x}) = \mathbf{1}$ but $x \star \overleftarrow{x}$ won't be tree-like in the sense of Hambly-Lyons. A tree-like path x (in the sense of Hambly-Lyons) has the property that there exists a continuous function $h : [0, 1] \to \mathbb{R}$, $h_t \geq 0$ for all $t \in [0, 1]$, $h_1 = h_0$ and

$$h_s = h_t = \inf_{s \leq u \leq t} h_u \implies x_s = x_t.$$

We will take an equivalent formulation of this property (see [13, 15]) as the definition of tree-like path, as follows.

Definition 1.1. Let E be a topological space. A continuous path $x : [0, T] \to E$ is *tree-like* if there exists a $\mathbb{R}$-tree τ, a continuous map $\phi : [0, T] \to \tau$ and a map $\psi : \tau \to E$ such that $\phi(0) = \phi(T)$ and $x = \psi \circ \phi$.

This definition of tree-like path is equivalent to Hambly-Lyons' definition when the path has bounded variation. The tree-like paths are also interesting in the context of homotopy. In Section 5.7 in [19], T. Levy showed that a bounded variation path is tree-like if and only if it is contractible to the constant path within its own image.

We will use our Definition 1.1 of tree-like path to extend Hambly-Lyons' result to weakly geometric rough paths in Banach spaces. The assumptions on the tensor product, which is only needed in the infinite dimensional case, and the issue of defining the map S for weakly geometric rough paths will be discussed in Section 2.1.

Theorem 1.1. *Let V be a Banach space such that the tensor powers of V is equipped with a family of tensor norms satisfying (2.2), (2.3) and (2.4). Let x be a weakly geometric rough path in V. Then $S(x) = \mathbf{1}$ if and only if x is tree-like in the sense of Definition 1.1.*

Remark 1.1. If $S(x) = \mathbf{1}$, then we even know what a corresponding τ,ϕ and ψ in Definition 1.1 are. Here τ is the set of signatures equipped with a special metric (see Theorem 4.1), ϕ is the path $t \to S(x)_{0,t}$ and ψ is the projection map $\pi^{(\lfloor p \rfloor)}$ if $x \in WG\Omega_p(V)$ (see Section 2.1 for notations).

Remark 1.2. Our proof does not make use of Hambly-Lyons' result and hence in the case $p = 1$, we have given a new and simple proof of Hambly-Lyons' theorem in [16].

Theorem 1.1 has the same implications for weakly geometric rough paths as Hambly-Lyons' for bounded variation paths. As part of the proof (see Lemma 4.6), we will also show that the map S is an isomorphism from the *reduced paths* (an analytic notion) to signatures (an algebraic structure). T. Lyons and W. Xu [23] have proposed inversion schemes for recovering the reduced paths from signatures for C^1 paths.

The more difficult implication in Theorem 1.1 is "$S(x) = \mathbf{1}$ implies x is tree-like". Our definition of tree-like weakly geometric rough paths gives rise to a natural strategy of proof, namely to first show that the space of signatures of paths has a $\mathbb{R}$-tree structure. A key Lemma in our approach is to identify signatures with injective paths on signatures (see Section 4.3). This Lemma translates the natural $\mathbb{R}$-tree structure of the latter to a $\mathbb{R}$-tree structure for the former (see Section 4.4 and Section 4.5). To identify signatures with injective paths on signatures, we use that for sufficiently smooth 1-form α and $N \in \mathbb{N}$, if paths x and y have the same signature and S_N denotes the truncated signature at degree N, then

$$\int \alpha\left(\mathrm{d}S_N(x)_{0,t}\right) = \int \alpha\left(\mathrm{d}S_N(y)_{0,t}\right)$$

(see Section 4.1 and 4.2). That we need to integrate against the truncated signature path $S_N(x)$ as opposed to merely integrating against the path x itself is an important new idea.

2 The signature of a path

We will follow Hambly-Lyons' [16] and view the map S as taking value in the formal series of tensors so that

$$S(x) = \mathbf{1} + \int_0^T \mathrm{d}x_{t_1} + \int_0^T \int_0^{t_2} \mathrm{d}x_{t_1} \otimes \mathrm{d}x_{t_2} + \ldots.$$

The formal series of tensors $S(x)$ is called the *signature* of the path x. The signature has a natural homomorphism property with respect the operations of concatenation and reversal. More precisely, given a Lie group G and continuous paths $x : [0, T_1] \to G$ and $y : [0, T_2] \to G$, we define the concatenation product $\star$ by

$$x \star y(t) = \begin{cases} x(t), & t \in [0, T_1]; \\ x(T_1)\, y(0)^{-1}\, y(t - T_1), & t \in (T_1, T_1 + T_2] \end{cases}$$

and the reversal operation $\overleftarrow{\cdot}$ by

$$\overleftarrow{x}(t) = x(T_1 - t), t \in [0, T_1].$$

K.T. Chen proved two fundamental algebraic properties of the map S, which in the language of tensors can be stated as :

1. (K.T. Chen, [8])The map S satisfies

$$S(x \star y) = S(x) \otimes (y); \; S(x) \otimes S(\overleftarrow{x}) = \mathbf{1}. \tag{2.1}$$

   In the rough path literature, the first identity in (2.1) is now known as *Chen's identity.*

2. (K.T. Chen, [9])The natural logarithm of S, in the space of non-commutative formal power series, is a Lie series.

Hambly-Lyons' characterisation of the kernel of the map S implies that the tree-like relation $\sim$, defined for continuous bounded variation paths x and y by

$$x \sim y \iff x \star \overleftarrow{y} \text{ is tree-like,}$$

is an equivalence relation. Moreover, the space of continuous bounded variation paths in $\mathbb{R}^d$, quotiented by the relation $\sim$, is a group with respect to the binary operation $\star$ and the inverse $\overleftarrow{\cdot}$. Our main result Theorem 1.1 implies that the same holds with bounded variation paths replaced by weakly geometric rough paths.

2.1 Setting for rough path theory

We briefly recall the notations and settings in rough path theory, which will be identical to that in Lyons-Qian's book [21].

Let V be a Banach space. Let $\otimes$ be a tensor product such that the tensor powers of V,$(V^{\otimes n} : n \geq 1)$, is equipped with a family $(\|\cdot\|_{V^{\otimes n}} : n \geq 1)$ of norms satisfying:

1. for $m, n \in \mathbb{N}$ and for all $u \in V^{\otimes m}$ and $v \in V^{\otimes n}$,

$$\|u \otimes v\|_{V^{\otimes(n+m)}} \leq \|u\|_{V^{\otimes m}} \|v\|_{V^{\otimes n}}; \tag{2.2}$$

2. for any permutation σ of $\{1, \ldots, n\}$,

$$\|v_1 \otimes \ldots \otimes v_n\|_{V^{\otimes n}} = \left\|v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}\right\|_{V^{\otimes n}}; \tag{2.3}$$

3. for any bounded linear functionals f on $V^{\otimes m}$ and g on $V^{\otimes n}$, there exists a unique bounded linear functional, denoted as $f \otimes g$, on $V^{\otimes(m+n)}$ such that for all $u \in V^{\otimes m}$ and $v \in V^{\otimes n}$,

$$f \otimes g(u \otimes v) = f(u) g(v). \tag{2.4}$$

A family of tensor norms satisfying conditions 1. and 2. are called a family of admissible tensor norms (see Definition 1.25 in [22]). By convention, we define $V^{\otimes 0}$ by $\mathbb{R}$. The projective tensor product $\|\cdot\|_{\vee}$, defined for $v \in V^{\otimes n}$ by

$$\|v\|_{\vee} = \inf \left\{ \sum_k \left|v_1^k\right| \dots \left|v_n^k\right| : v = \sum_k v_1^k \otimes \dots \otimes v_n^k,\ v_i^k \in V \right\},$$

satisfies conditions 1.-3 (Section 5.6.1 in [21]). We will use the shorthand $\|\cdot\|$ to denote $\|\cdot\|_{V^{\otimes i}}$ and ab to denote $a \otimes b$ where there is no confusion.

Let $T((V))$ be the formal series of tensors (Definition 2.4 [22]) and $T^{(n)}(V)$ be the truncated tensor algebra up to degree n (Definition 2.5 [22]). Let π_n and $\pi^{(n)}$ denote, respectively, the natural projection map from $T((V))$ onto $V^{\otimes n}$ and $T^{(n)}(V)$. Let $\mathcal{L}((V))$ denote the Lie formal series over V (Definition 2.2 [22]). Let $\tilde{T}((V))$ and $\tilde{T}^{(n)}(V)$ denote the subspaces of $T((V))$ and $T^{(n)}(V)$ respectively such that $\pi_0(a) = 1$ for all $a \in \tilde{T}((V))$ or $\tilde{T}^{(n)}(V)$. Let $G^{(*)} = \exp(\mathcal{L}((V)))$ be the space of group-like elements (p37 [22]) and $G^{(n)} = \pi^{(n)}\left(G^{(*)}\right)$ denote the free nilpotent Lie group of step n (p37 [22]). We will equip $G^{(n)}$ with the metric

$$d(a,b) = \max_{i \in \{1,\dots,n\}} \left\|\pi_i\left(a^{-1}b\right)\right\|^{\frac{1}{i}}.$$

Definition 2.1. Let $G^{(*)}_{p.r.c}$ (p.r.c. for positive radius of convergence) denote respectively the element a in $G^{(*)}$ such that

$$\max_{i \in \mathbb{N}} \|\pi_i(a)\|^{\frac{1}{i}} < \infty.$$

We will equip $G^{(*)}_{p.r.c.}$ with the metric

$$d(a,b) = \max_{i \in \mathbb{N}} \left\|\pi_i\left(a^{-1}b\right)\right\|^{\frac{1}{i}}$$

(see Lemma 1.1 in [5] for a proof of the symmetric property of d). Let E be a metric space. We say a continuous function $x : [0,T] \to E$ has finite p-variation if

$$\|x\|_{p-var} = \sup_{\mathcal{P}} \left(\sum_{t_i \in \mathcal{P}} d\left(x_{t_i}, x_{t_{i+1}}\right)^p \right)^{\frac{1}{p}} < \infty$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[0,T]$.

Definition 2.2. The space of weakly geometric rough paths, $WG\Omega_p(V)$, is the set of all continuous functions from a compact interval $[0,T]$ to $G^{(\lfloor p \rfloor)}$ with finite p-variation.

For $x, y \in WG\Omega_p(V)$ such that $x_0 = y_0$, we define the p-variation metric

$$d_{p-var}(x,y) = \max_{1 \le i \le \lfloor p \rfloor} \sup_{\mathcal{P}} \left(\sum_{t_j \in \mathcal{P}} \left\|\pi_i\left(x_{t_j}^{-1}x_{t_{j+1}} - y_{t_j}^{-1}y_{t_{j+1}}\right)\right\|^{\frac{p}{i}} \right)^{\frac{i}{p}}.$$

We will use $\mathbf{1}$ to denote the identity element with respect to $\otimes$ in $T((V))$.

Proposition 2.1. *(Extension Theorem, Theorem 2.2.1 [20] and Corollary 3.9 in [7]) Let $x \in WG\Omega_p(V)$. There exists a unique continuous path $S(x)_{0,\cdot} : [0,T] \to G^{(*)}_{p.r.c}$ with finite p-variation such that $S(x)_{0,0} = \mathbf{1}$ and $\pi^{(\lfloor p \rfloor)}\left(S(x)_{0,t}\right) = x_0^{-1}x_t$. We will call $S(x)_{0,T}$ the signature of x.*

We will often omit the subscript and use the shorthand $S(x)$ for the signature of x. We will also use $S_N(x)$ to denote $\pi^{(N)}(S(x))$.

3 Tree-like paths have trivial signature

3.1 Preliminary definitions

Definition 3.1. Let V be a topological space and let $x : [a,b] \to V$, $\tilde{x} : [c,d] \to V$ be continuous paths taking value in V. We say x is a *reparametrisation* of $\tilde{x}$ if there exists a homeomorphism σ of $[c,d]$ onto $[a,b]$ such that $x_{\sigma(t)} = \tilde{x}_t$ for all t.

Definition 3.2. Let τ be a $\mathbb{R}$-tree and $a, b \in \tau$, we will use the notation $[a,b]$ to denote the unique (up to reparametrisation) injective continuous function $x : [0,T] \to \tau$ on some compact interval $[0,T]$ such that $x_0 = a$ and $x_T = b$.

Definition 3.3. Let V be a topological space. A *rooted loop* in V is a continuous function $x : [s,t] \to V$ $(s \leq t)$ such that $x_s = x_t$. The element x_s is known as the *root* of x.

Definition 3.4. Let τ be a $\mathbb{R}$-tree and $r \in \tau$. We may define a partial order $\preceq$ with respect to r on τ by

$$a \preceq b \iff [r,a] \subseteq [r,b]. \tag{3.1}$$

3.2 The central case

Lemma 3.1. *Let τ be a $\mathbb{R}$-tree and $\phi : [0,T] \to \tau$ be a rooted loop. Suppose there exists a partition $\mathcal{P} = (t_0, \ldots, t_n)$ of $[0,T]$ such that if t_i, t_{i+1} are adjacent points in $\mathcal{P}$, then $\phi|_{[t_i,t_{i+1}]}$ is (not necessarily strictly) monotone with respect to the root of ϕ. If $\psi : \tau \to G^{\lfloor p \rfloor}$ is such that $\psi \circ \phi \in WG\Omega_p(V)$, then $\psi \circ \phi$ has trivial signature.*

Proof. We will prove by induction on $|\mathcal{P}|$. In the case $|\mathcal{P}| = 2$, as $\phi|_{[0,T]}$ is monotonic and $\phi(0) = \phi(T)$, ϕ is forced to be constant. In particular, $S(\psi \circ \phi) = \mathbf{1}$. For the induction step, let $\tau_{\max} \in \phi(\mathcal{P})$ be such that there does not exists $s \in \phi(\mathcal{P})$, $s \succ \tau_{\max}$. Let $t_i \in \mathcal{P}$ be such that $\phi(t_i) = \tau_{\max}$. Since $\phi(t_{i-1}) \preceq \phi(t_i)$, $\phi(t_{i+1}) \preceq \phi(t_i)$ and the set $\{t | t \preceq \phi(t_i)\}$ is totally ordered, we may assume, without loss of generality, that $\phi(t_{i-1}) \preceq \phi(t_{i+1})$. Let $t' \in [t_{i-1}, t_i]$ be such that $\phi(t') = \phi(t_{i+1})$. Then $\phi|_{[0,t'] \cup [t_{i+1},T]}$ is piecewise monotone with respect to the partition $\mathcal{P} \backslash \{t_i\}$. Therefore, by induction hypothesis,

$$S\left(\psi \circ \phi|_{[0,t'] \cup [t_{i+1},T]}\right) = \mathbf{1}. \tag{3.2}$$

As $\phi|_{[t',t_i]}$ and $\phi|_{[t_i,t_{i+1}]}$ are the unique injective curves connecting $\phi(t_i)$ and $\phi(t_{i+1})$ with opposite orientation, by the homomorphism property of signature (see Lemma 1.3 in [5]),

$$S\left(\psi \circ \phi|_{[t',t_i]}\right) \otimes S\left(\psi \circ \phi|_{[t_i,t_{i+1}]}\right) = \mathbf{1} \quad .$$

Hence $S\left(\psi \circ \phi|_{[t',t_{i+1}]}\right) = \mathbf{1}$ which implies, by (3.2) and Chen's identity, that $S(\psi \circ \phi) = \mathbf{1}$. □

3.3 Reducing to the central case

We will need the following result of general topology from [6], which allows us to erase loops from a continuous path to obtain an injective continuous path.

Lemma 3.2. *(R. Börger [6])Let X be a Hausdorff space and let $\varphi : [0,1] \to X$ be continuous with $\varphi(0) \neq \varphi(1)$. Then there exist a closed subset $A \subset [0,1]$, a continuous and order-preserving map $q : [0,1] \to [0,1]$, and an injective continuous map $\psi : [0,1] \to X$ with the following properties:*
(i)$\psi(0) = \varphi(0)$,$\psi(1) = \varphi(1)$.
(ii)$\psi \circ q|_A = \varphi|_A$.
(iii)$q|_A : A \to [0,1]$ is surjective.

Remark 3.1. The Lemma holds with $[0,1]$ replaced by any interval $[0,T]$, $T > 0$.

Proof of the "tree-like paths have trivial signature" part of Theorem 2.1. Using the notation in Definition 1.1, let the functions $\phi : [0,T] \to \tau$ and $\psi : \tau \to G^{(\lfloor p \rfloor)}$ be a factorisation for the tree-like path x. Let $\preceq$ be the natural partial order (see Definition 3.4) with respect to the root of ϕ. For any $a, b \in \tau$, we define $a \wedge b$ to be the unique element of τ such that

$$[\phi(0), a \wedge b] \quad = \quad [\phi(0), a] \cap [\phi(0), b]$$

(see Lemma 2.3 in [11] for the existence of $a \wedge b$). Let $\mathcal{P}$ be a partition of $[0,T]$. Let

$$B = \{\phi(t_{i_1}) \wedge \ldots \wedge \phi(t_{i_n}) : t_{i_1}, \ldots, t_{i_n} \in \mathcal{P},\ n \leq |\mathcal{P}|\}.$$

The set B can be interpreted as the set of branched points in the subtree of τ spanned by $\phi(\mathcal{P})$. Note that for any $b_1, b_2 \in B$ we have $b_1 \wedge b_2 \in B$. Define a sequence (s_i) by $s_0 = 0$,

$$s_{i+1} = \inf\{v > s_i : \phi(v) \in B \backslash \{\phi(s_i)\}\}.$$

By the continuity of ϕ and the finiteness of B, (s_i) is a finite sequence. Let $\mathcal{P}' = (s_i)$. For each s_i, we construct an injective path $x^{s_i,s_{i+1}}$ in τ in the following way: if we apply Lemma 3.2 to erase loops from the path

$$\varphi(t) = \left(\phi|_{[s_i,s_{i+1}]}, \psi \circ \phi|_{[s_i,s_{i+1}]}\right) \in \tau \times V, \tag{3.3}$$

we obtain a continuous injective path η in $\tau \times V$. The path $x_{\cdot}^{s_i,s_{i+1}}$ is the projection of η onto τ. Define $\phi' : [0,T] \to \tau$ so that for each $s_i \in \mathcal{P}'$

$$\begin{aligned} \phi'(t) &= \phi(t), \ t \in \mathcal{P}' \\ &= x_t^{s_i,s_{i+1}}, \ s_i \le t \le s_{i+1} \end{aligned}$$

and let $x^{\mathcal{P}'} := \psi \circ \phi'$. As the self-intersection of φ in (3.3) coincides with the self-intersection with of $\phi|_{[s_i,s_{i+1}]}$, the path $x^{\mathcal{P}'}|_{[s_i,s_{i+1}]}$ is a natural projection of η (see line below 3.3) onto V. Therefore, $x^{\mathcal{P}'}|_{[s_i,s_{i+1}]}$ is continuous for all i.

We now show that if s_{i-1}, s_i are adjacent points in $\mathcal{P}'$, then either $\phi(s_{i-1}) \preceq \phi(s_i)$ or $\phi(s_i) \preceq \phi(s_{i-1})$. As stated in Lemma 2.1 in [11], the image of the continuous path $\phi|_{[s_i,s_{i+1}]}$ in a $\mathbb{R}$-tree must contain $[\phi(s_i), \phi(s_{i+1})]$ and in particular the element $\phi(s_i) \wedge \phi(s_{i-1})$. As $\phi(s_i) \wedge \phi(s_{i-1}) \in B$, by the construction of the sequence (s_i), $\phi(s_i) \wedge \phi(s_{i-1})$ must be either equal to $\phi(s_i)$ or $\phi(s_{i+1})$.

In particular, ϕ' is piecewise monotone. Since the p-variation of $x^{\mathcal{P}'}$ is dominated by the p-variation of x, $x^{\mathcal{P}'} \in WG\Omega_p(V)$. Therefore, $x^{\mathcal{P}'}$ satisfies the assumptions of the central case, Lemma 3.1, and hence has trivial signature. Let $\mathcal{P}_n$ be a sequence of partitions such that $|\mathcal{P}_n| \to 0$ as $n \to \infty$. Let $\mathcal{P}'_n$ be the corresponding sequence constructed as above. Trivially, we have $\left\| x^{\mathcal{P}'_n} \right\|_{p-var} \le \|x\|_{p-var}$. As $|\mathcal{P}'_n| \to 0$, $x^{\mathcal{P}'_n}$ converges uniformly to x. By Lemma 1.5 in [5], there exists a subsequence $x^{\mathcal{P}'_{n_k}}$ such that $S\left(x^{\mathcal{P}'_{n_k}}\right) \to S(x)$ as $k \to \infty$. As $S\left(x^{\mathcal{P}'_{n_k}}\right) = \mathbf{1}$ for all k, the result follows. □

4 Paths with trivial signature are tree-like

4.1 A special functional on signature

The following is a key ingredient in the proof of our main result. It establishes a relation between a weakly geometric rough path $x \in WG\Omega_p(V)$ and its signature other than the one given in the Extension Theorem 2.1.

Lemma 4.1. *(Integration of 1-form is a functional of signature)Let $x, y \in WG\Omega_p(\mathbb{R}^d)$, $\pi_1(x_0) = \pi_1(y_0) = 0$ and $S(x) = S(y)$. Then for any $N \in \mathbb{N}$ and any C_c^K 1-form ψ on $\mathbb{R}^d$ with $K > p-1$, we have*

$$\int \psi(\mathrm{d}x_u) = \int \psi(\mathrm{d}y_u). \tag{4.1}$$

Proof. Suppose that

$$\psi(\mathrm{d}x) = \sum_{i=1}^{d} \psi_i(x)\, \mathrm{d}x^i.$$

Assume for now that $\{\psi_i\}$ were all polynomial functions. Then each $\psi_i(\pi_1(x_u))$ can be expressed as a linear functional (independent of path x and time u) of

$S(x)_{0,u}$ (see Theorem 2.15 [22]). Since

$$S(x) = \mathbf{1} + \int_0^T S(x)_{0,t} \otimes \mathrm{d}x_t,$$

the integral $\int \psi(\pi_1(x_u))\,\mathrm{d}x_u$ is also a linear functional, independent of x, of $S(x)$ and the desired result holds when ψ_i are polynomials for all i. It now suffices to note that functions in C_c^K can be approximated by polynomials in the Lip(K)-norm ([1]) and the map

$$\psi \to \int \psi(dz)$$

is continuous in the Lip(K)-norm for $K > p-1$ ([14], Theorem 10.50). $\square$

4.2 Finite dimensional projection of the signature path

As pointed out in the introduction, the class of functionals

$$\left\{ x \to \int \psi(\mathrm{d}x) : \psi \text{ smooth 1-forms} \right\}$$

is insufficient to separate signatures. Indeed, if x and y are any weakly geometric rough paths, then for any smooth 1-form, the additivity of integral will imply that

$$\int \psi(\mathrm{d}(x \star y)) = \int \psi(\mathrm{d}(y \star x)),$$

whereas in general $S(x \star y) \neq S(y \star x)$. Therefore, we will consider a larger class of functionals, namely

$$\left\{ x \to \int \psi\left(\mathrm{d}S(x)_{0,t}\right) : \psi \text{ smooth 1-forms} \right\},$$

if we could at all make sense of the integral. However, the finite dimensional nature of Lemma 4.1 forces us to use finite dimensional projection. We first truncate the path in $G^{(*)}_{p.r.c.}$ to a path in $G^{(n)}(V)$ and show that a path in $G^{(n)}(V)$ with finite p-variation can be lifted to a p-weakly geometric rough path. Then we project the infinite dimensional space V to $\mathbb{R}^d$ using a linear map Φ. In Section 4.2.1, we will make sense of the integral

$$\int \psi\left(\mathrm{d}\Phi \circ S_N(x)_{0,t}\right)$$

for a bounded linear functional Φ on $\bigoplus_{i=1}^N V^{\otimes i}$, a smooth 1-form ψ, a weakly geometric rough path x and $N \in \mathbb{N}$. In Section 4.2.2., we will show that for any two disjoint pieces of signature paths, there will be a finite dimensional projection so that their images remain separated.

4.2.1 Integration against the signature path

Let x be a weakly geometric rough path. In order to apply Lemma 4.1 to $S_N(x)$, we need to lift $S_N(x)_{0,\cdot}$ as a weakly geometric rough path. Let $W = \bigoplus_{i=1}^{N} V^{\otimes i}$. We will implicitly identify $W^{\otimes n}$ with $\bigoplus_{i_1,\ldots,i_n=1}^{N} V^{\otimes(i_1+\ldots+i_n)}$. Let $\pi_{i_1,\ldots,i_n}$ denote the projection of $\bigoplus_{i_1,\ldots,i_n=1}^{N} V^{\otimes(i_1+\ldots+i_n)}$ to the component $(i_1,\ldots,i_n)$. We will equip $W^{\otimes n}$ with the norm

$$\|v\|_{W^{\otimes n}} = \sum_{1\leq i_1,\ldots,i_n\leq N} \|\pi_{i_1,\ldots,i_n}(v)\|_{V^{\otimes(i_1+\ldots+i_n)}}. \tag{4.2}$$

It is easy to see that the family of tensor norms $\|\cdot\|_{W^{\otimes n}}$ is admissible. In this section, we will use the notation π_k to denote the projection from $T((W))$ onto $W^{\otimes k}$.

Lemma 4.2. *Let $N\in\mathbb{N}$. There exists a map $\mathcal{J} : WG\Omega_p(V) \to WG\Omega_p\left(\bigoplus_{i=0}^{N} V^{\otimes i}\right)$ such that for all $x \in WG\Omega_p(V)$:*

1. *$\pi_1(\mathcal{J}(x)) = S_N(x)_{0,\cdot}$;*
2. *If $x,y\in WG\Omega_p(V)$ is such that $S(x)=S(y)$, then $S(\mathcal{J}(x)) = S(\mathcal{J}(y))$.*

Proof. Suppose for now that x is a path with bounded variation and $S(x) = \left(1, X^1, X^2, \ldots\right)$ where $X^i \in V^{\otimes i}$. Then we may define $\mathcal{J}$ by condition 1. in the Lemma. We define:

1. for $v_1 \in V^{\otimes k_1},\ldots,v_n\in V^{\otimes k_n}$, a map $F_{m_1,\ldots,m_n}(v_1,\ldots,v_n)$ on $V^{\otimes(m_1+\ldots+m_n)}$ so that for all $w_1\in V^{\otimes m_1},\ldots,w_n\in V^{\otimes m_n}$,

$$\begin{aligned}
& F_{m_1,\ldots,m_n}(v_1,\ldots,v_n)\left[w_1\otimes\ldots\otimes w_{m_1+\ldots+m_n}\right] \\
= \;& v_1\otimes w_1\otimes\ldots\otimes w_{m_1}\otimes v_2\otimes w_{m_1+1}\otimes\ldots\otimes w_{m_1+m_2} \\
& \otimes\ldots\otimes v_n\otimes w_{m_1+\ldots+m_{n-1}+1}\otimes\ldots\otimes w_{m_1+\ldots+m_n};
\end{aligned}$$

2. for a permutation σ on $\{1,\ldots,n\}$, a map on $V^{\otimes n}$, also denoted by σ, by

$$\sigma(v_1\otimes\ldots\otimes v_n) = v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)};$$

3. for $j_1,\ldots,j_n\in\mathbb{N}$, $OS(j_1,\ldots,j_n)$ as the set of ordered shuffles (see p72 [22]).

By Chen's identity (2.1),

$$\begin{aligned}
& \pi_{i_1,\ldots,i_n}\left(S(\mathcal{J}(x))_{s,t}\right) && (4.3)\\
= \;& \int_{s<s_1<\ldots<s_n<t} \mathrm{d}X^{i_1}_{0,s_1}\otimes\ldots\otimes \mathrm{d}X^{i_n}_{0,s_n} && (4.4)\\
= \;& \sum_{j_n=1}^{i_n}\cdots\sum_{j_1=1}^{i_1} F_{j_1,\ldots,j_n}\left(X^{i_1-j_1}_{0,s},\ldots,X^{i_n-j_n}_{0,s}\right)\left[\int_{s<s_1<\ldots<s_n<t}\mathrm{d}X^{j_1}_{s,s_1}\otimes\ldots\otimes\mathrm{d}X^{j_n}_{s,s_n}\right] \\
= \;& \sum_{j_n=1}^{i_n}\cdots\sum_{j_1=1}^{i_1} F_{j_1,\ldots,j_n}\left(X^{i_1-j_1}_{0,s},\ldots,X^{i_n-j_n}_{0,s}\right)\left[\sum_{\pi\in OS(j_1,\ldots,j_n)}\pi\left(X^{j_1+\ldots+j_n}_{s,t}\right)\right]. && (4.5)
\end{aligned}$$

The final identity (4.5) uses identity (4.9) in [22].

We first restrict our attention to the finite dimensional case. For $x \in WG\Omega_p\left(\mathbb{R}^d\right)$, we will define $\mathcal{J}(x)$ so that (4.3) holds. To show that $\mathcal{J}(x)$ is a weakly geometric rough path, we observe that if x were to have bounded variation, then $S(\mathcal{J}(x))$ will satisfy Chen's identity and lie in $G^{(*)}_{p.r.c}$ ($G^{(*)}_{p.r.c}$ here refers to group-like elements over $\bigoplus_{i=0}^{N} V^{\otimes i}$ instead of V). For $x \in WG\Omega_p\left(\mathbb{R}^d\right)$, let $p' > p$ and let x_n be a sequence of bounded variation paths converging in $d_{p'-var}$ to x (see Corollary 8.26 in [14] for the existence of such sequence). As $S(x_n) \to S(x)$ as $n \to \infty$ (Corollary 9.11 in [14]), each $S(\mathcal{J}(x_n))$ satisfies Chen's identity (2.1) and lies in $G^{(*)}_{p.r.c.}$. We see by taking limit that $S(\mathcal{J}(x))$ will still satisfy Chen's identity and lie in $G^{(*)}_{p.r.c}$. Moreover, for all terms in the sum in (4.5) $j_1+\ldots+j_n \geq n$, which implies that $S(\mathcal{J}(x))$ has finite p-variation. Therefore, $\mathcal{J}$ maps $WG\Omega_p(V)$ to $WG\Omega_p\left(\bigoplus_{i=0}^{N} V^{\otimes i}\right)$. The expression (4.5) gives not just the expression for $\mathcal{J}(x)$ but also for the signature of $\mathcal{J}(x)$, which in particular implies the signature of the map $\mathcal{J}(x)$ is determined by the signature of x. The infinite dimensional case of this result is included as Lemma 1.2 in [5] $\square$

Lemma 4.3. *Let W be a Banach space and $\Phi : W \to \mathbb{R}^d$ be a continuous linear functional on W. Then there exists a map $\mathbf{F} : WG\Omega_p(W) \to WG\Omega_p\left(\mathbb{R}^d\right)$ such that for all $x \in WG\Omega_p(W)$:*

1. $\pi_1(\mathbf{F}(x)) = \Phi(\pi_1(x))$;

2. If $x, y \in WG\Omega_p(W)$ is such that $S(x) = S(y)$, then $S(\mathbf{F}(x)) = S(\mathbf{F}(y))$.

Proof. For any linear map $\Phi : W \to \mathbb{R}^d$, by the admissiblity conditions (2.2) and (2.4) of the tensor product, we may continuously extend $\mathbf{\Phi}$ to a bounded linear operator on $T^{(N)}(W)$ such that for $w_1, \ldots, w_N \in W$,

$$\mathbf{\Phi}(w_1 \otimes \ldots \otimes w_N) = \mathbf{\Phi}(w_1) \otimes \ldots \otimes \mathbf{\Phi}(w_N).$$

Let $x \in WG\Omega_p(W)$. As Φ is a bounded linear operator and the family of tensor norms on $(W^{\otimes n} : 1 \leq n \leq N)$ satisfies the admissibility conditions (2.2) and (2.4), $\Phi(x)$ has finite p-variation. As Φ is a homomorphism with respect to $\otimes$, for all $t \geq 0$, $\Phi(x_t)$ lies in the $\lfloor p \rfloor$-step free nilpotent Lie group $G^{\lfloor p \rfloor}$ over $\mathbb{R}^d$. Therefore, $\Phi(x) \in WG\Omega_p\left(\mathbb{R}^d\right)$. By construction, $\pi_1(\Phi(x)) = \Phi(\pi_1(x))$. Moreover, again by the homomorphism property of Φ and admissibility conditions (2.2) and (2.4) of the tensor norms, we have

$$\Phi(S_N(x)) = S_N(\Phi(x))$$

which implies property 2. in the Lemma. $\square$

4.2.2 Separation of signature paths

The following two lemmas together will tell us that with a careful choice of truncation or finite dimensional projection, the images of disjoint signature paths will remain disjoint.

Lemma 4.4. *Let $\Delta = \left\{(s,t) \in [0,T]^2 : s \leq t\right\}$. Let $S : [0,T] \to G^{(*)}_{p.r.c}$ be an injective path. Then for any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that $\pi^{(N)}(S_s) \neq \pi^{(N)}(S_t)$ for every $N \geq N(\varepsilon)$ and $(s,t) \in \Delta$ with $|t-s| \geq \varepsilon$.*

Proof. Let $\Delta_\varepsilon = \{(s,t) \in \Delta : \ t - s \geq \varepsilon\}$. For each $(s,t) \in \Delta_\varepsilon$, since $S(x)_{0,s} \neq S(x)_{0,t}$, there exists some $N_{s,t} \in \mathbb{N}$ such that

$$S_{N_{s,t}}(x)_{0,s} \neq S_{N_{s,t}}(x)_{0,t}. \tag{4.6}$$

By continuity, (4.6) holds in an open neighbourhood of (s,t). The result then follows easily from the compactness of Δ_ε. □

Lemma 4.5. *Let V be a Banach space, and K, L be two disjoint compact subsets of V. Then there exists $d \in \mathbb{N}$ and a continuous linear functional $\Phi : \ V \to \mathbb{R}^d$, such that $\Phi(K)$ and $\Phi(L)$ are disjoint in $\mathbb{R}^d$.*

Proof. By Hahn-Banach theorem, for each $\kappa \in K$ and $\lambda \in L$, there exists a bounded linear functional $f_{\kappa,\lambda} : V \to \mathbb{R}$ such that

$$f_{\kappa,\lambda}(\kappa) \neq f_{\kappa,\lambda}(\lambda).$$

Fix $\lambda \in L$. By the continuity of $f_{\kappa,\lambda}$ for each $\kappa \in K$ and the compactness of K, there exists $\kappa_1, \ldots, \kappa_j \in K$ such that the continuous map

$$\Phi_\lambda(\cdot) = (f_{\kappa_1,\lambda}(\cdot), \ldots, f_{\kappa_1,\lambda}(\cdot))$$

sends the set K and $\{\lambda\}$ to disjoint sets. By the continuity of Φ_λ for each $\lambda \in L$ and the compactness of L, there exists $\lambda_1, \ldots, \lambda_{j'} \in L$ such that

$$\Phi(\cdot) = \left(f_{\kappa_1,\lambda_1}(\cdot), \ldots, f_{\kappa_j,\lambda_1}(\cdot), f_{\kappa_1,\lambda_2}(\cdot), \ldots, f_{\kappa_j,\lambda_{j'}}(\cdot)\right)$$

maps K and L to disjoint sets. □

4.3 Identifying signatures with signature paths

The following Lemma lies at the heart of the proof of our main result, Theorem 1.1, and is interesting in its own right.

Lemma 4.6. *(Existence and uniqueness of reduced path)Let $S : [0,T] \to G^{(*)}_{p.r.c}$ be a continuous path with finite p-variation. There exists an injective path $\tilde{S} : \left[0, \tilde{T}\right] \to G^{(*)}_{p.r.c.}$, unique up to reparametrisation, such that $S_T = \tilde{S}_{\tilde{T}}$ and $S_0 = \tilde{S}_0$.*

Remark 4.1. In the case $p = 1$, the weakly geometric rough path $\pi_1\left(\tilde{S}\right)$ is *reduced* in the sense of Hambly-Lyons [16], meaning that it is the unique, up to translation and reparametrisation, minimiser of the set

$$\left\{\|x\|_{1-var} : x \in WG\Omega_1(V), S(x)_{0,1} = \tilde{S}_0^{-1}\tilde{S}_1\right\}.$$

For weakly geometric rough paths, we define reduced path to be a weakly geometric rough path x such that the path $t \to S(x)_{0,t}$ is injective.

Proof. For the existence part, let $\tilde{S} : [0,T] \to G^{(*)}_{p.r.c}$ be the injective path obtained by applying Lemma 3.2 to erase the loops in $S.$. Then by the order preserving property of q and (iii) in the loop erasing Lemma 3.2

$$\left\|\tilde{S}\right\|_{p-var} \quad \leq \quad \|S\|_{p-var}\,, \tag{4.7}$$

which implies that $\tilde{S}$ has finite p-variation.

For the uniqueness part, note the topological fact that two injective continuous paths are reparametrisation of each other if and only if they have the same starting point, ending point and image (see Lemma 26 in [3]). Assume, for contradiction, that $t \to S_t$ and $t \to \tilde{S}_t$ are injective, $S_0 = \tilde{S}_0$, $S_T = \tilde{S}_{\tilde{T}}$ but S and $\tilde{S}$ do not have the same image. We may assume without loss of generality that $S_0 = \tilde{S}_0 = \mathbf{1}$. Then there exists $s_1 < s_2 < t_2 < t_1$ such that

$$\begin{aligned} S_{[s_2,t_2]} \bigcap \left(S_{[0,s_1] \bigcup [t_1,T]} \bigcup \operatorname{Im}\left(\tilde{S}.\right)\right) &= \emptyset, \\ S_{[s_1,s_2]} \bigcap S_{[t_2,t_1]} &= \emptyset. \end{aligned} \tag{4.8}$$

It follows from the separation of finite dimensional projection results, Lemma 4.4 and Lemma 4.5, that there exists $N \in \mathbb{N}$ and a linear operator $\Phi : T^{(N)}(V) \to \mathbb{R}^d$ such that (4.8) holds with S and $\tilde{S}$ replaced by $\Phi\big(\pi^{(N)}(S)\big)$ and $\Phi\big(\pi^{(N)}\left(\tilde{S}\right)\big)$.

Let U_1, V_1, U_2, V_2 be bounded open neighborhoods of $\Phi\big(\pi^{(N)}(S)\big)_{[t_2,t_1]}$, $\Phi\big(\pi^{(N)}(S)\big)_{[s_1,s_2]}$, $\Phi\big(\pi^{(N)}(S)\big)_{[s_2,t_2]}$, $\Phi\big(\pi^{(N)}(S)\big)_{[0,s_1]\bigcup[t_1,T]} \bigcup \operatorname{Im}\left(\Phi\big(\pi^{(N)}(\tilde{S}.)\big)\right)$ respectively, such that

$$\overline{U_1} \bigcap \overline{V_1} = \emptyset,\ \overline{U_2} \bigcap \overline{V_2} = \emptyset.$$

Let $f_1, f_2 \in C_c^\infty\left(\mathbb{R}^d\right)$ be such that for $i = 1, 2$,

$$f_i(\mathbf{X}) = \begin{cases} 1, & \mathbf{X} \in \overline{U_i}; \\ 0, & \mathbf{X} \in \overline{V_i}. \end{cases}$$

Consider the 1-form $\varphi = f_2 \mathrm{d} f_1$. As the path $u \to \pi^{(\lfloor p \rfloor)}(S_u)$ is a weakly geometric rough path, Lemma 4.2 and Lemma 4.3 together states that $\Phi\big(\pi^{(N)}(S.)\big)$ can be canonically lifted as a weakly geometric rough path. Moreover, the signature of $\Phi\big(\pi^{(N)}(S.)\big)$ is a function of S_T. It follows that the integration of 1-form against $\Phi\big(\pi^{(N)}(S_u)\big)$ can be defined and

$$\int_0^T \varphi\left(\mathrm{d}\Phi\big(\pi^{(N)}(S_u)\big)\right) \quad = \quad \int_{s_2}^{t_2} \mathrm{d} f_1\left(\Phi\big(\pi^{(N)}(S_u)\big)\right) = 1,$$

while

$$\int_0^{\tilde{T}} \varphi\left(\mathrm{d}\Phi\big(\pi^{(N)}\left(\tilde{S}_u\right)\big)\right) = 0.$$

This leads to a contradiction to Lemma 4.1 which states that the integral of 1-form is a functional of the signature. $\square$

4.4 Completing the proof

Definition 4.1. Let $\mathcal{S}_p$ denote the set of injective continuous paths in $G^{(*)}_{p.r.c}$ with finite p-variation starting at $\mathbf{1}$. Define a relation $\preceq$ on $\mathcal{S}_p$ by

$$x \preceq y \iff \exists t \geq 0,\ x \text{ is a reparametrisation of } y|_{[0,t]}.$$

Lemma 4.7. *The space $(\mathcal{S}_p, \preceq)$ is a partially ordered set such that:*

1. *$\mathcal{S}_p$ has a least element $\mathbf{1}. : [0,0] \to \mathbf{1}$.*
2. *For all $S \in \mathcal{S}_p$, the set $\left\{\hat{S} \in \mathcal{S}_p : \hat{S} \preceq S\right\}$ is totally ordered.*
3. *For all $S, \tilde{S} \in \mathcal{S}_p$, there exists an element $S \wedge \tilde{S} \in \mathcal{S}_p$, unique up to reparametrisation, such that*

$$\left\{\hat{S} \in \mathcal{S}_p : \hat{S} \preceq S, \hat{S} \preceq \tilde{S}\right\} = \left\{\hat{S} \in \mathcal{S}_p : \hat{S} \preceq S \wedge \tilde{S}\right\}. \tag{4.9}$$

4. *The function $\|\cdot\|^p_{p-var} : S \to \|S\|^p_{p-var}$ has the property that $\|\mathbf{1}\|^p_{p-var} = 0$ and, for all S the restriction of $\|\cdot\|^p_{p-var}$ on the set $\left\{\hat{S} \in \mathcal{S}_p : \hat{S} \preceq S\right\}$ is strictly increasing.*

Proof. The only non-trivial statement is statement 3. The uniqueness follows trivially from (4.9). We now show the existence. Let

$$t = \sup\left\{\hat{t} \in [0,T] : S_{\hat{t}} \in \tilde{S}_{\left[0,\tilde{T}\right]}\right\}.$$

We first show that the inclusion $\supseteq$ in (4.9) holds with $S \wedge \tilde{S}$ replaced by $S|_{[0,t]}$. By the continuity of S and that $\tilde{S}_{\left[0,\tilde{T}\right]}$ is closed, there exists $\tilde{t}$ such that $S_t = \tilde{S}_{\tilde{t}}$. This implies $S|_{[0,t]}$ is a reparametrisation of $\tilde{S}|_{\left[0,\tilde{t}\right]}$ by the uniqueness of reduced path, Lemma 4.6. In particular, $S|_{[0,t]} \preceq \tilde{S}$ and the desired inclusion follows.

Conversely, if $\hat{S} \preceq S$ and $\hat{S} \preceq \tilde{S}$, then there exists $\hat{t}$ such that $\hat{S}$ is a reparametrisation of $S|_{\left[0,\hat{t}\right]}$ and hence $S_{\hat{t}} \in \tilde{S}_{\left[0,\tilde{T}\right]}$. In particular, $\hat{t} \leq t$ and $\hat{S} \preceq S|_{[0,t]}$. □

Proposition 4.1. *For $S^{(1)}, S^{(2)} \in \mathcal{S}_p$, define*

$$d\left(S^{(1)}, S^{(2)}\right) = \left\|S^{(1)}\right\|^p_{p-var} + \left\|S^{(1)}\right\|^p_{p-var} - 2\left\|S^{(1)} \wedge S^{(2)}\right\|^p_{p-var}.$$

Then $(\mathcal{S}_p, d)$ is a $\mathbb{R}$-tree.

Proof. By Proposition 3.10 in [12] (or see Lemma 1.7 in [5]), a partially ordered set satisfying 1.-4. in Lemma 4.7 is a $\mathbb{R}$-tree with metric d. □

Let $\left(\mathcal{G}_p, d_{\mathcal{G}_p}\right)$ denote the metric space defined as the pushforward, under the map $\mathbf{P} : S \to S_T$ (sending a path to its value at terminal time), of the metric space $(\mathcal{S}_p, d)$. As a set, Lemma 4.6 implies that

$$\mathcal{G}_p = \left\{S\left(x\right)_{0,T} : x \in WG\Omega_p\left(V\right)\right\}.$$

The metric space $\left(\mathcal{G}_p, d_{\mathcal{G}_p}\right)$ as the isometric image of a $\mathbb{R}$-tree is itself a $\mathbb{R}$-tree. The injective map $\mathbf{P}$ induces naturally a partial order, also denoted as $\preceq$, and an operation $\wedge$ on $\mathcal{G}_p$ satisfying (4.9) with $\mathcal{S}_p$ replaced by $\mathcal{G}_p$.

Lemma 4.8. *(Continuity estimate for the right concatenation)Let $S:[0,T]\to G^{(*)}_{p.r.c.}$ has finite p-variation. Then*

$$\left\|S\right\|_{p-var}^{p}-\left\|S|_{[0,s]}\right\|_{p-var}^{p}\leq\left(1+p\right)\left\|S\right\|_{p-var}^{p-1}\left\|S|_{[s,T]}\right\|_{p-var}. \tag{4.10}$$

Proof. Let $\mathcal{P}=\left(t_0<t_1<\ldots<t_n\right)$ be a partition of $[0,T]$. Let j be the last time in $\mathcal{P}$ such that $t_j\leq s$. Then

$$\begin{aligned}
&\sum_{i=0}^{n-1} d\left(S_{t_i},S_{t_{i+1}}\right)^p\\
\leq\;&\sum_{i=0}^{j-1} d\left(S_{t_i},S_{t_{i+1}}\right)^p+d\left(S_{t_j},S_s\right)^p+d\left(S_s,S_{t_{j+1}}\right)^p+\sum_{i=j+1}^{n-1} d\left(S_{t_i},S_{t_{i+1}}\right)^p\\
&+\left[d\left(S_{t_j},S_{t_{j+1}}\right)^p-d\left(S_{t_j},S_s\right)^p-d\left(S_s,S_{t_{j+1}}\right)^p\right].
\end{aligned} \tag{4.11}$$

By the mean value theorem and triangle inequality,

$$d\left(S_{t_j},S_{t_{j+1}}\right)^p-d\left(S_{t_j},S_s\right)^p\leq p\left\|S\right\|_{p-var}^{p-1} d\left(S_s,S_{t_{j+1}}\right),$$

which, together with (4.11), implies (4.10). $\square$

Lemma 4.9. *If $S:[0,T]\to G^{(*)}_{p.r.c}$ is continuous and has finite p-variation, then $S.$ is continuous in $\left(\mathcal{G}_p, d_{\mathcal{G}_p}\right)$.*

Proof. We first argue that for all $s<t$, there is a $u\in[s,t]$ such that $S_u=S_s\wedge S_t$. By applying Lemma 3.2 to erase loops from $S|_{[s,t]}$, we obtain an injective path $\tilde{\mathbf{S}}$ with finite p-variation connecting S_s and S_t. By the definition of $S_s\wedge S_t$, there is an injective path $\hat{\mathbf{S}}$ with finite p-variation connecting S_s to $S_s\wedge S_t$ and then to S_t. By the uniqueness of reduced path (Lemma 4.6), $\hat{\mathbf{S}}$ must coincide with $\tilde{\mathbf{S}}$, implying our claim. We now note that

$$\begin{aligned}
d_{\mathcal{G}_p}\left(S_s,S_t\right)\quad\leq\quad&\left\|\mathbf{P}^{-1}\left(S_u\right)\star\overleftarrow{S|_{[s,u]}}\right\|_{p-var}^{p}-\left\|\mathbf{P}^{-1}\left(S_u\right)\right\|_{p-var}^{p}\\
&+\left\|\mathbf{P}^{-1}\left(S_u\right)\star S_{[u,t]}\right\|_{p-var}^{p}-\left\|\mathbf{P}^{-1}\left(S_u\right)\right\|_{p-var}^{p}
\end{aligned}$$

which converges to 0 as $t\to s$ by the continuity estimate for the right concatenation, Lemma 4.8. $\square$

Proof of the "Paths with trivial signature are tree-like" part of Theorem 2.1. Let $x\in WG\Omega_p\left(V\right)$ be such that $S\left(x\right)_{0,T}=1$. By the Extension Theorem (Proposition 2.1), $S\left(x\right)_{0,t}\in G^{(*)}_{p.r.c.}$ for all t. In the Definition 1.1 of tree-like path, take τ to be $\left(\mathcal{G}_p, d_{\mathcal{G}_p}\right)$ (which is a $\mathbb{R}$-tree as shown earlier in Proposition 4.1), $\phi\left(t\right)=S\left(x\right)_{0,t}$ and $\psi\left(z\right)=\pi_{\lfloor p\rfloor}\left(z\right)$. Then by the definition of signature, $x_t=\pi_{\lfloor p\rfloor}\left(S\left(x\right)_{0,t}\right)$. The continuity of ϕ has been shown in Lemma 4.9. $\square$

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