Translation invariant realizability problem on the d-dimensional lattice: an explicit construction

Article

Published Version

It is advisable to refer to the publisher's version if you intend to cite from the work. See Guidance on citing.

www.reading.ac.uk/centaur

CentAUR

Central Archive at the University of Reading

Reading's research outputs online

Translation invariant realizability problem on the d-dimensional lattice: an explicit construction[*]

Emanuele Caglioti[†]   Maria Infusino[‡]   Tobias Kuna[§]

Abstract

We consider a particular instance of the truncated realizability problem on the $d-$dimensional lattice. Namely, given two functions $\rho_1(\mathbf{i})$ and $\rho_2(\mathbf{i}, \mathbf{j})$ non-negative and symmetric on $\mathbb{Z}^d$, we ask whether they are the first two correlation functions of a translation invariant point process. We provide an explicit construction of such a realizing process for any $d \geq 2$ when the radial distribution has a specific form. We also derive from this construction a lower bound for the maximal realizable density and compare it with the already known lower bounds.

Keywords: truncated moment problem; realizability; point processes; translation invariant; infinite dimensional moment problem.
AMS MSC 2010: 44A60; 60G55; 82B20.
Submitted to ECP on October 10, 2015, final version accepted on April 11, 2016.
Supersedes arXiv:1510.02954.

1 Introduction

Let d be a positive integer. Given a point process $P = \{P_\mathbf{i}\}_{\mathbf{i} \in \mathbb{Z}^d}$ on the $d-$dimensional lattice, $P_\mathbf{i} \in \{0, 1\}$, whose distribution is described by the probability measure μ, we define the first and second order *correlation function* as follows

$$\begin{cases} \rho_1(\mathbf{i}) := \langle P_\mathbf{i} \rangle \\ \rho_2(\mathbf{i}, \mathbf{j}) := \langle P_\mathbf{i} P_\mathbf{j} \rangle - \rho_1(\mathbf{i}) \delta(\mathbf{i} - \mathbf{j}) \end{cases},$$

where $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$, δ is the Dirac delta function and $\langle \cdot \rangle$ denotes the expectation w.r.t. μ.

The *truncated realizability problem* addresses the inverse question. Namely, given two functions $\rho_1(\mathbf{i})$ and $\rho_2(\mathbf{i}, \mathbf{j})$ non-negative and symmetric for all $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$, does there exist a point process P for which these are the correspondent first and second order correlation functions? Clearly, the truncated realizability problem can be posed for any finite sequence of non-negative and symmetric functions $(\rho_k(\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_k))_{k=1}^n$. When the question is asked for a given infinite sequence $(\rho_k)_{k \in \mathbb{N}}$, then the problem is addressed as

full realizability problem (see e.g. [15, 16] for a systematic study of the full realizability problem for point processes and [7, 8] for recent developments).

In the following we will consider the important special case of *translation invariant point processes*, which actually contains all the essential difficulties of the problem. In this case the realizability problem asks if, for given $\rho \in \mathbb{R}^+$ and $g : \mathbb{Z}^d \to \mathbb{R}^+$ symmetric, there exists a translation invariant point process such that its first two correlation functions are given by

$$\left\{ \begin{array}{l} \rho_1(\mathbf{i}) = \rho \\ \rho_2(\mathbf{i}, \mathbf{j}) = \rho^2 g(\mathbf{i} - \mathbf{j}) \end{array} \right. . \tag{1.1}$$

If such a process exists, then it is said to be *realizing* and the pair (ρ, g) is called *realizable* on $\mathbb{Z}^d$. Note that writing the second order correlation in this form is not an additional restriction beyond the assumption of translation invariance. The function g is known in classic fluid theory as *radial distribution*, [6].

The truncated realizability problem is in fact a longstanding problem in the classical theory of fluids (see e.g. [5, 19, 20]), but it has been investigated in many other contexts such as stochastic geometry [17], spatial statistics [3, 23], spatial ecology [18] and neural spike trains [1, 9], just to name a few. In particular, Stillinger, Torquato et al. developed fascinating applications in the study of heterogeneous materials and mesoscopic structures based on the solvability of the truncated realizability problem (see e.g. [4, 21, 22, 24, 25, 26]). A structural investigation of this problem was recently started in [13], where the authors identify the realizability problem as a particular instance of the infinite-dimensional truncated moment problem (see [2, 10, 11, 12, 14] for further recent developments about the truncated realizability problem for point processes). As far as we know, the only earlier reference about the truncated infinite-dimensional moment problem is [28].

In this paper, we will show how to explicitly construct a point process on the $d-$dimensional lattice with $d \geq 2$ such that, for given $\alpha \geq 0$, (1.1) holds for certain values of ρ and for $g = g^{(\alpha)}$ defined as follows:

$$g^{(\alpha)}(\mathbf{x}) := \left\{ \begin{array}{ll} 0 & \text{if } \mathbf{x} = 0 \\ \alpha & \text{if } |\mathbf{x}| = 1 \\ 1 & \text{if } |\mathbf{x}| > 1 \end{array} \right. . \tag{1.2}$$

Explicit constructions of point processes realizing this lattice problem in the case $d = 1$ were provided in [12, Appendix 1]. Such a problem has been extensively studied for the case $\alpha = 0$ by Stillinger and Torquato in [21] (see also [4, 25]). The function $g^{(0)}$ describes a model with on-site and nearest neighbour exclusion and with no correlation for pairs of sites separated by two or more lattice spacings.

From [12, Section 1], we know that for a fixed α the set of realizable densities ρ is an interval $[0, \bar{\rho}_\alpha(d)]$ with $0 < \bar{\rho}_\alpha(d) \leq 1$. Moreover, in [12] the authors discuss:

(i) certain general methods which, when applied to (1.2), yield lower bounds for $\bar{\rho}_\alpha(d)$ in any dimension d.

(ii) concrete upper and lower bounds for $\bar{\rho}_\alpha(1)$. In particular, the lower bounds improve those obtained from the general methods (i).

Our $d-$dimensional construction combined with the one-dimensional lower bounds (ii) provides a lower bound for $\bar{\rho}_\alpha(d)$ for any $d \geq 2$ and any $\alpha \geq 0$. We will briefly compare this with the lower bound obtained from the general methods (i). We also follow techniques from [12] to get an upper bound for $\bar{\rho}_\alpha(d)$.

2 An explicit realizing translation invariant point process on $\mathbb{Z}^d$

In the following, we explicitly construct a point process $P = \{P_\mathbf{i}\}_{\mathbf{i} \in \mathbb{Z}^d}$ on the $d-$dimensional lattice with $d \geq 2$ such that, for given $\alpha \geq 0$, the following hold for certain values of ρ (depending on α and on d):

$$\langle P_\mathbf{i} \rangle = \rho \tag{2.1}$$

and

$$\langle P_\mathbf{i} P_\mathbf{j} \rangle = \left\{ \begin{array}{ll} \rho & \text{if } \mathbf{i} = \mathbf{j} \\ \alpha\rho^2 & \text{if } |\mathbf{i} - \mathbf{j}| = 1 \\ \rho^2 & \text{if } |\mathbf{i} - \mathbf{j}| > 1 \end{array} \right. , \tag{2.2}$$

that is, the radial distribution is given by (1.2).

2.1 Construction in dimension 2

In order to build such a process on $\mathbb{Z}^2$ we start from a realizing one-dimensional process achieving density γ. Namely, given $\alpha \geq 0$, we consider a point process $\{A_i\}_{i \in \mathbb{Z}}$, $A_i \in \{0, 1\}$, defined on the one-dimensional lattice and such that for some $\gamma > 0$ we have

$$\langle A_i \rangle = \gamma,$$

and

$$\langle A_i A_j \rangle = \left\{ \begin{array}{ll} \gamma & \text{if } i = j \\ \alpha\gamma^2 & \text{if } |i - j| = 1 \\ \gamma^2 & \text{if } |i - j| > 1 \end{array} \right. .$$

We denote a process of this kind by $BP\gamma$ that stays for *basic process with density γ*. As pointed out in the introduction, there exists a good number of constructions of realizing processes in the one-dimensional case, see e.g. [12, Appendix 1]. The results in the one-dimensional case relevant to our investigation (in particular the range where γ can vary) are recalled in Section 3, which is devoted to the discussion of the maximal realizable density in any dimension.

Let us define two processes $B^{(1)} = \left\{ B^{(1)}_{i_1,i_2} \right\}_{(i_1,i_2) \in \mathbb{Z}^2}$ and $B^{(2)} = \left\{ B^{(2)}_{i_1,i_2} \right\}_{(i_1,i_2) \in \mathbb{Z}^2}$ on $\mathbb{Z}^2$ as follows. For a fixed $i_1 \in \mathbb{Z}$, the process $\left\{ B^{(1)}_{i_1,i_2} \right\}_{i_2 \in \mathbb{Z}}$ is a $BP\gamma$ in i_2. For any $i_1, j_1 \in \mathbb{Z}$ with $i_1 \neq j_1$, the processes $\left\{ B^{(1)}_{i_1,i_2} \right\}_{i_2 \in \mathbb{Z}}$ and $\left\{ B^{(1)}_{j_1,j_2} \right\}_{j_2 \in \mathbb{Z}}$ are independent. In particular, we have

$$\langle B^{(1)}_{i_1,i_2} B^{(1)}_{j_1,j_2} \rangle = \left\{ \begin{array}{ll} \gamma^2 & \text{if } i_1 \neq j_1 \\ \gamma & \text{if } i_1 = j_1 \text{ and } i_2 = j_2 \\ \alpha\gamma^2 & \text{if } i_1 = j_1 \text{ and } |i_2 - j_2| = 1 \\ \gamma^2 & \text{if } i_1 = j_1 \text{ and } |i_2 - j_2| > 1 \end{array} \right. . \tag{2.3}$$

In other words, the process $B^{(1)}$ can be seen as a sequence of vertical $BP\gamma$'s independent one from each other (see Figure 1 for an example).

Similarly, the process $B^{(2)}$ is defined as a sequence of horizontal $BP\gamma$'s independent one from each other (see Figure 2 for an example), i.e.

$$\langle B^{(2)}_{i_1,i_2} B^{(2)}_{j_1,j_2} \rangle = \left\{ \begin{array}{ll} \gamma^2 & \text{if } i_2 \neq j_2 \\ \gamma & \text{if } i_2 = j_2 \text{ and } i_1 = j_1 \\ \alpha\gamma^2 & \text{if } i_2 = j_2 \text{ and } |i_1 - j_1| = 1 \\ \gamma^2 & \text{if } i_2 = j_2 \text{ and } |i_1 - j_1| > 1 \end{array} \right. . \tag{2.4}$$
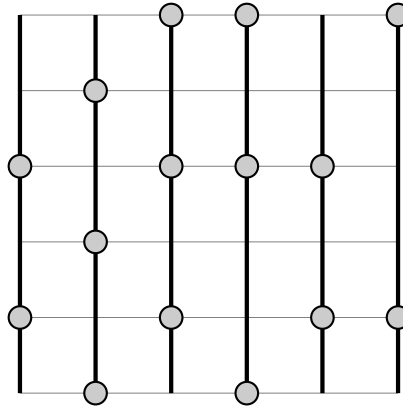
Figure 1: Example of process $B^{(1)}$ (with $\alpha = 0$)
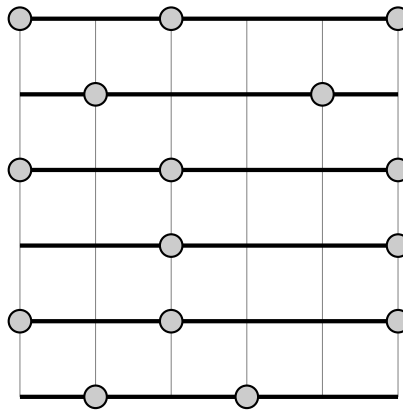
Figure 2: Example of process $B^{(2)}$ (with $\alpha = 0$)

Let us define now the process $P = \{P_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^2}$ as

$$P_{i_1,i_2} := B^{(1)}_{i_1,i_2} B^{(2)}_{i_1,i_2},$$

(see Figure 3 for P constructed from the basic processes in Figures 1 and 2). Since the processes $B^{(1)}$ and $B^{(2)}$ are independent, we get

$$\langle P_{i_1,i_2} \rangle = \langle B^{(1)}_{i_1,i_2} B^{(2)}_{i_1,i_2} \rangle = \langle B^{(1)}_{i_1,i_2} \rangle \langle B^{(2)}_{i_1,i_2} \rangle = \gamma \cdot \gamma = \gamma^2.$$

Hence, (2.1) holds for $\rho = \gamma^2$. From the independence of $B^{(1)}$ and $B^{(2)}$, we also get

$$\langle P_{i_1,i_2} P_{j_1,j_2} \rangle = \langle B^{(1)}_{i_1,i_2} B^{(2)}_{i_1,i_2} B^{(1)}_{j_1,j_2} B^{(2)}_{j_1,j_2} \rangle = \langle B^{(1)}_{i_1,i_2} B^{(1)}_{j_1,j_2} \rangle \langle B^{(2)}_{i_1,i_2} B^{(2)}_{j_1,j_2} \rangle. \tag{2.5}$$

We can easily check, by using (2.3) and (2.4) in (2.5), that (2.2) holds for $\rho = \gamma^2$. In fact, we need to consider only the following four cases, because all the others are equivalent to these ones by symmetry.

a) If $i_1 = j_1$ and $i_2 = j_2$ then

$$\langle P_{i_1,i_2} P_{j_1,j_2} \rangle = \gamma \cdot \gamma = \gamma^2.$$

b) If $i_1 = j_1$ and $i_2 \neq j_2$ then

$$\langle P_{i_1,i_2} P_{j_1,j_2} \rangle = \langle B^{(1)}_{i_1,i_2} B^{(1)}_{j_1,j_2} \rangle \gamma^2.$$

Therefore:

- if $i_2 = j_2 + 1$ then $\langle B_{i_1,i_2}^{(1)} B_{j_1,j_2}^{(1)} \rangle = \alpha\gamma^2$ and so $\langle P_{i_1,i_2} P_{j_1,j_2} \rangle = \alpha\gamma^4$

- if $|i_2 - j_2| > 1$ then $\langle B_{i_1,i_2}^{(1)} B_{j_1,j_2}^{(1)} \rangle = \gamma^2$ and so $\langle P_{i_1,i_2} P_{j_1,j_2} \rangle = \gamma^4$.

c) If $i_1 = j_1 + 1$ and $|i_2 - j_2| > 1$ then

$$\langle P_{i_1,i_2} P_{j_1,j_2} \rangle = \langle B_{i_1,i_2}^{(1)} B_{j_1,j_2}^{(1)} \rangle \langle B_{i_1,i_2}^{(2)} B_{j_1,j_2}^{(2)} \rangle = \gamma^4.$$

d) If $|i_1 - j_1| > 1$ and $i_2 \neq j_2$ then

$$\langle P_{i_1,i_2} P_{j_1,j_2} \rangle = \langle B_{i_1,i_2}^{(1)} B_{j_1,j_2}^{(1)} \rangle \langle B_{i_1,i_2}^{(2)} B_{j_1,j_2}^{(2)} \rangle = \gamma^4.$$
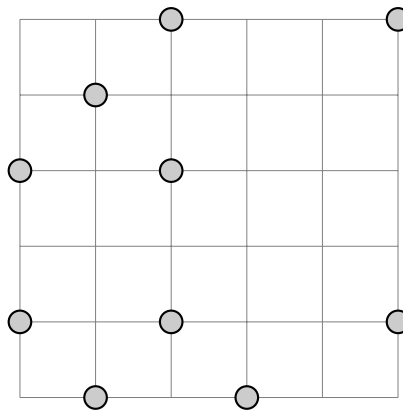

Figure 3: Process P constructed from the processes in Figure 1 and Figure 2

2.2 Construction in higher dimension

The construction presented in the previous subsection easily generalizes to any dimension $d > 2$ by defining $P_{i_1,\dots,i_d} := B_{i_1,\dots,i_d}^{(1)} \cdots B_{i_1,\dots,i_d}^{(d)}$ where, for any fixed $i_2,\dots,i_d \in \mathbb{Z}$, $\{B_{i_1,\dots,i_d}^{(1)}\}_{i_1 \in \mathbb{Z}}$ is a $BP\gamma$ in the variable i_1 with density γ and similarly for the other variables. Therefore, the point process P on $\mathbb{Z}^d$ defined as above satisfies (2.1) and (2.2) for $\rho = \gamma^d$.

3 Bounds for the maximal realizable density

In this section, we will discuss the problem of estimating the maximal realizable density $\bar{\rho}_\alpha(d)$. In particular, we will show a general upper bound for any $d \geq 1$ using the technique introduced in [12] for $d = 1$. As for the lower bound, we will recall the results in [12, Appendix 1] for the one-dimensional case and combine them with the explicit construction proposed in Section 2 to produce a lower bound for $\bar{\rho}_\alpha(d)$ for any $d \geq 2$. We will compare this with the lower bound obtained by applying the general methods of [12] to the case when the radial distribution is given by (1.2).

3.1 Upper bounds for $\bar{\rho}_\alpha(d)$

For $d \geq 1$ and $\alpha \geq 0$, the covariance matrix associated to a given pair $(\rho, g^{(\alpha)})$ realizable on $\mathbb{Z}^d$ must be positive semidefinite. This is equivalent to the non-negativity of the corresponding infinite volume structure function $\hat{S}$ on $\mathbb{R}^d$ (for more details see e.g. [12, Section 2]):

$$\hat{S}(\mathbf{k}) := \rho + \rho^2 \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k}\cdot\mathbf{x}} [g^{(\alpha)}(\mathbf{x}) - 1] \geq 0, \, \forall \, \mathbf{k} \in \mathbb{R}^d.$$

This leads to an explicit upper bound for the maximal realizable density $\bar{\rho}_\alpha(d)$. In fact, it is easy to see that for any $\mathbf{k} := (k_1, \ldots, k_d) \in \mathbb{R}^d$ we get

$$\hat{S}(\mathbf{k}) = \rho - \rho^2 + \rho^2 \sum_{\mathbf{x} \in \mathbb{Z}^d, |\mathbf{x}|=1} e^{i\mathbf{k}\cdot\mathbf{x}}(\alpha - 1)$$

$$= \rho - \rho^2 + \rho^2(\alpha - 1) \sum_{j=1}^{d} (e^{ik_j} + e^{-ik_j})$$

$$= \rho \left[1 - \rho \left(1 - 2(\alpha - 1) \sum_{j=1}^{d} \cos(k_j) \right) \right].$$

Then, using the non-negativity of $\hat{S}$ on $\mathbb{R}^d$, we get that

$$\rho \leq \frac{1}{f_\alpha(k_1, \ldots, k_d)}, \quad \forall (k_1, \ldots, k_d) \in \mathbb{R}^d,$$

where $f_\alpha(k_1, \ldots, k_d) := 1 - 2(\alpha - 1)\sum_{j=1}^{d} \cos(k_j)$. The best upper bound is then obtained for the points of $\mathbb{R}^d$ where f_α attains the maximum. Hence, we have that

$$\bar{\rho}_\alpha(d) \leq \frac{1}{\max_{\mathbf{k} \in \mathbb{R}^d} f_\alpha(\mathbf{k})} =: R_F(\alpha, d).$$

By computing the maximum of f_α over $\mathbb{R}^d$, we get our upper bound

$$R_F(\alpha, d) = \frac{1}{1 + 2d|1 - \alpha|}. \tag{3.1}$$

As mentioned above, this technique was employed in [12, Appendix 1] to get $R_F(\alpha, 1)$. Furthermore, the authors provided another upper bound $R_Y(\alpha, 1)$ in the one-dimensional case by using the Yamada condition (see [27]). Note that

$$\begin{cases} R_Y(\alpha, 1) = R_F(\alpha, 1), & \text{if } \alpha = \frac{1}{2} \text{ or } \alpha = \frac{k\pm1}{2k}, k \in \mathbb{N} \text{ or } \alpha \geq 1 \\ R_Y(\alpha, 1) < R_F(\alpha, 1) & \text{otherwise.} \end{cases}$$

3.2 Lower bounds for $\bar{\rho}_\alpha(d)$

Applying [12, Theorem 3.2] for $g \equiv g^{(\alpha)}$ when $0 \leq \alpha < 1$ and [12, Theorem 5.1] for $G_2(\mathbf{x}, \mathbf{y}) = g^{(\alpha)}(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ such that $\mathbf{x} \neq \mathbf{y}$ when $\alpha \geq 1$, we get that

$$\bar{\rho}_\alpha(d) \geq r_A(\alpha, d) := \begin{cases} \frac{1}{e(2d+1-2d\alpha)}, & \text{if } 0 \leq \alpha < 1, \\ \frac{1}{\alpha^{2d}}, & \text{if } \alpha \geq 1. \end{cases} \tag{3.2}$$

For $d = 1$, this lower bound has been improved in [12, Appendix 1] by explicitly constructing a translation invariant realizing process at some value of ρ and α. Let us summarize in one formula the lower bounds coming from the two main constructions considered in [12, Appendix 1]:

$$\bar{\rho}_\alpha(1) \geq \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^2}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{1+\sqrt{2-2\alpha}}, & \text{if } \frac{1}{2} \leq \alpha \leq 1, \\ \frac{1}{2\alpha-1}, & \text{if } \alpha \geq 1. \end{cases} \tag{3.3}$$

In [2] a further explicit construction is provided for the case $\alpha = 0$, which slightly improves this lower bound to $\bar{\rho}_0(1) > 0.265$. In the same work also the upper bound is improved to $\bar{\rho}_0(1) < (326 - \sqrt{3115})/822 \approx 0.3287$. However, it remains an open problem to reduce the gap between lower and upper bounds for $\bar{\rho}_\alpha(1)$ for any $\alpha \geq 0$.

Exactly as in the one-dimensional case, also for $d \geq 2$, one can try to obtain better lower bounds than (3.2) by using explicit constructions. In the following, we will use the construction we proposed in Section 2 combined with the one-dimensional lower bound (3.3) to compute a new lower bound for $\bar{\rho}_\alpha(d)$, which we will briefly compare with (3.2).

If we apply the construction given for $d \geq 2$ in Section 2 starting with a basic process with density $\bar{\rho}_\alpha(1)$, then we get a point process on $\mathbb{Z}^d$ which realizes the pair $\left((\bar{\rho}_\alpha(1))^d, g^{(\alpha)}\right)$ for any $\alpha \geq 0$. This explicit construction guarantees that for any $\alpha \geq 0$,

$$\bar{\rho}_\alpha(d) \geq (\bar{\rho}_\alpha(1))^d.$$

Using the lower bounds (3.3) in the latter inequality, we directly have the following

$$\bar{\rho}_\alpha(d) \geq r_C(\alpha, d) := \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^{2d}}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{(1+\sqrt{2-2\alpha})^d}, & \text{if } \frac{1}{2} \leq \alpha \leq 1, \\ \frac{1}{(2\alpha-1)^d}, & \text{if } \alpha \geq 1. \end{cases} \tag{3.4}$$

Note that:

- if $0 \leq \alpha < \frac{1}{2}$ then $r_C(\alpha, d) \leq r_A(\alpha, d)$

- if $\alpha \geq 1$ then $r_A(\alpha, d) \leq r_C(\alpha, d)$

- if $\frac{1}{2} \leq \alpha \leq 1$ then the relation between the two bounds depends on the dimension d. Actually, for each $d \geq 2$ there exists $\alpha_C(d) \in \left[\frac{1}{2}, 1\right]$ such that $r_A(\alpha, d) \leq r_C(\alpha, d)$ for any $\alpha_C(d) \leq \alpha \leq 1$.

The comparison between the lower bounds $r_C(\alpha, d)$ and $r_A(\alpha, d)$ is illustrated in Figure 4 for $d = 2, \ldots, 6$ and for $0 \leq \alpha < 1$.
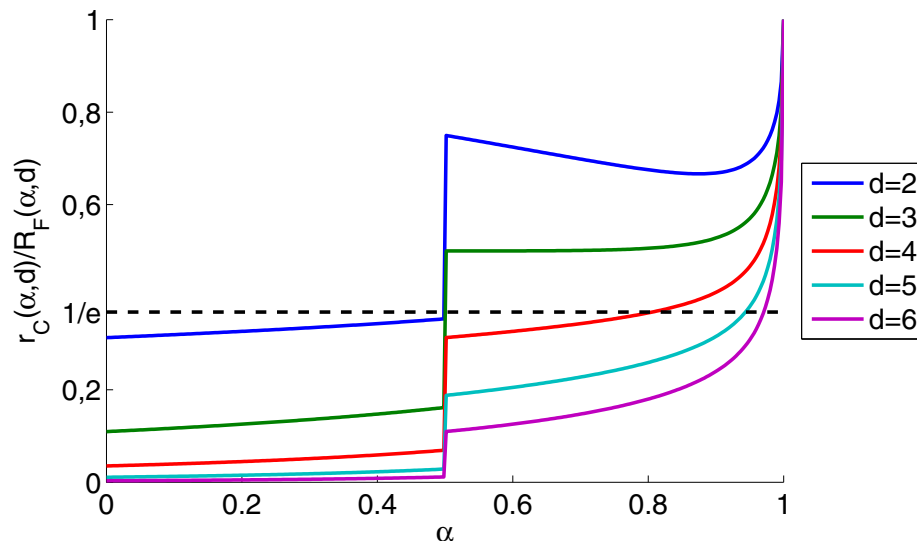

Figure 4: Comparison between the lower bounds $r_C(\alpha, d)$ and $r_A(\alpha, d)$ plotted relatively to the upper bound $R_F(\alpha, d)$ as functions of α with $0 \leq \alpha < 1$. The coloured lines correspond to $\frac{r_C(\alpha, d)}{R_F(\alpha, d)}$ for $d = 2, \ldots, 6$ and the dotted line to $\frac{r_A(\alpha, d)}{R_F(\alpha, d)}$ for any d. For the definitions of $R_F(\alpha, d)$, $r_A(\alpha, d)$, $r_C(\alpha, d)$ see (3.1), (3.2), (3.4), respectively.

References

[1] Brown, E. N.; Kass, R. E. and Mitra, P. P.: Multiple neural spike train data analysis: state-of-the-art and future challenges. *Nature Neuroscience* **7**, (2004), 456–471.

[2] Caglioti, E.; Kuna, T.; Lebowitz, J. L. and Speer, E. R.: Point Processes with Specied Low Order Correlations. *Markov Processes Relat. Fields* **12 (2)**, (2006), 257–272. MR-2249631

[3] Cocco, S. and Monasson, R.: Adaptive cluster expansion for the inverse Ising Problem: convergence, algorithm and tests *J. Stat. Phys.*, **147(2)**, (2012), 252–314. MR-2922769

[4] Crawford, J.; Torquato, S. and Stillinger, F. H.: Aspects of correlation function realizability. *J. Chem. Phys.* **119**, (2003), 7065–7074.

[5] Garrod, C. and Percus, J. K.: Reduction of the N-particle variational problem, *J. Math. Phys.* **5**, (1964), 1756–1776. MR-0170658

[6] Hansen, J. P. and McDonald, I. R.: Theory of simple liquids. *Academic Press, New York*, 2nd edition, 1987.

[7] Infusino, M. and Kuna, T.: The full moment problem on subsets of configurations, in preparation.

[8] Infusino, M.; Kuna, T. and Rota, A.: The full infinite dimensional moment problem on semialgebraic sets of generalized functions, *J. Funct. Analysis*, **267(5)**, (2014), 1382–1418. MR-3229795

[9] Jarvis, M. R. and Mitra, P. P.: Sampling properties of the spectrum and coherency of sequences of action potentials. *Neural Comp.* **13**, (2004), 717–749.

[10] Koralov, L.: Existence of pair potential corresponding to specified density and pair correlation. *Lett. Math. Phys.*, **71(2)**, (2005), 135–148. MR-2134693

[11] Koralov, L.: An inverse problem for Gibbs fields with hard core potential. *Lett. Math. Phys.*, **48(5)**, (2007), 053301, 13pp. MR-2329857

[12] Kuna, T.; Lebowitz, J. L. and Speer, E. R.: Realizability of point processes, *J. Stat. Phys.* **129(3)**, (2007), 417–439. MR-2351408

[13] Kuna, T.; Lebowitz, J. L. and Speer, E. R.: Necessary and sufficient conditions for realizability of point processes, *Ann. Appl. Prob.* **21(4)**, (2011), 1253–1281. MR-2857448

[14] Lachieze-Rey, R. and Molchanov, I.: Regularity conditions in the realisability problem in applications to point processes and random closed sets. *Ann. Appl. Probab.* **25(1)**, (2015), 116–149. MR-3297768

[15] Lenard, A.: States of classical statistical mechanical systems of infinitely many particles I. *Arch. Rational Mech. Anal.*, **59(3)**, (1975), 219–239. MR-0391830

[16] Lenard, A.: States of classical statistical mechanical systems of infinitely many particles II. *Arch. Rational Mech. Anal.*, **59(3)**, (1975), 241–256. MR-0391831

[17] Molchanov, I.: Theory of random sets. *Probability and its applications. Springer, New York*, 2005. MR-2132405

[18] Murrell, D. J.; Dieckmann, U. and Law, R.; On moment closure for population dynamics in continuous space. *J. Theo. Bio.*, **229**, (2004), 421–432.

[19] Percus, J. K.: The pair distribution function in classical statistical mechanics. In: Frisch, H.L., Lebowitz, J.L. (eds.) *The Equilibrium Theory of Classical Fluids*, pp. II33-II170, Benjamin, New York, 1964.

[20] Percus, J. K.: Kinematic restrictions on the pair density-prototype, Unpublished Lecture Notes, Courant Institute of Mathematical Sciences.

[21] Stillinger, F. H. and Torquato, S.: Pair correlation function realizability: Lattice model implications. *J. Phys. Chem. B*, **108**, (2004), 19589–19594.

[22] Stillinger, F. H. and Torquato, S.: Realizability issues for iso-g(2) processes. *Mol. Phys.*, **103**, (2005), 2943–2949.

[23] Stoyan, D.: Basic ideas of spatial statistics. In *Statistical physics and spatial statistics (Wuppertal, 1999)*, 3–21, *Lecture Notes in Phys.*, **554**, Springer, Berlin, 2000. MR-1870942

[24] Torquato, S.: Random Heterogeneous Materials: Microstructure and macroscopic properties, *Interdisciplinary Applied Mathematics 16, Springer-Verlag, NewYork*, 2002. MR-1862782

[25] Torquato, S. and Stillinger, F. H.: Local density fluctuations, hyperuniformity, and order metrics. *Phys. Rev. E.(3)* **68(4)**, (2003), 041113: 25 pp. MR-2060843

[26] Torquato, S. and Stillinger, F. H.: New conjectural lower bounds on the optimal density of sphere packings. *Exp. Math.* **15(3)**, (2006), 307–331. MR-2264469

[27] Yamada, M.: Geometrical study of the pair distribution function in the many-body problem. *Progr. Theoret. Phys.* **25**, (1961), 579–594. MR-0127288 MR-0127288

[28] Us, G. F.: A truncated symmetric generalized power moment problem. *Ukrain. Mat. Z.* **26**, (1974), 348–358, 429. MR-0348550

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards

- Free for authors, free for readers

- Quick publication (no backlog)

- Secure publication (LOCKSS[1])

- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4] , ProjectEuclid[5]

- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)

- Submit your best articles to EJP-ECP

- Choose EJP-ECP over for-profit journals

---

[1] LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2] EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
[3] IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4] BS: Bernoulli Society http://www.bernoulli-society.org/
[5] Project Euclid: https://projecteuclid.org/
[6] IMS Open Access Fund: http://www.imstat.org/publications/open.htm