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On the average value of the least common multiple of k positive integers

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Abstract

We deduce an asymptotic formula with error term for the sum $\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k])$, where $[n_1, \dots, n_k]$ stands for the least common multiple of the positive integers n_1, \dots, n_k ($k \geq 2$) and f belongs to a large class of multiplicative arithmetic functions, including, among others, the functions $f(n) = n^r$, $\varphi(n)^r$, $\sigma(n)^r$ ($r > -1$ real), where φ is Euler's totient function and σ is the sum-of-divisors function. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables, starting with the observation that given a multiplicative function f , the function of k variables $f([n_1, \dots, n_k])$ is multiplicative.

Keywords: greatest common divisor, least common multiple, arithmetic function of several variables, multiplicative function, Dirichlet series, asymptotic formula

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1. Introduction

We use the following notation: $\mathbb{N} = \{1, 2, \dots\}$, $*$ is the Dirichlet convolution of arithmetic functions, id_r ($r \in \mathbb{R}$) is the function $\text{id}_r(n) = n^r$ ($n \in \mathbb{N}$), $\mathbf{1} = \text{id}_0$, $\text{id} = \text{id}_1$, μ denotes the Möbius function, λ is the Liouville function,

$\sigma_r = \mathbf{1} * \text{id}_r$, $\sigma = \sigma_1$ is the sum-of-divisors function, $\tau = \sigma_0$ is the divisor function, $\beta_r = \lambda * \text{id}_r$, $\beta = \beta_1$ is the alternating sum-of-divisors function (cf. [19]), $\varphi_r = \mu * \text{id}_r$ is the generalized Euler function, $\varphi = \varphi_1$ is Euler's totient function, $\psi_r = \mu^2 * \text{id}_r$ is the generalized Dedekind function, $\psi = \psi_1$ is the classical Dedekind function. If $n \in \mathbb{N}$, then $n = \prod_p p^{\nu_p(n)}$ is its prime power factorization, the product being over the primes p , where all but a finite number of the exponents $\nu_p(n)$ are zero.

Furthermore, let (n_1, \dots, n_k) and $[n_1, \dots, n_k]$ denote the greatest common divisor (gcd) and the least common multiple (lcm) of $n_1, \dots, n_k \in \mathbb{N}$ ($k \geq 2$), respectively.

It is easy to see that for any arithmetic function f we have the identity

$$\sum_{n_1, \dots, n_k \leq x} f((n_1, \dots, n_k)) = \sum_{d \leq x} (\mu * f)(d) \left\lfloor \frac{x}{d} \right\rfloor^k, \quad (1)$$

which leads to asymptotic formulas for this sum. For example, if $f = \text{id}$ and $k \geq 3$, then we have

$$\sum_{n_1, \dots, n_k \leq x} (n_1, \dots, n_k) = \frac{\zeta(k-1)}{\zeta(k)} x^k + O(R_k(x)), \quad (2)$$

where $R_3(x) = x^2 \log x$ and $R_k(x) = x^{k-1}$ for $k \geq 4$. The case $f = \text{id}$, $k = 2$ can be treated separately by writing

$$\begin{aligned} \sum_{m, n \leq x} (m, n) &= 2 \sum_{m \leq n \leq x} (m, n) - \sum_{n \leq x} n \\ &= 2 \sum_{n \leq x} (\mu * \text{id} \tau)(n) - \frac{x^2}{2} + O(x), \end{aligned}$$

giving, by using elementary arguments, the formula

$$\sum_{m, n \leq x} (m, n) = \frac{x^2}{\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(x^{1+\theta+\varepsilon}), \quad (3)$$

valid for every $\varepsilon > 0$, where γ is Euler's constant and θ is the exponent appearing in Dirichlet's divisor problem.

For the lcm of k positive integers there is no formula similar to (1). However, in the case $k = 2$, the lcm of the integers $m, n \in \mathbb{N}$ can be written using

their gcd as $[m, n] = mn/(m, n)$, which enables to establish the following asymptotic formula, valid for any positive real number r :

$$\sum_{m, n \leq x} [m, n]^r = \frac{\zeta(r+2)}{\zeta(2)} \cdot \frac{x^{2(r+1)}}{(r+1)^2} + O(x^{2r+1} \log x). \quad (4)$$

If $r \in \mathbb{N}$, then the error term in (4) can be improved into $O(x^{2r+1}(\log x)^{2/3}(\log \log x)^{4/3})$, which is a consequence of the result of Walfisz [23, Satz 1, p. 144] for $\sum_{n \leq x} \varphi(n)$.

For $k = 2$ the asymptotic formulas concerning $\sum_{m, n \leq x} (m, n)^r$ and $\sum_{m, n \leq x} [m, n]^r$ are equivalent to those for $\sum_{n \leq x} g_r(n)$ and $\sum_{n \leq x} \ell_r(n)$, respectively, where $g_r(n) = \sum_{1 \leq j \leq n} (j, n)^r$ is the gcd-sum function and $\ell_r(n) = \sum_{1 \leq j \leq n} [j, n]^r$ is the lcm-sum function. The function $g_1(n) = \sum_{1 \leq j \leq n} (j, n)$, investigated by S. S. Pillai [16], is also called Pillai's function in the literature.

The above and related results go back, in chronological order, to the work of E. Cesàro [6], E. Cohen [9, 10, 11], K. Alladi [1], P. Diaconis and P. Erdős [12], J. Chidambaraswamy and R. Sitaramachandrarao [7], K. A. Broughan [5], O. Bordellès [2, 3, 4], Y. Tanigawa and W. Zhai [17], S. Ikeda and K. Mat-suoka [15], and others.

For example, formula (3) with the weaker error $O(x^{3/2} \log x)$ was given in [12, Th. 2, Eq. (1.4)] and was recovered in [5, Th. 4.7]. Formula (3) with the above error term was established in [7, Th. 3.1] and recovered in [2, Th. 1.1] (in both papers for Pillai's function). Formula (4) was established in [12, Th. 2, Eq. (1.6)]. The better error term for (4) in the case $r \in \mathbb{N}$ was obtained in [15, Th. 2]. Asymptotic formulas for (1) in the case $k = 2$ and for various choices of the function f , including $f = \sigma$ and $f = \varphi$ were deduced in [4, 9, 10, 11]. See also the survey paper [18].

The result

$$\sum_{m, n, q \leq x} [m, n, q]^r \sim c_r \frac{x^{3(r+1)}}{(r+1)^3} \quad (x \rightarrow \infty),$$

valid for $r \in \mathbb{N}$, without any error term and with a computable constant c_r given in an implicit form, was obtained by J. L. Fernández and P. Fernández [13, Th. 3(b)]. Their proof is by an ingenious method based on the identity $[m, n, q](m, n)(m, q)(n, q) = mnq(m, n, q)$ ($m, n, q \in \mathbb{N}$) and using the dominated convergence theorem. As far as we know, there are no other asymptotic results in the literature for the sum

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]), \quad (5)$$

in the case $k \geq 3$, where f is an arithmetic function. It seems that the method of [13] can not be extended for $k \geq 3$, even in the case $f = \text{id}_r$. Also, it is not possible to reduce the estimation of the sum (5) to sums of a single variable, like in (1).

In this paper we deduce an asymptotic formula with remainder term for the sum (5), where $k \geq 2$ and f belongs to a large class of multiplicative arithmetic functions, including the functions id_r with $r > -1$ real and σ_r , β_r , φ_r , ψ_r with $r \geq 1/2$ real. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables starting with the observation that given a multiplicative function f , the function of k variables $f([n_1, \dots, n_k])$ is multiplicative and the associated multiple Dirichlet series factorizes as an Euler product. The same method was used by the second author [21] for a different problem. See the survey paper [20] of the second author for basic properties of multiplicative functions of several variables and related convolutions.

We also extend to the k dimensional case the formula

$$\sum_{m, n \leq x} \frac{[m, n]}{(m, n)} = \frac{\pi^2}{60} x^4 + O(x^3 \log x), \quad (6)$$

which can be obtained in a similar manner to the results (2) and (4). Properties of the operation $m \circ n = [m, n]/(m, n)$ were investigated by the first author [14].

Note that the following recent result of different type, concerning the lcm of several positive integers, was obtained by J. Cilleruelo, J. Rué, P. Šarka and A. Zumalacárregui [8]: $\text{lcm}\{a : a \in A\} = 2^{n(1+o(1))}$ for almost all subsets $A \subset \{1, \dots, n\}$.

2. Main results

Let $r \in \mathbb{R}$ be a fixed number. Let \mathcal{A}_r denote the class of complex valued multiplicative arithmetic functions satisfying the following properties: there exist real constants C_1, C_2 such that

$$|f(p) - p^r| \leq C_1 p^{r-1/2} \quad \text{for every prime } p, \quad (\text{i})$$

and

$$|f(p^\nu)| \leq C_2 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geq 2. \quad (\text{ii})$$

Note that conditions (i) and (ii) imply that

$$|f(p^\nu)| \leq C_3 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geq 1, \quad (\text{iii})$$

where $C_3 = \max(C_1 + 1, C_2)$.

Observe that $\text{id}_r \in \mathcal{A}_r$ for every $r \in \mathbb{R}$, while $\sigma_r, \beta_r, \varphi_r, \psi_r \in \mathcal{A}_r$ for every $r \in \mathbb{R}$ with $r \geq 1/2$. The functions $f(n) = \sigma(n)^r, \beta(n)^r, \varphi(n)^r, \psi(n)^r$ also belong to the class \mathcal{A}_r for every $r \in \mathbb{R}$. As other examples of functions in the class \mathcal{A}_r , with $r \in \mathbb{R}$, we mention $\varphi^*(n)^r, \sigma^*(n)^r$ and $\sigma^{(e)}(n)^r$, where $\varphi^*(n) = \prod_{p|n} (p^{\nu_p(n)} - 1)$ is the unitary Euler totient, $\sigma^*(n) = \prod_{p|n} (p^{\nu_p(n)} + 1)$ is the sum-of-unitary-divisors function and $\sigma^{(e)}(n) = \prod_{p|n} \sum_{d|\nu_p(n)} p^d$ denotes the sum of exponential divisors of n . Furthermore, if f is a bounded multiplicative function such that $f(p) = 1$ for every prime p , then $f \in \mathcal{A}_0$. In particular, $\mu^2 \in \mathcal{A}_0$.

We prove the following results.

Theorem 2.1. *Let $k \geq 2$ be a fixed integer and let $f \in \mathcal{A}_r$ be a function, where $r > -1$ is real. Then for every $\varepsilon > 0$,*

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) = C_{f,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1) - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right), \quad (7)$$

and

$$\sum_{n_1, \dots, n_k \leq x} \frac{f([n_1, \dots, n_k])}{(n_1 \cdots n_k)^r} = C_{f,k} x^k + O\left(x^{k - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right), \quad (8)$$

where

$$C_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

Formula (7) shows that the average order of $f([n_1, \dots, n_k])$ is $C_{f,k}(n_1 \cdots n_k)^r$, in the sense that

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) \sim \sum_{n_1, \dots, n_k \leq x} C_{f,k} (n_1 \cdots n_k)^r \quad (x \rightarrow \infty).$$

From (8) we deduce that

$$\lim_{x \rightarrow \infty} \frac{1}{x^k} \sum_{n_1, \dots, n_k \leq x} \frac{f([n_1, \dots, n_k])}{(n_1 \cdots n_k)^r} = C_{f,k},$$

representing the mean value of the function $f([n_1, \dots, n_k])/(n_1 \cdots n_k)^r$. See N. Ushiroya [22, Th. 4] and the second author [20, Prop. 19] for general results on mean values of multiplicative arithmetic functions of several variables.

Theorem 2.2. *Let $k \geq 2$ be a fixed integer and let $f \in \mathcal{A}_r$ be a function, where $r \geq 0$ is real. Then for every $\varepsilon > 0$,*

$$\sum_{n_1, \dots, n_k \leq x} f\left(\frac{[n_1, \dots, n_k]}{(n_1, \dots, n_k)}\right) = D_{f,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon}\right), \quad (9)$$

where

$$D_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

In the case $f = \text{id}_r$ we obtain from Theorem 2.1 the next result:

Corollary 1. *Let $k \geq 3$ and $r > -1$ be a real number. Then for every $\varepsilon > 0$,*

$$\sum_{n_1, \dots, n_k \leq x} [n_1, \dots, n_k]^r = C_{r,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right), \quad (10)$$

and

$$\sum_{n_1, \dots, n_k \leq x} \left(\frac{[n_1, \dots, n_k]}{n_1 \cdots n_k}\right)^r = C_{r,k} x^k + O\left(x^{k-\frac{1}{2}\min(r+1,1)+\varepsilon}\right),$$

where

$$C_{r,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{p^{r \max(\nu_1, \dots, \nu_k)}}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$C_{r,3} = \zeta(r+2)\zeta(2r+3) \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} + \frac{2}{p^{r+2}} - \frac{3}{p^{r+3}} + \frac{1}{p^{r+5}}\right), \quad (11)$$

$$\begin{aligned} C_{r,4} = & \zeta(r+2)\zeta(2r+3)\zeta(3r+4) \prod_p \left(1 - \frac{6}{p^2} + \frac{8}{p^3} - \frac{3}{p^4} + \frac{5}{p^{r+2}} - \frac{12}{p^{r+3}} + \frac{6}{p^{r+4}} + \frac{4}{p^{r+5}} \right. \\ & \left. - \frac{3}{p^{r+6}} + \frac{3}{p^{2r+3}} - \frac{4}{p^{2r+4}} - \frac{6}{p^{2r+5}} + \frac{12}{p^{2r+6}} - \frac{5}{p^{2r+7}} + \frac{3}{p^{3r+5}} - \frac{8}{p^{3r+6}} + \frac{6}{p^{3r+7}} - \frac{1}{p^{3r+9}}\right). \end{aligned} \quad (12)$$

In the case $f = \text{id}_r$ we deduce from Theorem 2.2:

Corollary 2. *Let $k \geq 3$ and $r > 0$ be a real number. Then for every $\varepsilon > 0$,*

$$\sum_{n_1, \dots, n_k \leq x} \left(\frac{[n_1, \dots, n_k]}{(n_1, \dots, n_k)} \right)^r = D_{r,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon}\right), \quad (13)$$

where

$$D_{r,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{p^{r(\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k))}}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$D_{r,3} = C_{r,3} \frac{\zeta(3r+3)}{\zeta(2r+3)}, \quad D_{r,4} = C_{r,4} \frac{\zeta(4r+4)}{\zeta(3r+4)}.$$

We remark that in the case $k = 2$ asymptotic formulas (10) and (13) reduce to (4) and (6) (case $r = 1$), respectively, but the latter ones have better error terms. Note that $D_{r,2} = \zeta(2r+2)/\zeta(2)$.

Among other special cases we consider the functions $\sigma, \varphi \in \mathcal{A}_1$ and $\mu^2 \in \mathcal{A}_0$.

Corollary 3. *Let $k \geq 2$. Then for every $\varepsilon > 0$,*

$$\sum_{n_1, \dots, n_k \leq x} \sigma([n_1, \dots, n_k]) = C_{\sigma,k} \frac{x^{2k}}{2^k} + O\left(x^{2k-1/2+\varepsilon}\right),$$

and

$$\sum_{n_1, \dots, n_k \leq x} \frac{\sigma([n_1, \dots, n_k])}{n_1 \cdots n_k} = C_{\sigma,k} x^k + O\left(x^{k-1/2+\varepsilon}\right),$$

where

$$C_{\sigma,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{\sigma(p^{\max(\nu_1, \dots, \nu_k)})}{p^{2(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$C_{\sigma,2} = \zeta(3)\zeta(4) \prod_p \left(1 + \frac{1}{p^2} - \frac{2}{p^3} - \frac{2}{p^5} + \frac{2}{p^6}\right).$$

Corollary 4. *Let $k \geq 2$. Then for every $\varepsilon > 0$,*

$$\sum_{n_1, \dots, n_k \leq x} \varphi([n_1, \dots, n_k]) = C_{\varphi, k} \frac{x^{2k}}{2^k} + O(x^{2k-1/2+\varepsilon}),$$

and

$$\sum_{n_1, \dots, n_k \leq x} \frac{\varphi([n_1, \dots, n_k])}{n_1 \cdots n_k} = C_{\varphi, k} x^k + O(x^{k-1/2+\varepsilon}),$$

where

$$C_{\varphi, k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{\varphi(p^{\max(\nu_1, \dots, \nu_k)})}{p^{2(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$C_{\varphi, 2} = \zeta(3) \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} - \frac{1}{p^4} + \frac{2}{p^5} - \frac{1}{p^6}\right).$$

Corollary 5. *Let $k \geq 2$. Then for every $\varepsilon > 0$,*

$$\sum_{n_1, \dots, n_k \leq x} \mu^2([n_1, \dots, n_k]) = \frac{x^k}{\zeta(2)^k} + O(x^{k-1/2+\varepsilon}).$$

Remark 1. It would be interesting to find the best possible error, especially in particular cases. For example, for $r = 1$ in Corollary 1, the relative error is $O(x^{-1/2+\varepsilon})$. Can we improve the exponent further and if so, by how much?

3. Proofs

An arithmetic function g of k variables is called multiplicative if

$$g(m_1 n_1, \dots, m_k n_k) = g(m_1, \dots, m_k) g(n_1, \dots, n_k),$$

provided that $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. Hence

$$g(n_1, \dots, n_k) = \prod_p g(p^{\nu_p(n_1)}, \dots, p^{\nu_p(n_k)})$$

for every $n_1, \dots, n_k \in \mathbb{N}$. In this case the multiple Dirichlet series of the function g can be expanded into an Euler product:

$$\sum_{n_1, \dots, n_k=1}^{\infty} \frac{g(n_1, \dots, n_k)}{n_1^{z_1} \cdots n_k^{z_k}} = \prod_p \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{g(p^{\nu_1}, \dots, p^{\nu_k})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}}.$$

We need the following lemmas.

Lemma 3.1. *If $k \geq 2$ and $f \in \mathcal{A}_r$ with $r > -1$ real, then*

$$L_{f,k}(z_1, \dots, z_k) := \sum_{n_1, \dots, n_k=1}^{\infty} \frac{f([n_1, \dots, n_k])}{n_1^{z_1} \dots n_k^{z_k}} = \zeta(z_1-r) \dots \zeta(z_k-r) H_{f,k}(z_1, \dots, z_k),$$

where the multiple Dirichlet series $H_{f,k}(z_1, \dots, z_k)$ is absolutely convergent for

$$\Re z_1, \dots, \Re z_k > A := \begin{cases} r + \frac{1}{2}, & \text{if } r \geq 0, \\ \frac{r+1}{2}, & \text{if } -1 < r < 0. \end{cases} \quad (14)$$

Proof. If f is a multiplicative function of a single variable, then the arithmetic function of k variables $f([n_1, \dots, n_k])$ is multiplicative. It follows that

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \quad (15)$$

Case I. Assume that $r \geq 0$. Grouping the terms of the sum in (15) according to the values $\nu_1 + \dots + \nu_k$ we have

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \left(1 + \frac{f(p)}{p^{z_1}} + \dots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1 + \dots + \nu_k \geq 2} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right). \quad (16)$$

Let $\Re z_1, \dots, \Re z_k \geq \delta > r$. By using condition (i) from the definition of the class \mathcal{A}_r ,

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) \quad (1 \leq j \leq k).$$

Also, by condition (iii) following the definition of the class \mathcal{A}_r and by using that $r \geq 0$ we deduce that

$$\left| \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right| \leq C_3 \frac{p^{r \max(\nu_1, \dots, \nu_k)}}{p^{\delta(\nu_1 + \dots + \nu_k)}} \leq C_3 \frac{1}{p^{(\delta-r)(\nu_1 + \dots + \nu_k)}}.$$

Thus the sum in (16) over $\nu_1 + \dots + \nu_k \geq 2$ is $O(p^{-2(\delta-r)})$. We obtain

$$\begin{aligned} & L_{f,k}(z_1, \dots, z_k) \zeta^{-1}(z_1-r) \dots \zeta^{-1}(z_k-r) \\ &= \prod_p \left(1 - \frac{1}{p^{z_1-r}} \right) \dots \left(1 - \frac{1}{p^{z_k-r}} \right) \left(1 + \frac{1}{p^{z_1-r}} + \dots + \frac{1}{p^{z_k-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) \right) \end{aligned}$$

$$+O\left(\frac{1}{p^{2(\delta-r)}}\right)\Bigg)=\prod_p\left(1+O\left(\frac{1}{p^{\delta-r+1/2}}\right)+O\left(\frac{1}{p^{2(\delta-r)}}\right)\right),$$

since $\Re z_j \geq \delta$ ($1 \leq j \leq k$), where the terms $\pm \frac{1}{p^{z_j-r}}$ ($1 \leq j \leq k$) cancel out. Here the latter product converges absolutely when $\delta - r + 1/2 > 1$ and $2(\delta - r) > 1$, that is, for $\delta > r + 1/2$.

Case II. Assume that $-1 < r < 0$. Now we group the terms of the sum in (15) according to the values $\max(\nu_1, \dots, \nu_k)$:

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \left(1 + \sum_{\max(\nu_1, \dots, \nu_k)=1} \frac{f(p)}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} + \sum_{\max(\nu_1, \dots, \nu_k) \geq 2} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right). \quad (17)$$

Let $\Re z_1, \dots, \Re z_k \geq \delta \geq 0$. Consider the sum in (17) over $\max(\nu_1, \dots, \nu_k) = 1$ and suppose that $\nu_i = 1$ for m ($1 \leq m \leq k$) distinct values of i . If $m = 1$, then by condition (i) from the definition of the class \mathcal{A}_r we have

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) \quad (1 \leq j \leq k).$$

If $m \geq 2$, then

$$\left| \frac{f(p)}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right| \leq \frac{(C_1 + 1)p^r}{p^{m\delta}} = O\left(\frac{1}{p^{2\delta-r}}\right).$$

This shows that the sum in (17) over $\max(\nu_1, \dots, \nu_k) = 1$ is

$$\frac{1}{p^{z_1-r}} + \dots + \frac{1}{p^{z_k-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right).$$

Furthermore, by condition (ii) we deduce that for $\max(\nu_1, \dots, \nu_k) \geq 2$,

$$\left| \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right| \leq C_2 \frac{p^{r \max(\nu_1, \dots, \nu_k)}}{p^{\delta(\nu_1 + \dots + \nu_k)}} \leq C_2 \frac{1}{p^{(\delta-r) \max(\nu_1, \dots, \nu_k)}}$$

($\delta \geq 0$) and it follows that the sum in (17) over $\max(\nu_1, \dots, \nu_k) \geq 2$ is $O(p^{-2(\delta-r)}) = O(p^{-(2\delta-r)})$, since $r < 0$.

We obtain that

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \left(1 + \frac{1}{p^{z_1-r}} + \dots + \frac{1}{p^{z_k-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right) \right)$$

and

$$\begin{aligned}
& L_{f,k}(z_1, \dots, z_k) \zeta^{-1}(z_1 - r) \cdots \zeta^{-1}(z_k - r) \\
&= \prod_p \left(1 - \frac{1}{p^{z_1 - r}} \right) \cdots \left(1 - \frac{1}{p^{z_k - r}} \right) \prod_p \left(1 + \frac{1}{p^{z_1 - r}} + \cdots + \frac{1}{p^{z_k - r}} \right. \\
&\quad \left. + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) + O\left(\frac{1}{p^{2\delta - r}}\right) \right) \\
&= \prod_p \left(1 + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) + O\left(\frac{1}{p^{2\delta - r}}\right) \right),
\end{aligned}$$

since $\Re z_j \geq \delta$ ($1 \leq j \leq k$), where the terms $\pm \frac{1}{p^{z_j - r}}$ ($1 \leq j \leq k$) cancel out, similar to Case I. Here the latter product converges absolutely when $\delta - r + 1/2 > 1$ and $2\delta - r > 1$, that is, for $\delta > (r + 1)/2 > 0$. \square

Lemma 3.2. *If $k \geq 2$ and $f \in \mathcal{A}_r$ with $r \geq 0$, then*

$$\bar{L}_{f,k}(z_1, \dots, z_k) := \sum_{n_1, \dots, n_k=1}^{\infty} \frac{f\left(\frac{[n_1, \dots, n_k]}{(n_1, \dots, n_k)}\right)}{n_1^{z_1} \cdots n_k^{z_k}} = \zeta(z_1 - r) \cdots \zeta(z_k - r) \bar{H}_{f,k}(z_1, \dots, z_k),$$

where the multiple Dirichlet series $\bar{H}_{f,k}(z_1, \dots, z_k)$ is absolutely convergent for $\Re z_1, \dots, \Re z_k > r + 1/2$.

Proof. Similar to the proof of Lemma 3.1, Case I. If f is multiplicative, then the function $f([n_1, \dots, n_k]/(n_1, \dots, n_k))$ is also multiplicative and we have

$$\begin{aligned}
\bar{L}_{f,k}(z_1, \dots, z_k) &= \prod_p \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} \\
&= \prod_p \left(1 + \frac{f(p)}{p^{z_1}} + \cdots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1 + \cdots + \nu_k \geq 2} \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} \right). \tag{18}
\end{aligned}$$

If $\Re z_1, \dots, \Re z_k \geq \delta > r$, then it follows that

$$\left| \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}} \right| \leq C \frac{p^{r(\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k))}}{p^{\delta(\nu_1 + \cdots + \nu_k)}} \leq C \frac{1}{p^{(\delta - r)(\nu_1 + \cdots + \nu_k)}},$$

thus the sum in (18) over $\nu_1 + \cdots + \nu_k \geq 2$ is $O(p^{-2(\delta - r)})$. Furthermore, we use the same arguments as in the previous proof. \square

Proof of Theorem 2.1. From Lemma 3.1 we deduce the convolutional identity

$$f([n_1, \dots, n_k]) = \sum_{j_1 d_1 = n_1, \dots, j_k d_k = n_k} j_1^r \cdots j_k^r h_{f,k}(d_1, \dots, d_k),$$

where

$$\sum_{n_1, \dots, n_k=1}^{\infty} \frac{h_{f,k}(n_1, \dots, n_k)}{n_1^{z_1} \cdots n_k^{z_k}} = H_{f,k}(z_1, \dots, z_k).$$

Therefore

$$\begin{aligned} \sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) &= \sum_{j_1 d_1 \leq x, \dots, j_k d_k \leq x} j_1^r \cdots j_k^r h_{f,k}(d_1, \dots, d_k) \\ &= \sum_{d_1, \dots, d_k \leq x} h_{f,k}(d_1, \dots, d_k) \sum_{j_1 \leq x/d_1} j_1^r \cdots \sum_{j_k \leq x/d_k} j_k^r \\ &= \sum_{d_1, \dots, d_k \leq x} h_{f,k}(d_1, \dots, d_k) \left(\frac{x^{r+1}}{(r+1)d_1^{r+1}} + O\left(\frac{x^R}{d_1^R}\right) \right) \cdots \left(\frac{x^{r+1}}{(r+1)d_k^{r+1}} + O\left(\frac{x^R}{d_k^R}\right) \right), \end{aligned}$$

where $R := \max(r, 0)$. We deduce that

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) = \frac{x^{k(r+1)}}{(r+1)^k} \sum_{d_1, \dots, d_k \leq x} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \cdots d_k^{r+1}} + S_{k,r}(x), \quad (19)$$

with

$$S_{k,r}(x) \ll \sum_{u_1, \dots, u_k} x^{u_1 + \cdots + u_k} \sum_{d_1, \dots, d_k \leq x} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{u_1} \cdots d_k^{u_k}}, \quad (20)$$

where the first sum is over $u_1, \dots, u_k \in \{r+1, R\}$ such that at least one u_i is R . Let u_1, \dots, u_k be fixed and assume that $u_i = R$ for t ($1 \leq t \leq k$) values of i , we take the first t values of i . Then $x^{u_1 + \cdots + u_k}$ times the inner sum of (20) is, using the notation A given by (14),

$$\begin{aligned} &\ll x^{(k-t)(r+1)+tR} \sum_{d_1, \dots, d_k \leq x} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^R \cdots d_t^R d_{t+1}^{r+1} \cdots d_k^{r+1}} \\ &= x^{(k-t)(r+1)+tR} \sum_{d_1, \dots, d_k \leq x} \frac{|h_{f,k}(d_1, \dots, d_k)| d_1^{A-R+\varepsilon} \cdots d_t^{A-R+\varepsilon}}{d_1^{A+\varepsilon} \cdots d_t^{A+\varepsilon} d_{t+1}^{r+1} \cdots d_k^{r+1}} \end{aligned}$$

$$\begin{aligned}
&\leq x^{(k-t)(r+1)+tR} x^{t(A-R+\varepsilon)} \sum_{d_1, \dots, d_k=1}^{\infty} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{A+\varepsilon} \dots d_t^{A+\varepsilon} d_{t+1}^{r+1} \dots d_k^{r+1}} \\
&= x^{k(r+1)-t(r+1-A)+t\varepsilon} H_{f,k}(A+\varepsilon, \dots, A+\varepsilon, r+1, \dots, r+1) \\
&\ll x^{k(r+1)-t(r+1-A)+t\varepsilon},
\end{aligned}$$

since the latter series is convergent by Lemma 3.1. Using that $r+1-A = \frac{1}{2} \min(r+1, 1) > 0$, the obtained error is maximal for $t=1$ giving

$$O\left(x^{k(r+1)-\frac{1}{2} \min(r+1, 1)+\varepsilon}\right).$$

Furthermore, for the sum in the main term of (19) we have

$$\begin{aligned}
&\sum_{d_1, \dots, d_k \leq x} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \dots d_k^{r+1}} \\
&= \sum_{d_1, \dots, d_k=1}^{\infty} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \dots d_k^{r+1}} - \sum_{\emptyset \neq I \subseteq \{1, \dots, k\}} \sum_{\substack{d_i > x, i \in I \\ d_j \leq x, j \notin I}} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \dots d_k^{r+1}}, \quad (21)
\end{aligned}$$

where the series is convergent by Lemma 3.1, and its sum is $H_{f,k}(r+1, \dots, r+1)$.

Let I be fixed and assume that $I = \{1, 2, \dots, s\}$, that is $d_1, \dots, d_s > x$ and $d_{s+1}, \dots, d_k \leq x$, where $s \geq 1$. We deduce, by noting that $A - (r+1) = -\frac{1}{2} \min(r+1, 1) < 0$,

$$\begin{aligned}
&\sum_{\substack{d_1, \dots, d_s > x \\ d_{s+1}, \dots, d_k \leq x}} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{r+1} \dots d_k^{r+1}} \\
&= \sum_{\substack{d_1, \dots, d_s > x \\ d_{s+1}, \dots, d_k \leq x}} \frac{|h_{f,k}(d_1, \dots, d_k)| d_1^{A-(r+1)+\varepsilon} \dots d_s^{A-(r+1)+\varepsilon}}{d_1^{A+\varepsilon} \dots d_s^{A+\varepsilon} d_{s+1}^{r+1} \dots d_k^{r+1}} \\
&\leq x^{s(A-(r+1)+\varepsilon)} \sum_{d_1, \dots, d_k=1}^{\infty} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{A+\varepsilon} \dots d_s^{A+\varepsilon} d_{s+1}^{r+1} \dots d_k^{r+1}} \\
&= x^{s(A-(r+1)+\varepsilon)} H_{f,k}(A+\varepsilon, \dots, A+\varepsilon, r+1, \dots, r+1) \\
&\ll x^{-\frac{s}{2} \min(r+1, 1)+s\varepsilon},
\end{aligned}$$

the latter series (the same as before) being convergent, and the obtained error is maximal for $s = 1$ giving, according to (19) and (21), the same error

$$O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right).$$

This proves asymptotic formula (7) with the constant $C_{f,k} = H_{f,k}(r + 1, \dots, r + 1)$. Here, according to Lemma 3.1,

$$C_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

The proof of (8) is similar, based on Lemma 3.1 and the convolutional identity

$$\frac{f([n_1, \dots, n_k])}{(n_1 \cdots n_k)^r} = \sum_{j_1 d_1 = n_1, \dots, j_k d_k = n_k} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^r \cdots d_k^r},$$

which implies that

$$\sum_{n_1, \dots, n_k \leq x} \frac{f([n_1, \dots, n_k])}{(n_1 \cdots n_k)^r} = \sum_{d_1, \dots, d_k \leq x} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^r \cdots d_k^r} \sum_{j_1 \leq x/d_1} 1 \cdots \sum_{j_k \leq x/d_k} 1.$$

□

Proof of Theorem 2.2. Formula (9) is obtained by using Lemma 3.2, in exactly the same way as (7) (here $r \geq 0$ and $R = \max(r, 0) = r$), with the constant $D_{f,k} = \overline{H}_{f,k}(r + 1, \dots, r + 1)$. □

Proof of Corollary 1. Apply Theorem 2.1 for $f = \text{id}_r$. Here

$$\begin{aligned} C_{r,3} &= \prod_p \left(1 - \frac{1}{p}\right)^3 \sum_{a,b,c=0}^{\infty} \frac{p^{r \max(a,b,c)}}{p^{(r+1)(a+b+c)}} \\ &= \prod_p \left(1 - \frac{1}{p}\right)^3 (6S_1 + 3S_2 + 3S_3 + S_4), \end{aligned}$$

with

$$S_1 = \sum_{0 \leq a < b < c} \frac{p^{rc}}{p^{(r+1)(a+b+c)}}, \quad S_2 = \sum_{0 \leq a=b < c} \frac{p^{rc}}{p^{(r+1)(2a+c)}},$$

$$S_3 = \sum_{0 \leq a < b=c} \frac{p^{rc}}{p^{(r+1)(a+2c)}}, \quad S_4 = \sum_{0 \leq a=b=c} \frac{p^{rc}}{p^{(r+1)3c}},$$

which gives (11). Formula (12) for the constant $C_{r,4}$ can be computed in a similar manner. \square

Proof of Corollary 2. Apply Theorem 2.2 for $f = \text{id}_r$. The constants $D_{r,3}$ and $D_{r,4}$ can be evaluated like above. \square

Proof of Corollaries 3, 4, 5. Apply Theorem 2.1 for $f = \sigma$, $f = \varphi$ with $r = 1$, resp. $f = \mu^2$ with $r = 0$. \square

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