Generalised solutions for fully nonlinear PDE systems and existence-uniqueness theorems

GENERALISED SOLUTIONS FOR FULLY NONLINEAR PDE SYSTEMS AND EXISTENCE-UNIQUENESS THEOREMS

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Abstract. We introduce a new theory of generalised solutions which applies to fully nonlinear PDE systems of any order and allows for merely measurable maps as solutions. This approach bypasses the standard problems arising by the application of Distributions to PDEs and is not based on either integration by parts or on the maximum principle. Instead, our starting point builds on the probabilistic representation of derivatives via limits of difference quotients in the Young measures over a toric compactification of the space of jets. After developing some basic theory, as a first application we consider the Dirichlet problem and we prove existence-uniqueness-partial regularity of solutions to fully nonlinear degenerate elliptic 2nd order systems and also existence of solutions to the ∞-Laplace system of vectorial Calculus of Variations in L^∞.

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1. Introduction

It is well known that PDEs, either linear or nonlinear, in general do not possess classical solutions, in the sense that not all derivatives that appear in the equation

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may actually exist. The standard approach to this problem consists of looking for appropriately defined *generalised solutions* for which at least existence can be proved given certain boundary conditions. Subsequent considerations typically include uniqueness, regularity, qualitative properties and numerics. This approach has been enormously successful but unfortunately only PDEs with fairly special structure have been considered so far. A standing idea in this regard consists of using integration-by-parts in order to interpret derivatives "weakly" by "passing them to test functions". This duality method which dates back to the 1930s ([S1, S2, So]) is basically restricted to divergence structure equations and systems. A more recent approach discovered in the 1980s is that of viscosity solutions ([CL]) and builds on the maximum principle as a device to define "weak" solutions. Although it applies mostly to the scalar case, it has been hugely successful since it includes fully nonlinear single equations.

In this paper we introduce a new theory of generalised solutions which applies to nonlinear PDE systems of any order. Our approach allows for *merely measurable* maps to be rigorously interpreted and studied as solutions of systems which are possibly fully nonlinear and with discontinuous coefficients. More precisely, let $p, n, N, M \in \mathbb{N}$, let also $\Omega \subseteq \mathbb{R}^n$ be an open set and

$$\mathcal{F} \ : \ \Omega \times \Big(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s \times \cdots \times \mathbb{R}^{Nn^p}_s\Big) \longrightarrow \mathbb{R}^M \tag{1.1}$$

a Carathéodory map. The theory we propose herein applies to measurable solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ of the system

$$\mathcal{F}\Big(x, u(x), \mathrm{D}u(x), ..., \mathrm{D}^p u(x)\Big) \, = \, 0, \quad x \in \Omega, \tag{1.2}$$

without any further restrictions on $\mathcal{F}$ and u. In (1.1)-(1.2), $\mathbb{R}^{Nn}$ symbolises the space of $N \times n$ matrices and $\mathbb{R}^{Nn^p}_s$ symbolises the space of symmetric tensors

$$\Big\{ \mathbf{X} \in \mathbb{R}^{Nn^p} \ \big| \ \mathbf{X}_{\alpha i_1 \dots i_p} = \mathbf{X}_{\alpha \sigma(i_1 \dots i_p)} \ , \ \alpha = 1, ..., N, \\ i_k = 1, ..., n, \ k = 1, ..., p, \ \sigma \text{ permutation of size } p \Big\}$$

wherein the gradient matrix $\mathrm{D}u = (\mathrm{D}_i u_\alpha(x))^{\alpha=1,\dots,N}_{i=1,\dots,n}$ and the pth order derivative

$$\mathrm{D}^p u(x) \ = \ \Big(\mathrm{D}^p_{i_1 \dots i_p} u_\alpha(x)\Big)^{\alpha=1,\dots,N}_{i_1,\dots,i_p \in \{1,\dots,n\}}$$

of (smooth) maps are respectively valued. Evidently, $\mathrm{D}_i = \partial / \partial x_i$, $x = (x_1, ..., x_n)^\top$, $u = (u_1, ..., u_N)^\top$ and $\mathbb{R}^{Nn^1}_s = \mathbb{R}^{Nn}$. Since we will not assume that the solutions are locally integrable on Ω, the derivatives $\mathrm{D}u$, ..., $\mathrm{D}^p u$ may not have classical meaning, *not even in the sense of distributions*.

The starting point of our approach in not based either on duality or on the maximum principle. Instead, it builds on the probabilistic representation of infinitesimal limits of difference quotients by using *Young measures*. This concept was introduced in the 1930s ([Y]) in order to show existence of "relaxed" solutions to nonconvex variational problems for which the minimum may not be attained. Nowadays Young measures form a full-blown active area of general topology ([CFV, FG, V]), whilst their utility in Calculus of Variations and PDE theory renders them indispensable tools for applications ([E, P, FL, M, DPM, KR]), especially in the quantification of the failure of strong convergence due to oscillations and/or concentrations.

In the present framework, Young measures valued into *compact tori and spheres* (instead of Euclidean spaces as in the aforementioned applications) are utilised in order to define generalised solutions of (1.2) by applying them to the *difference quotients* of the candidate solution. The notion is pedagogically derived later, but the idea of the definition when $p = 1$ in (1.1) can be briefly motivated as follows: let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a strong $W^{1,\infty}(\Omega, \mathbb{R}^N)$ solution to

$$\mathcal{F}\big(\cdot, u, \mathrm{D}u\big) = 0, \quad \text{a.e. on } \Omega. \tag{1.3}$$

We aim at finding a "weak" formulation of (1.3) which makes sense when u is merely measurable. To this end, we restate (1.3) as

$$\int_{\mathbb{R}^{Nn}} \Phi(X)\,\mathcal{F}\big(x, u(x), X\big)\, d[\delta_{\mathrm{D}u(x)}](X) = 0, \qquad \text{a.e. } x \in \Omega, \tag{1.4}$$

for any compactly supported $\Phi \in C_c\big(\mathbb{R}^{Nn}\big)$. Namely, we switch from the classical viewpoint of the gradient as a map $\mathrm{D}u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}$ by seeing it a probability-valued map given by the Dirac mass at $\mathrm{D}u$:

$$\delta_{\mathrm{D}u} \;:\; \Omega \subseteq \mathbb{R}^n \longrightarrow \mathscr{P}(\mathbb{R}^{Nn}), \quad x \longmapsto \delta_{\mathrm{D}u(x)}.$$

Further, we may restate that $\mathrm{D}u$ is the limit in measure of the difference quotients $\mathrm{D}^{1,h}u$ as $h \to 0$ by writing

$$\delta_{\mathrm{D}^{1,h}u} \xrightharpoonup{\;*\;} \delta_{\mathrm{D}u}, \quad \text{as } h \to 0. \tag{1.5}$$

The weak* convergence above is meant in the Young measures valued into $\mathbb{R}^{Nn}$, that is the set of measurable probability-valued mappings $\Omega \subseteq \mathbb{R}^n \longrightarrow \mathscr{P}(\mathbb{R}^{Nn})$ (for details we refer to Section 2). The rationale of the reformulation (1.4)-(1.5) of (1.3) is that *we may thus allow for general probability-valued "diffuse gradients" of measurable maps which may not be concentration measures.* This is indeed possible if we replace $\mathbb{R}^{Nn}$ by its 1-point spherical compactification $\overline{\mathbb{R}}^{Nn} := \mathbb{R}^{Nn} \cup \{\infty\}$. By considering instead the maps $(\delta_{\mathrm{D}^{1,h}u})_{h\neq 0}$ as *Young measures valued into* $\overline{\mathbb{R}}^{Nn}$, we obtain the necessary compactness and we *always* have subsequential weak* limits in the set of Young measures $\Omega \subseteq \mathbb{R}^n \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn}\big)$:

$$\delta_{\mathrm{D}^{1,h_i}u} \xrightharpoonup{\;*\;} \mathcal{D}u, \quad \text{as } h_i \to 0. \tag{1.6}$$

Then, we interpret (1.3) for just measurable maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ as

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi(X)\,\mathcal{F}\big(x, u(x), X\big)\, d[\mathcal{D}u(x)](X) = 0, \qquad \text{a.e. } x \in \Omega, \tag{1.7}$$

for any "test function" $\Phi \in C_c\big(\mathbb{R}^{Nn}\big)$ and any "diffuse gradient" $\mathcal{D}u$. Up to a minor technical adaptation (we may need to expand derivatives with respect to non-standard frames determined by $\mathcal{F}$) (1.6) and (1.7) essentially constitute the definition of **diffuse derivatives** and **$\mathcal{D}$-solutions** in the special case of (1.3) and will be the central notion of solution in this paper.

Our motivation to introduce and study generalised solutions for nonlinear PDE systems is primarily sparked by the recently discovered systems associated to vectorial Calculus of Variations in the space L^∞ and in particular the model ∞-Laplace system. Calculus of Variations in L^∞ has a long history which started in the 1960s ([A1]-[A5]). Aronsson was the first to consider variational problems for the supremal functional

$$E_\infty(u, \Omega') := \big\|H(\cdot, u, \mathrm{D}u)\big\|_{L^\infty(\Omega')}, \quad u \in W^{1,\infty}(\Omega, \mathbb{R}^N),\ \Omega' \Subset \Omega. \tag{1.8}$$

He studied the case $N = 1$ and introduced the appropriate L^∞-notion of minimisers, derived the PDE which is the L^∞-analogue of the Euler-Lagrange equation and studied its classical solutions. In the simplest case of $H(p) = |p|^2$, the L^∞-equation is called the ∞-Laplacian and reads

$$\Delta_\infty u := \mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u = 0. \tag{1.9}$$

Since then, the field has undergone huge development due to both the intrinsic mathematical interest and the important for applications: minimisation of the maximum provides more *realistic* models when compared to the classical case of integral functionals. A basic difficulty is that (1.9) possesses singular solutions. Aronsson himself exhibited this in [A6, A7] and the field remained dormant until the 1990s when the development of viscosity solutions led to an explosion of interest (e.g. [C, BEJ, E, E2] and for a pedagogical introduction see [K8]).

Until recently, the study of supremal functionals in conjunction to their associated PDEs was essentially restricted to $N = 1$. The principal obstruction appears to be the absence of an efficient theory of generalised solutions allowing the study of general systems, including those arising in L^∞. For instance, the deep contributions [BJW1, BJW2] essentially aimed at studying only the functional when $N \geq 2$. The foundations of the vector case, including the discovery of the appropriate system counterpart of (1.9), the correct vectorial L^∞-minimality notion and the study of classical solutions have been laid in a series of recent papers of the author ([K1]-[K6]). In the model case of

$$E_\infty(u, \Omega') = \big\| |\mathrm{D}u|^2 \big\|_{L^\infty(\Omega')}, \quad u \in W^{1,\infty}(\Omega, \mathbb{R}^N),\ \Omega' \Subset \Omega \tag{1.10}$$

(where $|\mathrm{D}u|$ is the Euclidean norm of the gradient on $\mathbb{R}^{Nn}$), the analogue of the Euler-Lagrange equation is the ∞-Laplace system:

$$\Delta_\infty u := \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes \mathrm{I}\Big) : \mathrm{D}^2 u = 0. \tag{1.11}$$

In the above, $[\![\mathrm{D}u(x)]\!]^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of the matrix $\mathrm{D}u(x)$. In index form (1.11) reads

$$\sum_{\beta=1}^N \sum_{i,j=1}^n \Big(\mathrm{D}_i u_\alpha \, \mathrm{D}_j u_\beta + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp_{\alpha\beta} \delta_{ij}\Big) \mathrm{D}^2_{ij} u_\beta = 0, \qquad \alpha = 1, ..., N$$

and $[\![\mathrm{D}u]\!]^\perp = \mathrm{Proj}_{(R(\mathrm{D}u))^\perp}$. An additional difficulty of (1.11) which is not present in the scalar case of (1.9) is that the nonlinear operator may have *discontinuous coefficients* even when applied to smooth maps because the new term involving $[\![\mathrm{D}u(x)]\!]^\perp$ depends on the dimension of the tangent space of $u(\Omega)$ at x ([K1, K6]). Let us also note that almost simultaneously to [K1], Sheffield and Smart [SS] studied the relevant problem of vectorial optimal Lipschitz extensions and derived a different singular version of "∞-Laplacian", which in the present setting amounts to changing in (1.10) from the Euclidean to the operator norm on $\mathbb{R}^{Nn}$.

A further motivation to introduce generalised solutions stems from the insufficiency of the current PDE approaches to handle even elliptic linear systems with rough coefficients. For example, if $\mathbf{A}$ is a continuous symmetric 4th order tensor on $\mathbb{R}^{Nn}$ satisfying the strict Legendre-Hadamard condition, for the divergence system

$$\sum_{\beta=1}^N \sum_{i,j=1}^n \mathrm{D}_i \Big(\mathbf{A}_{\alpha i \beta j}(x) \, \mathrm{D}_j u_\beta(x)\Big) = 0, \quad \alpha = 1, ..., N,$$

"everything" is known: existence-uniqueness of weak solutions, regularity, etc (see e.g. [GM]). On the other hand, for its non-divergence counterpart

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n}\mathbf{A}_{\alpha i\beta j}(x)\mathrm{D}^2_{ij}u_\beta(x)\,=\,0,\quad \alpha=1,...,N, \tag{1.12}$$

"nothing" is known, not even what is a meaningful notion of generalised solution, unless $\mathbf{A}$ is $C^{0,\alpha}$ and *strictly elliptic* in which case a priori estimates guarantee that solutions of (1.12), if they exist, must be classical ([GM]). To the best of our knowledge there are *no results* for (1.12) in the general case. If $\mathbf{A}$ is monotone (i.e. $\mathbf{A}_{\alpha i\beta j}=\delta_{\alpha\beta}A_{ij}$), the system decouples to N independent equations and can be treated in the viscosity sense.

In the present paper, after motivating, introducing and developing some basic theory of $\mathcal{D}$-solutions for general systems (Section 2), we apply it to two important problems. Accordingly, we first consider the Dirichlet problem

$$\left\{\begin{aligned} \Delta_\infty u\,&=\,0, &&\text{in } \Omega,\\ u\,&=\,g, &&\text{on } \partial\Omega, \end{aligned}\right. \tag{1.13}$$

when $\Omega\subseteq\mathbb{R}^n$ is an open domain with finite measure, $n=N$ and $g\in W^{1,\infty}(\Omega,\mathbb{R}^n)$. In Section 3 we prove existence of $\mathcal{D}$-solutions to (1.13) in $W^{1,\infty}_g(\Omega,\mathbb{R}^n)$ with extra geometric properties (Theorem 27, Corollary 29). *The question of uniqueness for vectorial L^∞ problems has already been answered negatively in* [K2] even in the class of **smooth** solutions (Remark 28).

The idea of the proof has two main steps (see Subsection 3.1). We first apply the Dacorogna-Marcellini Baire Category method ([DM]) which is an analytic counterpart of Gromov's Convex Integration and prove existence of a $W^{1,\infty}$ map solving a first order differential inclusion associated to (1.13). Next, we characterise this map as a $\mathcal{D}$-solution to (1.13) by utilising the machinery of Section 2. Along the way we establish a general tool of independent interest which goes far beyond the ∞-Laplacian and provides a method of constructing $\mathcal{D}$-solutions to "tangent equations" (Theorem 30).

The second main question we consider in this paper concerns the existence, uniqueness and (partial) regularity of $\mathcal{D}$-solutions to the Dirichlet problem

$$\left\{\begin{aligned} \mathcal{F}(\cdot,\mathrm{D}^2u)\,&=\,f, &&\text{in } \Omega,\\ u\,&=\,0, &&\text{on } \partial\Omega, \end{aligned}\right. \tag{1.14}$$

when $\Omega\Subset\mathbb{R}^n$ is a C^2 convex domain, $F:\Omega\times\mathbb{R}^{Nn^2}_s\longrightarrow\mathbb{R}^N$ is a Carathéodory map and $f\in L^2(\Omega,\mathbb{R}^N)$. The essential hypothesis guaranteeing well posedness is a *degenerate ellipticity* condition which requires $\mathcal{F}$ to be "controllably away" from a degenerate linear operator (Definition 32). Our condition is relatively strong, but classical examples (see e.g. [LU]) show that *even in the scalar case*, the Dirichlet problem for the uniformly elliptic equation $A(x):\mathrm{D}^2u(x)=f(x)$ is *not well posed* if A is discontinuous and extra conditions are required. (1.14) has first been considered by Campanato [C1]-[C3] under a strict ellipticity assumption of Cordes type which implies (1.14) is well posed in $(W^{2,2}\cap W^{1,2}_0)(\Omega,\mathbb{R}^N)$. Very recently, the author ([K9, K11] and [K7]) generalised these results by proving well posedness in the same space under a weaker condition. The latter results for strong solutions of strictly elliptic systems were stepping stones to the approach we develop herein for $\mathcal{D}$-solutions of degenerate systems.

In Section 4 we prove existence of a unique $\mathcal{D}$-solution to (1.14) which satisfies a *new type of partial regularity*, possessing differentiable projections only along certain rank-one lines (Theorem 34). This regularity is *optimal* (Remark 35). In particular, the solution may not be even $W^{1,1}_{\text{loc}}$ and does not enjoy any conventional partial regularity of the type of being more regular on a set of full measure. An extra difficulty is the satisfaction of the boundary condition since under this low regularity there is no trace operator.

The proof is rather long and is based on the study of (1.14) for linear degenerate systems with constant coefficients *in the $\mathcal{D}$-sense* and on a "perturbation device". The solvability of the linear problem involves approximation and a priori *partial estimates* (Theorem 37). Well posedness of (1.14) is established via fixed point in an appropriate functional "fibre space" tailored to the degenerate case ((4.4),(4.5)). The fibre space is an extension of the classical Sobolev space and consists of *partially* regular maps being weakly differentiable only along certain "elliptic" rank-one directions. We then characterise the fixed point in the fibre space as the unique $\mathcal{D}$-solution of (1.14).

We conclude this introduction by noting that in the companion paper [K12] of this work we have utilised the present framework to prove existence and partial regularity of absolutely minimising $\mathcal{D}$-solutions to the system of equations arising from (1.8) when $n = 1$ (see also the joint works [AK, KP1] with Abugirda and Pryer). Moreover, in the most recent papers [K13]-[K15] and also jointly with Croce, Pisante and Pryer in [CKP, KP2] we have obtained various explorative results by utilising $\mathcal{D}$-solutions. We hope that the systematic theory proposed herein will be the starting point for future developments.

2. Theory of $\mathcal{D}$-solutions for fully nonlinear systems

2.1. **Preliminaries.** We begin with some introductory material which will be used throughout freely, perhaps without explicit reference to this subsection.

Basics. Let $n, N \in \mathbb{N}$ be fixed, which in this paper will always be the dimensions of domain and range respectively of our candidate solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$. By Ω we will always mean an open subset of $\mathbb{R}^n$. Unless indicated otherwise, Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{1, ..., N\}$ and latin indices $i, j, k, ...$ (perhaps indexed $i_1, i_2, ...$) will run in $\{1, ..., n\}$, even when the range is not given explicitly. The norms $|\cdot|$ appearing throughout will always be the Euclidean, while the Euclidean inner products will be denoted by either "$\cdot$" on $\mathbb{R}^n, \mathbb{R}^N$ or by ":" on tensor spaces, e.g. on $\mathbb{R}^{Nn}$ we have $|X|^2 = \sum_{\alpha,i} X_{\alpha i} X_{\alpha i} \equiv X : X$ and on $\mathbb{R}^{Nn^2}_s$ we have $|\mathbf{X}|^2 = \sum_{\alpha,i,j} \mathbf{X}_{\alpha ij} \mathbf{X}_{\alpha ij} \equiv \mathbf{X} : \mathbf{X}$, etc. The standard bases on $\mathbb{R}^n$, $\mathbb{R}^N$, $\mathbb{R}^{Nn}$ will be denoted by $\{e^i\}$, $\{e^\alpha\}$ and $\{e^\alpha \otimes e^i\}$. By introducing the *symmetrised tensor product*

$$a \vee b := \frac{1}{2}\Big(a \otimes b + b \otimes a\Big), \quad a, b \in \mathbb{R}^n, \tag{2.1}$$

we will write $\big\{e^\alpha \otimes (e^{i_1} \vee ... \vee e^{i_p})\big\}$ for the standard basis of the $\mathbb{R}^{Nn^p}_s$. We will follow the convention of denoting vector subspaces of Euclidean spaces as well as the orthogonal projections on them by the same symbol. For example, if $\Sigma \subseteq \mathbb{R}^N$ is a subspace, we denote the projection map $\text{Proj}_\Sigma : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by just Σ and we have $\Sigma^2 = \Sigma^\top = \Sigma \in \mathbb{R}^{N^2}_s$. We will also systematically use the Alexandroff 1-point compactification $\mathbb{R}^{Nn^p}_s \cup \{\infty\}$ of the space $\mathbb{R}^{Nn^p}_s$. Its metric will be the standard one which makes it homeomorphic to the sphere of the same dimension (via the stereographic projection which identifies $\{\infty\}$ with the north pole). We will denote it by $\overline{\mathbb{R}}^{Nn^p}_s$. We note that all balls and distances taken in $\mathbb{R}^{Nn^p}_s$ (which we will view as a metric vector space isometrically contained into $\overline{\mathbb{R}}^{Nn^p}_s$) will be the Euclidean. Similar consideration apply to the torus $\overline{\mathbb{R}}^{Nn} \times ... \times \overline{\mathbb{R}}^{Nn^p}_s$ and its densely and compactly contained metric vector space $\mathbb{R}^{Nn} \times ... \times \mathbb{R}^{Nn^p}_s$. Our measure theoretic and function space notation is either standard as e.g. in [E2, D, EG] or self-explanatory. For example, the modifier "measurable" will always mean "Lebesgue measurable", the Lebesgue measure on $\mathbb{R}^n$ will be denoted by $|\cdot|$, the s-Hausdorff measure by $\mathcal{H}^s$, the characteristic function of a set A by χ_A, the standard L^p and Sobolev spaces of maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \Sigma \subseteq \mathbb{R}^N$ by $L^p(\Omega, \Sigma), W^{m,p}(\Omega, \Sigma)$, etc.

General frames, derivative expansions, difference quotients. In what follows we will need to consider non-standard orthonormal frames of $\mathbb{R}^{Nn^p}_s$ and express derivatives $\mathrm{D}^p u$ with respect to them. Let $\{E^1, ..., E^N\}$ be an orthonormal frame of $\mathbb{R}^N$ and suppose that for each $\alpha = 1, ..., N$ we have an orthonormal frame $\{E^{(\alpha)1}, ..., E^{(\alpha)n}\}$ of $\mathbb{R}^n$. For these orthonormal frames, we will equip the spaces $\mathbb{R}^{Nn} = \mathrm{span}[\{E^{\alpha i}\}]$ and $\mathbb{R}^{Nn^p}_s = \mathrm{span}[\{E^{\alpha i_1 \cdots i_p}\}]$ with:

$$E^{\alpha i} := E^\alpha \otimes E^{(\alpha)i}, \qquad E^{\alpha i_1 \dots i_p} := E^\alpha \otimes \Big(E^{(\alpha)i_1} \vee ... \vee E^{(\alpha)i_p}\Big). \tag{2.2}$$

For these frames, let $\mathrm{D}_{E^{(\alpha)i}}$ and $\mathrm{D}^p_{E^{(\alpha)i_p} \dots E^{(\alpha)i_1}} = \mathrm{D}_{E^{(\alpha)i_p}} \cdots \mathrm{D}_{E^{(\alpha)i_1}}$ denote the directional derivatives of 1st and pth order along the respective directions. The gradient $\mathrm{D}u$ of a map $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ can be expressed as

$$\mathrm{D}u = \sum_{\alpha,i} \Big(E^{\alpha i} : \mathrm{D}u\Big) E^{\alpha i} = \sum_{\alpha,i} \Big(\mathrm{D}_{E^{(\alpha)i}}(E^\alpha \cdot u)\Big) E^{\alpha i} \tag{2.3}$$

and in general the pth order derivative $\mathrm{D}^p u$ as

$$\mathrm{D}^p u = \sum_{\alpha, i_1, \dots, i_p} \Big(E^{\alpha i_1 \dots i_p} : \mathrm{D}^p u\Big) E^{\alpha i_1 \dots i_p} = \sum_{\alpha, i_1, \dots, i_p} \Big(\mathrm{D}^p_{E^{(\alpha)i_1} \dots E^{(\alpha)i_p}}(E^\alpha \cdot u)\Big) E^{\alpha i_1 \dots i_p}. \tag{2.4}$$

We will also use the following notation for the *pth order Jet of u*:

$$\mathrm{D}^{[p]} u := \big(\mathrm{D}u, \mathrm{D}^2 u, ..., \mathrm{D}^p u\big).$$

Given $a \in \mathbb{R}^n$ with $|a| = 1$ and $h \in \mathbb{R} \setminus \{0\}$, when $x, x + ah \in \Omega$ the 1st difference quotient of u along the direction a at x will be denoted by

$$\mathrm{D}^{1,h}_a u(x) := \frac{u(x + ha) - u(x)}{h}. \tag{2.5}$$

By iteration, if $h_1, ..., h_p \neq 0$ the pth order difference quotient along $a_1, ..., a_p$ is

$$\mathrm{D}^{p,h_p \dots h_1}_{a_p \dots a_1} u := \mathrm{D}^{1,h_p}_{a_p}\Big(\cdots \big(\mathrm{D}^{1,h_1}_{a_1} u\big)\Big). \tag{2.6}$$

Young Measures into compact spaces. This subsection collects basic material that can be found in different guises and greater generality e.g. in [CFV, FG, V]. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $\mathbb{K}$ a compact subset of some Euclidean space $\mathbb{R}^d$. Later we will take $\mathbb{K}$ to be either the sphere $\overline{\mathbb{R}}^{Nn^p}_s$ or the torus $\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s$. Consider the space $L^1\big(E, C(\mathbb{K})\big)$ of strongly measurable maps in the standard Bochner sense, where $C(\mathbb{K})$ is the space of real continuous functions on $\mathbb{K}$ (for details we refer e.g. to [Ed, FL, F] and references therein). The elements of $L^1\big(E, C(\mathbb{K})\big)$ are Carathéodory functions $\Phi : E \times \mathbb{K} \longrightarrow \mathbb{R}$ (i.e. $\Phi(\cdot, X)$ is measurable for $X \in \mathbb{K}$ and $\Phi(\cdot, X)$ is continuous for a.e. $x \in E$) for which $\|\Phi\|_{L^1(E,C(\mathbb{K}))} := \int_E \max_{X\in\mathbb{K}} \big|\Phi(x,X)\big|\,dx < \infty$. It is well-known that (see e.g. [FL]) that

$$\big(L^1\big(E, C(\mathbb{K})\big)\big)^* = L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big).$$

The dual space above consists of measure-valued maps $E \ni x \longmapsto \vartheta(x) \in \mathcal{M}(\mathbb{K})$ which are weakly* measurable, that is for any fixed open set $\mathcal{U} \subseteq \mathbb{K}$, the function $E \ni x \longmapsto [\vartheta(x)](\mathcal{U}) \in \mathbb{R}$ is measurable. The norm of the space is $\|\vartheta\|_{L^\infty_{w^*}(E,\mathcal{M}(\mathbb{K}))} := \operatorname{ess\,sup}_{x\in E} \|\vartheta(x)\|$, where "$\|\cdot\|$" denotes the total variation. Since $L^1\big(E, C(\mathbb{K})\big)$ is separable, the closed unit ball of $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ is sequentially weakly* compact. The duality pairing $\langle\cdot,\cdot\rangle : L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big) \times L^1\big(E, C(\mathbb{K})\big) \longrightarrow \mathbb{R}$ is given by

$$\langle \vartheta, \Phi\rangle := \int_E \int_{\mathbb{K}} \Phi(x,X)\, d[\vartheta(x)](X)\, dx.$$

Definition 1 (Young Measures)**.** The subset of the unit sphere of $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ which consists of probability-valued maps is called the set of Young measures:

$$\mathscr{Y}(E,\mathbb{K}) := \Big\{\vartheta \in L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big) : \vartheta(x) \in \mathscr{P}(\mathbb{K}),\ \text{for a.e. } x \in E\Big\}.$$

Remark 2 (Properties of $\mathscr{Y}(E,\mathbb{K})$)**.** The following well known facts will be extensively used hereafter (for proofs see e.g. [FG]):

(i) [**weak* compactness**] The set $\mathscr{Y}(E,\mathbb{K})$ is convex and (by the compactness of $\mathbb{K}$, it can be shown) it is sequentially weakly* compact in $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$. Hence, for any $(\vartheta^m)_1^\infty$, there is a ϑ and a subsequence along which $\vartheta^{m_j} \overset{*}{\rightharpoonup} \vartheta$ as $j \to \infty$.

(ii) [**Young measures induced by functions**] Every measurable map $v : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{K}$ induces a Young measure $\delta_v \in \mathscr{Y}(E,\mathbb{K})$ given by $\delta_v(x) := \delta_{v(x)}$.

(iii) [**weak* LSC**] We have the following one-sided characterisation of weak* convergence: $\vartheta^m \overset{*}{\rightharpoonup} \vartheta$ in $\mathscr{Y}(E,\mathbb{K})$ if and only if $\langle\vartheta,\Psi\rangle \le \liminf_{m\to\infty}\langle\vartheta^m,\Psi\rangle$ for any bounded from below function $\Psi : E \times \mathbb{K} \longrightarrow (-\infty,+\infty]$ measurable in x for all $X \in \mathbb{K}$ and lower semicontinuous in X for a.e. $x \in E$.

The next result is a minor variant of a classical result which we give together with its short proof because it plays a fundamental role in our setting.

Lemma 3. *Suppose $E \subseteq \mathbb{R}^n$ is measurable and $v^m, v^\infty : E \longrightarrow \mathbb{K}$ are measurable maps, $m \in \mathbb{N}$. Then, up to the passage to subsequences, we have $v^m \longrightarrow v^\infty$ a.e. on E if and only if $\delta_{v^m} \overset{*}{\rightharpoonup} \delta_{v^\infty}$ in $\mathscr{Y}(E,\mathbb{K})$.*

Proof of Lemma 3. ($\Rightarrow$) If $v^m \longrightarrow v^\infty$ a.e. on E, by Remark 2 there is $(v^{m_k})_1^\infty$ such that $\delta_{v^{m_k}} \overset{*}{\rightharpoonup} \vartheta^\infty$ in $\mathscr{Y}(E,\mathbb{K})$. If $\Phi \in L^1\big(E, C(\mathbb{K})\big)$, we have

$$\int_E \Phi\big(x, v^{m_k}(x)\big)\, dx \longrightarrow \int_E\int_{\mathbb{K}} \Phi(x,X)\, d[\vartheta^\infty(x)](X)\, dx$$

and also, the L^1 bound $|\Phi(\cdot, v^{m_k})| \le \max_{X\in\mathbb{K}} |\Phi(\cdot,X)|$ gives $\Phi(\cdot, v^{m_k}) \longrightarrow \Phi(\cdot, v^\infty)$ in $L^1(E)$. Hence, by uniqueness of limits $\vartheta^\infty = \delta_{v^\infty}$ a.e. on E.

($\Leftarrow$) If $\delta_{v^m} \overset{*}{\rightharpoonup} \delta_{v^\infty}$ in $\mathscr{Y}(E,\mathbb{K})$, we choose $\Phi(x,X) := |X - v^\infty(x)|$ where $|\cdot|$ denotes the norm of $\mathbb{R}^d$ restricted to the compact set $\mathbb{K}$. Then, for any $\varepsilon > 0$

$$0 = \int_E \Phi(\cdot, v^\infty) = \lim_{m\to\infty}\int_E \Phi(\cdot, v^m) \ge \varepsilon \limsup_{m\to\infty}\Big|\big\{|v^m - v^\infty| > \varepsilon\big\}\Big|.$$

Hence, $v^m \longrightarrow v^\infty$ in measure on E which gives $v^{m_l} \longrightarrow v^\infty$ a.e. on E. $\square$

2.2. **Motivation of the notions.** We seek to find a meaningful notion of generalised solution for fully nonlinear PDE systems which does not require any more regularity apart from measurability. We derive it in the instructive case of 2nd order systems. Suppose $\mathcal{F}$ is as in (1.1) with $p = 2$ and suppose $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is a $W^{2,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ strong a.e. solution to the system

$$\mathcal{F}(\cdot, u, \mathrm{D}u, \mathrm{D}^2 u) = 0, \quad \text{ in } \Omega. \tag{2.7}$$

By the standard equivalence between weak and strong derivatives, the difference quotients converge along subsequence a.e. on Ω to the weak derivatives. Hence,

$$\mathcal{F}\Big(\cdot, u, \lim_{m\to\infty} \mathrm{D}^{1,h_m} u, \lim_{m',m''\to\infty} \mathrm{D}^{2,h_{m'}h_{m''}} u\Big) = 0,$$

a.e. on Ω. Here $\mathrm{D}^{1,h}$, $\mathrm{D}^{2,kh}$ stand for the usual difference quotient operators whose components with respect to standard basis $\mathrm{D}^{1,h}_{e^i}$, $\mathrm{D}^{2,kh}_{e^i e^j}$ are given by (2.5), (2.6). Since $\mathcal{F}$ is a Carathéodory map, the limits commute with the nonlinearity:

$$\lim_{m,m',m''\to\infty} \mathcal{F}\Big(\cdot, u, \mathrm{D}^{1,h_m} u, \mathrm{D}^{2,h_{m'}h_{m''}} u\Big) = 0, \tag{2.8}$$

a.e. on Ω. The crucial observation is that (2.8) *is independent of the weak differentiability of* u and makes sense if u is merely measurable. How can we represent these limits and turn them into a handy definition? Going back to (2.7), we observe that u is a strong solution of (2.7) if and only if it satisfies

$$\int_{\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s} \Phi(X,\mathbf{X})\mathcal{F}(\cdot,u,X,\mathbf{X})\, d\big[\delta_{(\mathrm{D}u,\mathrm{D}^2u)}\big](X,\mathbf{X}) = 0, \quad \text{ a.e. on } \Omega,$$

for any "test" function $\Phi \in C_c\big(\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s\big)$. This gives the idea that we can view the difference quotients as Young measures arising from functions, that is $\delta_{\mathrm{D}^{1,h_m}u} : \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn}\big)$ and $\delta_{\mathrm{D}^{2,h_{m'}h_{m''}}u} : \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)$. The reason we compactify the space is to obtain weak* compactness. This compensates the possible loss of mass to ∞ since the difference quotients may not converge in any classical sense for just measurable maps. However, when considered in the Young measures valued into spheres they have subsequential weak* limits. It is also more fruitful to take limits *separately* (regardless of order), because the resulting object will be a (fibre) product Young measure valued in the compact torus $\overline{\mathbb{R}}^{Nn}\times\overline{\mathbb{R}}^{Nn^2}_s$:

$$\delta_{\left(\mathrm{D}^{1,h_m}u,\mathrm{D}^{2,h_{m'}h_{m''}}u\right)} \overset{*}{\longrightarrow} \mathcal{D}u \times \mathcal{D}^2 u \quad \text{ in } \mathscr{Y}\Big(\Omega, \overline{\mathbb{R}}^{Nn}\times\overline{\mathbb{R}}^{Nn^2}_s\Big), \tag{2.9}$$

subsequentially as $m, m', m'' \to \infty$ separately. Then, (2.8) is equivalent to

$$\int_{\overline{\mathbb{R}}^{Nn}\times\overline{\mathbb{R}}^{Nn^2}_s} \Phi(X,\mathbf{X})\mathcal{F}(\cdot,u,X,\mathbf{X})\, d\Big[\delta_{\left(\mathrm{D}^{1,h_m}u,\mathrm{D}^{2,h_{m'}h_{m''}}u\right)}\Big](X,\mathbf{X}) \longrightarrow 0,$$

subsequentially as $m, m', m'' \to \infty$, a.e. on Ω, for any $\Phi \in C_c\big(\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s\big)$. By using Lemma 16 that follows, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}\times\overline{\mathbb{R}}^{Nn^2}_s} \Phi(X,\mathbf{X})\mathcal{F}(\cdot,u,X,\mathbf{X})\, d\big[\mathcal{D}u\times\mathcal{D}^2u\big](X,\mathbf{X}) = 0, \quad \text{ a.e. on } \Omega,$$

for any Φ. Note that this statement is *independent* of the regularity of the solution. If $u \in W^{1,1}_{\text{loc}}(\Omega,\mathbb{R}^N)$ by Lemma 3 we have that $\mathcal{D}u = \delta_{\mathrm{D}u}$ a.e. on Ω and the above statement simplifies to (2.11). If further $\mathrm{D}^2 u$ exists weakly on Ω, by Lemma 3 we have $\mathcal{D}^2 u = \delta_{\mathrm{D}^2 u}$ a.e. on Ω thus recovering strong solutions.

2.3. **Main definitions and analytic properties.** We begin by introducing difference quotients taken with respect to frames as in (2.2), (2.3), (2.4). The only difficulty is the complexity in the notation so for pedagogical reasons we give the 1st order case separately from the general pth order case.

Definition 4 (Difference quotients)**.** Suppose $\{E^1,...,E^N\}$ is an orthonormal frame of $\mathbb{R}^N$ and for each $\alpha=1,...,N$ we have an orthonormal frame $\{E^{(\alpha)1},...,E^{(\alpha)n}\}$ of $\mathbb{R}^n$ whilst the spaces $\mathbb{R}^{Nn^p}_s$ are equipped with the frames of (2.2), $p\in\mathbb{N}$. Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be a measurable map, extended by zero on $\mathbb{R}^n\setminus\Omega$. Given infinitesimal sequences $(h_m)_{m\in\mathbb{N}}$ and $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^p}\subseteq\big(\mathbb{R}\setminus\{0\}\big)^p$ such that

$$h_m\to 0 \text{ as } m\to\infty\ ,\quad h_{\underline{m}}=(h_{m^1},...,h_{m^p}),\quad h_{m^q}\to 0\ \text{ as } m^q\to\infty,$$

we define the 1**st and** p**th order difference quotients of** u (with respect to the fixed reference frames) arising from $(h_m)_{m\in\mathbb{N}}$ and $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^p}$ as

$$\begin{aligned}
&\mathrm{D}^{1,h_m}u\ :\quad \mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{Nn}, && m\in\mathbb{N},\\
&\mathrm{D}^{p,h_{\underline{m}}}u\ :\quad \mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{Nn^p}_s, && \underline{m}=(m^1,...,m^p)\in\mathbb{N}^p,
\end{aligned}$$

given respectively by

$$\begin{aligned}
\mathrm{D}^{1,h_m}u &:= \sum_{\alpha,i}\Big[\mathrm{D}^{1,h_m}_{E^{(\alpha)i}}(E^\alpha\cdot u)\Big]\,E^{\alpha i},\\
\mathrm{D}^{p,h_{\underline{m}}}u &:= \sum_{\alpha,i_1,...,i_p}\Big[\mathrm{D}^{p,h_{m^p}...h_{m^1}}_{E^{(\alpha)i_p}...E^{(\alpha)i_1}}(E^\alpha\cdot u)\Big]\,E^{\alpha i_1...i_p}.
\end{aligned}$$

In the above, the notation in the brackets is as in (2.5), (2.6). Further, given an infinitesimal sequence with a trigonal matrix of indices

$$(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^{p^2}}\subseteq\big(\mathbb{R}\setminus\{0\}\big)^{p^2},\quad \underline{m}=\left[\begin{array}{cccc} m^1_1 & 0 & 0\ ... & 0\\ m^1_2 & m^2_2 & 0\ ... & 0\\ \vdots & & \ddots & \vdots\\ m^1_p & m^2_p & ... & m^p_p\end{array}\right],\quad h_{m^q_p}\to 0 \text{ as } m^q_p\to\infty,$$

we will denote its nonzero row elements by $\underline{m}_q:=(m^1_q,...,m^q_q)\in\mathbb{N}^q$, $q=1,...,p$. We define the p**th order Jet** $\mathrm{D}^{[p],h_{\underline{m}}}u$ **of difference quotients of** u (with respect to the fixed reference frames) arising from $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^{p^2}}$ as

$$\mathrm{D}^{[p],h_{\underline{m}}}u:=\Big(\mathrm{D}^{1,h_{\underline{m}_1}}u,...,\mathrm{D}^{p,h_{\underline{m}_p}}u\Big)\quad:\quad \mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s.$$

Definition 5 (Multi-indexed convergence)**.** Let $\underline{m}$ be either a vector of indices in $\mathbb{N}^p$ or a lower trigonal matrix of indices in $\mathbb{N}^{p^2}$. The expression $\underline{m}\longrightarrow\infty$ will symbolise separate successive convergence with respect to each entry in the order:

$$\begin{aligned}
&m^1\to\infty,\ ...,\ m^p\to\infty && (\underline{m}\text{ vector}),\\
&m^1_1\to\infty,\ m^1_2\to\infty,\ m^2_2\to\infty,\ ...,\ m^{p-1}_p\to\infty,\ m^p_p\to\infty && (\underline{m}\text{ matrix}).
\end{aligned}$$

Definition 6 (Diffuse derivatives and Jets)**.** Suppose we have fixed some reference frames as in Definition 4. For any measurable $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$, we define **diffuse gradients** $\mathcal{D}u$, **diffuse** p**th order derivatives** $\mathcal{D}^pu$ and **diffuse** p**th order Jets** $\mathcal{D}^{[p]}u$ **of** u as the subsequential limits of difference quotients arising along infinitesimal sequences in the spaces of Young measures valued in the respective spherical/toric compactifications:

$$\begin{aligned}
&\delta_{\mathrm{D}^{1,h_m}u} \overset{*}{\rightharpoonup} \mathcal{D}u, && \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\big), && \text{as } m\to\infty,\\
&\delta_{\mathrm{D}^{p,h_{\underline{m}}}u} \overset{*}{\rightharpoonup} \mathcal{D}^p u, && \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^p}_s\big), && \text{as } \underline{m}\to\infty,\ \underline{m}\in\mathbb{R}^p,\\
&\delta_{\mathrm{D}^{[p],h_{\underline{m}}}u} \overset{*}{\rightharpoonup} \mathcal{D}^{[p]} u, && \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s\big), && \text{as } \underline{m}\to\infty,\ \underline{m}\in\mathbb{R}^{p^2}.
\end{aligned}$$

Remark 7. As a consequence of the separate convergence, the pth order Jet is always a (fibre) product Young measure: $\mathcal{D}^{[p]}u \ = \ \mathcal{D}u\times\cdots\times\mathcal{D}^p u$.

We now record that Remark 2(i) implies the existence of diffuse derivatives.

Lemma 8 (Existence of diffuse derivatives)**.** *Every measurable mapping $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ possesses diffuse derivatives of all orders, actually at least one for every choice of infinitesimal sequence.*

Remark 9 (Nonexistence of distributional derivatives)**.** Since we do not require our maps to be in $L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, they may not possess distributional derivatives.

In general diffuse derivatives *may not be unique* for nonsmooth maps. However, they are compatible with weak derivatives:

Lemma 10 (Compatibility of weak and diffuse derivatives)**.** *If $u\in W^{1,1}_{loc}(\Omega,\mathbb{R}^N)$, then the diffuse gradient $\mathcal{D}u$ is unique and $\delta_{\mathrm{D}u}=\mathcal{D}u$, a.e. on Ω. More generally, if $q\in\{1,...,p-1\}$ and $u\in W^{q,1}_{loc}(\Omega,\mathbb{R}^N)$, then $\mathcal{D}^{[q]}u$ is unique and*

$$\mathcal{D}^{[p]}u = \delta_{(\mathrm{D}u,\dots,\mathrm{D}^q u)}\times\mathcal{D}^{q+1}\times\cdots\times\mathcal{D}^p u, \quad a.e. \ on \ \Omega.$$

Proof of Lemma 10. It suffice to establish only the 1st order case. For any fixed $e\in\mathbb{R}^n$ we have $\mathrm{D}^{1,h}_e u\longrightarrow D_e u$ in $L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ as $h\to 0$. We choose $e:=E^{(\alpha)i}$ and $h:=h_m$ to get $\mathrm{D}^{1,h_m}_{E^{(\alpha)i}}(E^\alpha\cdot u)\longrightarrow \mathrm{D}_{E^{(\alpha)i}}(E^\alpha\cdot u)$ in $L^1_{\mathrm{loc}}(\Omega)$ as $m\to\infty$. Thus, by (2.3), (2.4) and Definition 4 we have $\mathrm{D}^{1,h_m}u\longrightarrow\mathrm{D}u$ a.e. on Ω as $m\to\infty$ along a subsequence. Application of Lemma 3 completes the proof. ☐

Next we show that the diffuse gradient is a Dirac mass if and only if u is "differentiable in measure", a notion introduced and studied by Ambrosio-Malý in [AM]. This notion arose in the study of the regularity of the flow map of ODEs driven by Sobolev vector fields (see [BL]).

Definition 11 (Differentiability in measure, cf. [AM])**.** Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be measurable. We say that u is differentiable in measure on Ω with derivative the measurable map $\mathcal{L}Du:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{Nn}$ if for any $\varepsilon>0$ and $E\subseteq\Omega$ with $|E|<\infty$,

$$\lim_{y\to 0}\left|\left\{x\in E\,:\,\left|\frac{u(x+y)-u(x)-\mathcal{L}\mathrm{D}u(x)y}{|y|}\right|>\varepsilon\right\}\right| = 0.$$

In [AM] it is shown that this notion is strictly weaker than the classical notion of approximate differentiability ([EG]).

Lemma 12 (Gradient in measure vs diffuse gradient)**.** *Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be measurable and suppose we have fixed some reference frames as in Definition 4.*
(a) If u is differentiable is measure with derivative $\mathcal{L}\mathrm{D}u$, then the diffuse gradient $\mathcal{D}u\in\mathscr{Y}(\Omega,\overline{\mathbb{R}}^{Nn})$ is unique and $\mathcal{D}u=\delta_{\mathcal{L}\mathrm{D}u}$ a.e. on Ω.
(b) If there exists a measurable map $U:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{Nn}$ such that for any diffuse gradient $\mathcal{D}u\in\mathscr{Y}(\Omega,\overline{\mathbb{R}}^{Nn})$ we have $\mathcal{D}u=\delta_U$ a.e. on Ω, then it follows that u is differentiable in measure and $U=\mathcal{L}\mathrm{D}u$ a.e. on Ω.

Proof of Lemma 12. (a) By choosing $y := hE^{(\alpha)i}$ in Definition 11 applied to the projection $E^\alpha \cdot u$ we get that $\mathrm{D}^{1,h}_{E^{(\alpha)i}}(E^\alpha \cdot u) \longrightarrow E^{\alpha i} : (\mathcal{L}\mathrm{D}u)$ as $h \to 0$ locally in measure on Ω. Thus, for any $h_m \to 0$, there is $h_{m_k} \to 0$ such that the convergence is a.e. on Ω, whence $\mathcal{D}u = \delta_{\mathcal{L}\mathrm{D}u}$ by Lemma 3.

(b) We begin by observing a triviality: for any map $f : \mathbb{R}^n \to \mathbb{R}^N$ we have $f(y) \to l$ as $y \to 0$ if and only if for any $y_m \to 0$, there is $y_{m_k} \to 0$ such that $f(y_{m_k}) \to l$ as $k \to \infty$. We continue by noting that by Lemma 3 and our assumption we have that for any $h_m \to 0$ there is $h_{m_k} \to 0$ such that $\mathrm{D}^{1,h_{m_k}}u \longrightarrow U$ a.e. on Ω, as $k \to \infty$. Hence, we obtain that $\mathrm{D}^{1,h}u \longrightarrow U$ as $h \to 0$ (full limit), a.e. on Ω. Since a.e. convergence implies convergence locally in measure, we deduce that $U = \mathcal{L}\mathrm{D}u$ a.e. on Ω, as desired. ☐

The next notion of solution will be central in this work. For pedagogical reasons, we give it first for $W^{1,1}_{\mathrm{loc}}$ solutions of 2nd order systems and then in the general case.

Definition 13 (Weakly differentiable $\mathcal{D}$-solutions of 2nd order PDE systems)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open, $\mathcal{F} : \Omega \times \big(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s\big) \longrightarrow \mathbb{R}^M$ a Carathéodory map and $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ a map in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Suppose we have fixed some reference frames as in Definition 4 and consider the PDE system

$$\mathcal{F}\big(\cdot, u, \mathrm{D}u, \mathrm{D}^2u\big) = 0, \quad \text{in } \Omega. \tag{2.10}$$

We will say that u is a **$\mathcal{D}$-solution of** (2.10) when for any diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}(\Omega, \overline{\mathbb{R}}^{Nn^2}_s)$ of u (Definition 6) and any $\Phi \in C_c\big(\mathbb{R}^{Nn^2}_s\big)$ we have

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X})\,\mathcal{F}\big(\cdot, u, \mathrm{D}u, \mathbf{X}\big)\, d[\mathcal{D}^2u](\mathbf{X}) = 0, \quad \text{a.e. on } \Omega. \tag{2.11}$$

We note that $\mathcal{F}$ is not actually necessary to be continuous with respect to $(u, \mathrm{D}u)$ and merely Borel measurable suffices. Now we consider the general case. For brevity we will write $\underline{\mathbf{X}} = (\mathbf{X}_1, ..., \mathbf{X}_p)$ for the points of the torus $\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s$.

Definition 14 ($\mathcal{D}$-solutions for pth order PDE systems)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and $\mathcal{F}$ a Carathéodory map as in (1.1). Suppose $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is measurable and we have fixed some reference frames as in Definition 4. Consider the system

$$\mathcal{F}\Big(x, u(x), \mathrm{D}^{[p]}u(x)\Big) = 0, \quad x \in \Omega. \tag{2.12}$$

We will say that u is a **$\mathcal{D}$-solution of** (2.12) when for any $\mathcal{D}^{[p]}u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s\big)$ of u (Definition 6) and any $\Phi \in C_c\big(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\big)$, we have

$$\int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} \Phi(\underline{\mathbf{X}})\,\mathcal{F}\big(x, u(x), \underline{\mathbf{X}}\big)\, d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}}) = 0, \quad \text{a.e. } x \in \Omega.$$

The following result asserts the fairly obvious fact that $\mathcal{D}$-solutions and strong solutions are compatible.

Proposition 15 (Compatibility of strong with $\mathcal{D}$-solutions)**.** *Let $\mathcal{F}$ a Carathéodory map as in* (1.1) *and $u \in W^{p,1}_{loc}(\Omega, \mathbb{R}^N)$ (or merely p-times differentiable in measure, Definition 11). Then, u is a $\mathcal{D}$-solution of* (2.12) *on Ω if and only if u is a strong solution of* (2.12) *a.e. on Ω.*

Proof of Proposition 15. It is an immediate consequence of Lemma 10 (or Lemma 12) and the motivation of the notions (Subsection 2.2). ☐

Our next result is a simple yet powerful convergence tool which we give in the generality of Young measures and will play an important role in later sections.

Lemma 16 (Convergence lemma). *Suppose that $u^\infty, (u^\mu)_1^\infty$ are measurable maps $\mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ satisfying $u^\mu \longrightarrow u^\infty$ a.e. on Ω. Let $\mathbb{W}$ be a finite dimensional metric vector space, isometrically contained into a compactification $\mathbb{K}$ of $\mathbb{W}$. Suppose we have Carathéodory maps $\mathcal{F}^\infty$, $\mathcal{F}^\mu : \Omega \times \big(\mathbb{R}^N \times \mathbb{W}\big) \longrightarrow \mathbb{R}^M$, $\mu \in \mathbb{N}$ such that for a.e. $x \in \Omega$, $\mathcal{F}^\mu(x,\cdot,\cdot) \longrightarrow \mathcal{F}^\infty(x,\cdot,\cdot)$ in $C(\mathbb{R}^N \times \mathbb{W})$ as $\mu \to \infty$. Suppose further we have Young measures $\vartheta^\infty, (\vartheta^\mu)_1^\infty \in \mathscr{Y}\big(\Omega,\mathbb{K}\big)$ such that $\vartheta^\mu \overset{*}{\rightharpoonup} \vartheta^\infty$ in $\mathscr{Y}\big(\Omega,\mathbb{K}\big)$ as $\mu \to \infty$. Then, if for a given $\Phi \in C_c(\mathbb{W})$ we have*

$$\int_{\mathbb{K}} \Phi(\boldsymbol{X})\,\mathcal{F}^\mu\big(x,u^\mu(x),\boldsymbol{X}\big)\,d[\vartheta^\mu(x)](\boldsymbol{X}) \,=\, 0, \quad a.e.\ x \in \Omega,$$

for all $\mu \in \mathbb{N}$, it follows that the same conclusion holds for $\mu = \infty$ as well.

Proof of Lemma 16. We fix $\Phi \in C_c(\mathbb{W})$ and set

$$\phi^m(x) \,:=\, \Big\|\Phi(\cdot)\Big(\mathcal{F}^m\big(x,u^m(x),\cdot\big) \,-\, \mathcal{F}^\infty\big(x,u^\infty(x),\cdot\big)\Big)\Big\|_{C(\mathbb{W})}$$

and we claim that $\phi^m(x) \longrightarrow 0$ for a.e. $x \in \Omega$. To see this, fix $x \in \Omega$ such that $u^m(x) \longrightarrow u^\infty(x)$ (the set of such points has full measure in Ω). Fix also $U \Subset \mathbb{R}^N$ and $W \Subset \mathbb{W}$ such that $u^m(x), u^\infty(x) \in U$ and $\mathrm{supp}(\Phi) \subseteq W$ for large $m \in \mathbb{N}$. By our assumptions, $\|\mathcal{F}^m(x,\cdot,\cdot) - \mathcal{F}^\infty(x,\cdot,\cdot)\|_{C(U\times W)} \longrightarrow 0$ as $m \to \infty$. If $\omega_x^\infty \in C[0,\infty)$ symbolises the modulus of continuity of $U \ni \xi \longmapsto \mathcal{F}^\infty(x,\xi,\mathbf{X}) \in \mathbb{R}^M$ which is uniform in $\mathbf{X} \in W$, we have

$$\begin{aligned}|\phi^m(x)| \,\leq\, \|\Phi\|_{C(W)} \Bigg(&\Big\|\mathcal{F}^\infty\big(x,u^m(x),\cdot\big) \,-\, \mathcal{F}^\infty\big(x,u^\infty(x),\cdot\big)\Big\|_{C(W)} \\ &+ \Big\|\mathcal{F}^m\big(x,u^m(x),\cdot\big) \,-\, \mathcal{F}^\infty\big(x,u^m(x),\cdot\big)\Big\|_{C(W)}\Bigg) \\ \leq \|\Phi\|_{C(\mathbb{W})}\Big(\omega_x^\infty\big(\big|u^m(x) - u^\infty(x)\big|\big) \,+\, &\big\|\mathcal{F}^m(x,\cdot,\cdot) - \mathcal{F}^\infty(x,\cdot,\cdot)\big\|_{C(U\times W)}\Big),\end{aligned}$$

giving that $|\phi^m(x)| = o(1)$, as $m \to \infty$. We now fix $R > 0$ and set

$$\Omega_R \,:=\, \Big\{x \in \Omega \,:\, \big\|\Phi(\cdot)\,\mathcal{F}^\infty\big(x,u^\infty(x),\cdot\big)\big\|_{C(\mathbb{W})} < R\Big\} \cap \mathbb{B}_R(0).$$

Since $|\Omega_R| < \infty$, by the Egoroff theorem we can find measurable sets $\{E_i\}_1^\infty \subseteq \Omega_R$ such that $|E_i| \to 0$ as $i \to \infty$ and for each $i \in \mathbb{N}$ we have $\phi^m \longrightarrow 0$ in $L^\infty(\Omega_R \setminus E_i)$ as $m \to \infty$. Since $|\Omega_R| < \infty$, we have $\phi^m \longrightarrow 0$ in $L^1(\Omega_R \setminus E_i)$ as well. Further, the functions $\Psi^m(x,\mathbf{X}) := \big|\Phi(\mathbf{X})\,\mathcal{F}^m\big(x,u^m(x),\mathbf{X}\big)\big|$, $m \in \mathbb{N} \cup \{\infty\}$, are elements of $L^1\big(\Omega_R \setminus E_i, C(\mathbb{K})\big)$ because

$$\|\Psi^m - \Psi^\infty\|_{L^1(\Omega_R\setminus E_i, C(\mathbb{K}))} \,\leq\, \|\phi^m\|_{L^1(\Omega_R\setminus E_i)}$$

and for m large we have

$$\|\Psi^m\|_{L^1(\Omega_R\setminus E_i, C(\mathbb{K}))} \,\leq\, 1 + \|\Psi^\infty\|_{L^1(\Omega_R\setminus E_i, C(\mathbb{K}))} \,\leq\, 1 + |\Omega_R|\,R.$$

Hence, $\Psi^m \longrightarrow \Psi^\infty$ in $L^1\big(\Omega_R \setminus E_i, C(\mathbb{K})\big)$ and also by assumption $\vartheta^m \overset{*}{\rightharpoonup} \vartheta^\infty$ in $\mathscr{Y}\big(\Omega_R \setminus E_i, \mathbb{K}\big)$. By the weak*-strong continuity of the pairing

$$L^\infty_{w^*}\Big(\Omega_R \setminus E_i, \mathcal{M}(\mathbb{K})\Big) \,\times\, L^1\Big(\Omega_R \setminus E_i, C(\mathbb{K})\Big) \longrightarrow \mathbb{R}$$

we may pass to the limit in our hypothesised identity to obtain

$$\int_{\mathbb{K}} \Phi(\mathbf{X})\mathcal{F}^\infty\big(x, u^\infty(x), \mathbf{X}\big)\, d[\vartheta^\infty(x)](\mathbf{X}) \,=\, 0,$$

for a.e. $x \in \Omega_R \setminus E_j$. We conclude by letting $j \to \infty$ and then taking $R \to \infty$. ☐

The next result is a direct consequence of Lemma 16 and establishes that $\mathcal{D}$-solutions are well behaved under weak* convergence.

Corollary 17 (Convergence of $\mathcal{D}$-solutions)**.** *Let $(u^\mu)_1^\infty$ be a sequence of maps where each $u^\mu : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is measurable and $u^\mu \longrightarrow u^\infty$ a.e. on Ω. Let also $(\mathcal{F}^\mu)_1^\infty$ be Carathéodory maps as in (1.1). Suppose each u^μ is a $\mathcal{D}$-solution of*

$$\mathcal{F}^\mu\left(x, u^\mu(x), \mathrm{D}^{[p]}u^\mu(x)\right) \,=\, 0, \quad x \in \Omega,$$

and $\mathcal{F}^\mu(x,\cdot,\cdot) \longrightarrow \mathcal{F}^\infty(x,\cdot,\cdot)$ uniformly on compact subsets as $\mu \to \infty$, for a.e. $x \in \Omega$. If every jet $\mathcal{D}^{[p]}u^\infty$ can be weakly approximated by a subsequence of the respective Jets $\mathcal{D}^{[p]}u^{\mu_\nu}$, then u^∞ is a $\mathcal{D}$-solution of the limit system for $\mu = \infty$.*

Remark 18. Note that Corollary 17 is *not* a stability result, in the sense that we do not have compactness of diffuse jets as part of the conclusion. In fact, such a result is not possible without extra assumptions which would entail some sort of *a priori estimates*: for instance, consider the sequence $u^\mu(x) := \mu^{-1}\sin(\mu x)$, $x \in \mathbb{R}$. Then, $u^\mu \overset{*}{\longrightarrow} u^\infty$ in $W^{1,\infty}(\mathbb{R})$ where $u^\infty \equiv 0$. However, $\mathcal{D}u^\mu = \delta_{\mathrm{D}u^\mu} \overset{*}{\longrightarrow} \vartheta$ in $\mathscr{Y}(\mathbb{R},\mathbb{R})$ as $\mu \to \infty$, where for a.e. $x \in \mathbb{R}$ $\mathrm{supp}(\vartheta(x)) = [-1,1]$ while $\vartheta(x) \neq \mathcal{D}u^\infty(x) = \delta_{\{0\}}$.

The next result gives equivalent formulations of the definition of $\mathcal{D}$-solutions. To this end we need some further terminology.

Definition 19 (Reduced support)**.** Given a probability $\vartheta \in \mathscr{P}\big(\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s\big)$, we define its **reduced support** as

$$\mathrm{supp}_*(\vartheta) \,:=\, \mathrm{supp}(\vartheta) \cap \left(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\right).$$

Definition 20 (Cut offs associated to a map)**.** Let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be measurable and $\mathcal{F}$ as in (1.1). For any measurable $U : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s$ and $R > 0$, we define the **cut off of U associated to $\mathcal{F}$** as:

$$[U]^R \,:=\, \left\{ \begin{array}{ll} U, & \text{on } \big\{|U| \le R\big\}, \\ \mathbf{0}^R, & \text{on } \big\{|U| > R\big\}. \end{array} \right.$$

Here $\mathbf{0}^R$ is a measurable selection of the set-valued mapping

$$\Omega \ni x \longmapsto \Big\{\mathcal{F}\big(x, u(x), \cdot\big) = 0\Big\} \cap \mathbb{B}_R(0) \ \subseteq \ \left(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\right) \setminus \{\emptyset\},$$

that is, $\mathbf{0}^R : \Omega \longrightarrow \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s$ satisfies $\mathcal{F}\big(x, u(x), \mathbf{0}^R(x)\big) \,=\, 0$ and $|\mathbf{0}^R(x)| \le R$ for a.e. $x \in \Omega$.

The existence of selections is a consequence of Aumann's theorem (see e.g. [FL]). If $\mathcal{F}\big(x, u(x), \cdot\big)$ is linear, we may choose $\mathbf{0}^R(x) \equiv 0$ with no (R, x)-dependence.

Proposition 21 (Equivalent definitions for $\mathcal{D}$-solutions)**.** *Let $\mathcal{F}$ be as in (1.1) and $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ a measurable map. Then, the following are equivalent:*

(1) *The map u is a $\mathcal{D}$-solution of the PDE system*

$$\mathcal{F}\left(x, u(x), \mathrm{D}^{[p]}u(x)\right) = 0, \quad x \in \Omega.$$

(2) *For any diffuse pth order Jet of u, we have*

$$\sup_{\underline{\boldsymbol{X}}\in \mathrm{supp}_*(\mathcal{D}^{[p]}u(x))} \big|\mathcal{F}\big(x,u(x),\underline{\boldsymbol{X}}\big)\big| = 0, \quad a.e.\ x\in\Omega.$$

(3) *For any diffuse pth order Jet of u, we have the inclusion*

$$\mathrm{supp}_*\big(\mathcal{D}^{[p]}u(x)\big) \subseteq \Big\{\mathcal{F}\big(x,u(x),\cdot\big)=0\Big\}, \quad a.e.\ x\in\Omega.$$

(4) *For any diffuse pth order Jet of u, we have*

$$\int_{\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s} \big|\mathcal{F}\big(x,u(x),\underline{\boldsymbol{X}}\big)\big|\, d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\boldsymbol{X}}) = 0, \quad a.e.\ x\in\Omega.$$

(5) *For any pth order Jet of difference quotients of u and any $R>0$, we have*

$$\mathcal{F}\left(\cdot,u,\big[\mathrm{D}^{[p],h_{\underline{m}}}u\big]^R\right) \longrightarrow 0,$$

for a.e. $x\in\Omega$, as $\underline{m}\to\infty$ along subsequences.

(6) *For any pth order Jet of difference quotients of u and any $R>0$, we have*

$$\mathrm{dist}\left(\big[\mathrm{D}^{[p],h_{\underline{m}}}u\big]^R(x)\ ,\ \Big\{\mathcal{F}\big(x,u(x),\cdot\big)=0\Big\}\cap\mathbb{B}_R(0)\right)\longrightarrow 0,$$

for a.e. $x\in\Omega$, as $\underline{m}\to\infty$ along subsequences.

The presence of reduced supports and cut offs is informally interpreted as that the mass which does not escape to infinity actually lies on the zero level set of the coefficients. The proof of Proposition 21 does not rely on the particular structure of diffuse Jets and is a consequence of the next more general result.

Lemma 22. *All the equivalences of Proposition 21 remains true if more generally one replaces the jet $\mathrm{D}^{[p],h_{\underline{m}}}u$ by any measurable sequence*

$$U^m\ :\ \mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s,\quad m\in\mathbb{N},$$

and the respective Jet $\mathcal{D}^{[p]}u$ by any Young measure $\vartheta\in\mathscr{Y}\left(\Omega,\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s\right)$ such that $\delta_{U^m}\overset{*}{\longrightarrow}\vartheta$ as $m\to\infty$.

Proof of Lemma 22 & Proposition 21. We begin by showing (1)⇔(3)⇔(2) and then we establish that (6)⇒(5)⇒(4)⇒(3)⇒(6).

(1)⇒(3): Suppose $\delta_{U^m}\overset{*}{\longrightarrow}\vartheta$ and that for any $\Phi\in C_c\big(\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s\big)$ we have

$$\int_{\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s}\Phi(\underline{\mathbf{X}})\,\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\,d[\vartheta(x)](\underline{\mathbf{X}}) = 0,\quad \text{a.e. } x\in\Omega,$$

whilst for some of these $x\in\Omega$ we have $\mathrm{supp}\big(\vartheta_*(x)\big)\not\subseteq\big\{\mathcal{F}\big(x,u(x),\cdot\big)=0\big\}$. Then, there exists $\underline{\mathbf{X}}_0$ with $\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}_0\big)\neq 0$ and $[\vartheta(x)]\big(\mathbb{B}_R(\underline{\mathbf{X}}_0)\big)>0$ for $R>0$. By continuity, there exist $c_0,R_0>0$ and $\mu\in\{1,...,M\}$ such that $\big|\mathcal{F}_\mu\big(x,u(x),\cdot\big)\big|\geq c_0$ on $\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)$. By choosing Φ such that $\chi_{\mathbb{B}_{R_0/2}(\underline{\mathbf{X}}_0)}\leq\Phi\leq\chi_{\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)}$, we get

$$\begin{aligned}
0 &= \left|\int_{\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s}\Phi(\underline{\mathbf{X}})\,\mathcal{F}_\mu\big(x,u(x),\underline{\mathbf{X}}\big)\,d[\vartheta(x)](\underline{\mathbf{X}})\right|\\
&= \int_{\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)}\Phi(\underline{\mathbf{X}})\,\big|\mathcal{F}_\mu\big(x,u(x),\underline{\mathbf{X}}\big)\big|\,d[\vartheta(x)](\underline{\mathbf{X}})\\
&\geq c_0\,[\vartheta(x)]\big(\mathbb{B}_{R_0/2}(\underline{\mathbf{X}}_0)\big).
\end{aligned}$$

The above contradiction establishes that the desired inclusion holds a.e. on Ω.

(3)$\Rightarrow$(1): Suppose $\mathrm{supp}_*\big(\vartheta(x)\big) \subseteq \big\{\mathcal{F}\big(x,u(x),\cdot\big)=0\big\}$ for a.e. $x\in\Omega$. Then, for any $\Phi \in C_c\big(\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s\big)$ and any such x, $\Phi(\cdot)\mathcal{F}\big(x,u(x),\cdot\big)$ vanishes $[\vartheta(x)]$-a.e. on $\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s$. Thus,

$$\int_{\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s} \Phi(\underline{\mathbf{X}})\, F\big(x,u(x),\underline{\mathbf{X}}\big)\, d[\vartheta(x)](\underline{\mathbf{X}}) \,=\, 0.$$

(3)$\Leftrightarrow$(2): Effectively, they are just restatements of each other and either of them states that for any diffuse pth order Jet, for a.e. $x\in\Omega$ and for all $\underline{\mathbf{X}}\in\mathrm{supp}_*(\mathcal{D}^{[p]}u(x))$, we have $\big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\big|=0$.

(6)$\Rightarrow$(5): If suffices to show that for a.e. $x\in\Omega$ and any $R>0$ there is a strictly increasing modulus of continuity $\omega_{R,x}\in C[0,\infty)$ such that

$$\Big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\Big| \,\leq\, \omega_{R,x}\left(\mathrm{dist}\left(\underline{\mathbf{X}}\ ,\ \Big\{\mathcal{F}\big(x,u(x),\cdot\big)=0\Big\}\cap\mathbb{B}_R(0)\right)\right),$$

when $\underline{\mathbf{X}}\in\overline{\mathbb{B}_R(0)}$. In such an event we conclude by choosing $\underline{\mathbf{X}} := [U^m]^R(x)$. To see the claim, note that for a.e. $x\in\Omega$ there is such an $\omega_{R,x}$ with

$$\Big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big) \,-\, \mathcal{F}\big(x,u(x),\underline{\mathbf{Y}}\big)\Big| \,\leq\, \omega_{R,x}\big(\big|\underline{\mathbf{X}}-\underline{\mathbf{Y}}\big|\big)$$

for all $\underline{\mathbf{X}},\underline{\mathbf{Y}}\in\overline{\mathbb{B}_R(0)}$. By choosing $\underline{\mathbf{Y}}\in\big\{\mathcal{F}\big(x,u(x),\cdot\big)=0\big\}$, we have

$$\begin{aligned}\Big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\Big| \,&\leq \inf_{\mathcal{F}(x,u(x),\underline{\mathbf{Y}})=0,|\underline{\mathbf{Y}}|\leq R}\omega_{R,x}\big(\big|\underline{\mathbf{X}}-\underline{\mathbf{Y}}\big|\big)\\ &=\omega_{R,x}\left(\inf_{\mathcal{F}(x,u(x),\underline{\mathbf{Y}})=0,|\underline{\mathbf{Y}}|\leq R}\big|\underline{\mathbf{X}}-\underline{\mathbf{Y}}\big|\right),\end{aligned}$$

as desired.

(5)$\Rightarrow$(4): We fix $R>0$ and $\Phi\in C_c\big(\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s\big)$ such that $\chi_{\mathbb{B}_{R/2}(0)}\leq\Phi\leq\chi_{\mathbb{B}_R(0)}$. For any $k\in\mathbb{N}$, we set

$$\Omega_k \,:=\, \left\{x\in\Omega\cap\mathbb{B}_k(0)\,:\, \sup_{\underline{\mathbf{X}}\in\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s}\Phi(\underline{\mathbf{X}})\big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\big|\leq k\right\}.$$

Then, $\Omega_k\subseteq\Omega_{k+1}$ and $|\Omega\setminus\Omega_k|\longrightarrow 0$ as $k\to\infty$. We also define

$$\Psi^k(x,\underline{\mathbf{X}}) \,:=\, \Phi(\underline{\mathbf{X}})\big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\big|\chi_{\Omega_k}(x), \quad k\in\mathbb{N}.$$

Since $\delta_{U^m}\overset{*}{\rightharpoonup}\vartheta$ as $m\to\infty$, we have

$$\int_\Omega \Psi^k\big(x,U^m(x)\big)\,dx \,\longrightarrow\, \int_\Omega\int_{\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s}\Psi^k\big(x,\underline{\mathbf{X}}\big)d[\vartheta(x)](\underline{\mathbf{X}})\,dx.$$

By assumption, we have $\mathcal{F}\big(\cdot,u,[U^m]^R\big)\longrightarrow 0$ a.e. on Ω as $m\to\infty$ and also the identity

$$\Phi\left([U^m]^R\right)\mathcal{F}\big(\cdot,u,[U^m]^R\big) \,=\, \Phi\left(U^m\right)\mathcal{F}\big(\cdot,u,U^m\big)$$

which is valid a.e. on Ω. From the above we infer that $\Psi^k(\cdot,U^m)\longrightarrow 0$ a.e. on Ω. Moreover, by using the bound $|\Phi^k|\leq k$ and that $|\Omega_k|<\infty$, the Dominated convergence theorem implies $\Psi^k(\cdot,U^m)\longrightarrow 0$ in $L^1(\Omega)$ as $m\to\infty$. Hence, for a.e. $x\in\Omega_k$ we have

$$0 \,=\, \int_{\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s}\Psi^k\big(x,\underline{\mathbf{X}}\big)\,d[\vartheta(x)](\underline{\mathbf{X}})$$

$$= \int_{\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s} \Phi(\underline{\mathbf{X}})\big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\big|\, d[\vartheta(x)](\underline{\mathbf{X}})$$

$$\geq \int_{\mathbb{B}_{R/2}(0)} \big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\big|\, d[\vartheta(x)](\underline{\mathbf{X}}).$$

The conclusion follows by letting $k \to \infty$ and then $R \to \infty$.

(4)⇒(3): We argue as in the case "(1)⇒(3)". Suppose that

$$\int_{\mathbb{R}^{Nn}\times\cdots\times\mathbb{R}^{Nn^p}_s} \big|\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}\big)\big|\, d[\vartheta(x)](\underline{\mathbf{X}}) = 0, \quad \text{a.e. } x \in \Omega,$$

whilst for some of these $x \in \Omega$ we have $\operatorname{supp}\big(\vartheta_*(x)\big) \not\subseteq \big\{\big|\mathcal{F}\big(x,u(x),\cdot\big)\big| = 0\big\}$. Then, there exists $\underline{\mathbf{X}}_0$ with $\mathcal{F}\big(x,u(x),\underline{\mathbf{X}}_0\big) = 0$ with $[\vartheta(x)]\big(\mathbb{B}_R(\underline{\mathbf{X}}_0)\big) > 0$ for $R > 0$. Thus, there exist $c_0, R_0 > 0$ such that $\big|\mathcal{F}\big(x,u(x),\cdot\big)\big| \geq c_0 > 0$ on $\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)$. Hence, $c_0\,[\vartheta(x)]\big(\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)\big) \leq 0$ and this contradiction establishes the desired inclusion.

(3)⇒(6): We fix $R > 0$ and define $\Psi : \Omega \times \overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s \longrightarrow [0,\infty)$ by

$$\Psi(x,\underline{\mathbf{X}}) := \chi_{\overline{\mathbb{B}_R(0)}}(\underline{\mathbf{X}})\,\mathrm{dist}\left(\underline{\mathbf{X}}\ ,\, \mathbb{B}_R(0) \cap \Big\{\big|\mathcal{F}\big(x,u(x),\cdot\big)\big| = 0\Big\}\right).$$

Then, Ψ is measurable in x for all $\underline{\mathbf{X}}$ (this is a consequence of Aumann's theorem, see e.g. [FL]), upper semicontinuous in $\underline{\mathbf{X}}$ for a.e. x and also bounded. Hence, since $\delta_{U^m} \overset{*}{\longrightarrow} \vartheta$ as $m \to \infty$, by Remark 2iii), we have

$$\limsup_{m\to\infty} \int_\Omega \Psi\big(x,U^m(x)\big)\,dx \leq \int_\Omega \int_{\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s} \Psi\big(x,\underline{\mathbf{X}}\big)\, d[\vartheta(x)](\underline{\mathbf{X}})\,dx$$

$$= \int_\Omega \int_{\overline{\mathbb{B}_R(0)}} \mathrm{dist}\left(\underline{\mathbf{X}}\ ,\ \Big\{\big|\mathcal{F}\big(x,u(x),\cdot\big)\big| = 0\Big\} \cap \mathbb{B}_R(0)\right)\, d[\vartheta(x)](\underline{\mathbf{X}})\,dx.$$

By assumption and by Definition 19, we have the inclusions $\operatorname{supp}\big(\vartheta(x)\big) \cap \overline{\mathbb{B}_R(0)} \subseteq \big\{\big|\mathcal{F}\big(x,u(x),\cdot\big)\big| = 0\big\} \cap \overline{\mathbb{B}_R(0)} \subseteq \{\Psi(x,\cdot) = 0\}$, for a.e. $x \in \Omega$. Hence, the last integral above vanishes and we obtain that $\Psi(\cdot,U^m) \longrightarrow 0$ in $L^1(\Omega)$ as $m \to \infty$. Further, in view of Definition 20, we have the identity

$$\Psi\big(x,U^m(x)\big) = \mathrm{dist}\left([U^m]^R(x)\ ,\ \Big\{\big|\mathcal{F}\big(x,u(x),\cdot\big)\big| = 0\Big\} \cap \mathbb{B}_R(0)\right),$$

which holds for a.e. $x \in \Omega$ and by using it we obtain that

$$\int_\Omega \mathrm{dist}\left([U^m]^R(x)\ ,\ \Big\{\big|\mathcal{F}\big(x,u(x),\cdot\big)\big| = 0\Big\} \cap \mathbb{B}_R(0)\right)\,dx \longrightarrow 0,$$

as $m \to \infty$. The conclusion follows by passing to a subsequence. □

2.4. **Nonlinear nature of diffuse derivatives.** (This subsection is not needed for remainder of the paper.) In the context of classical PDE approaches (classical, strong, weak, distributional solutions), it is standard that the generalised derivative is a linear operator. However, this is generally *false for diffuse derivatives. Our approach is genuinely nonlinear and not a variant of classical developments.* Below we give a condition which guarantees that the sum of two $\mathcal{D}$-solutions to a certain linear equation is a $\mathcal{D}$-solution itself; this happens if at least one of the solutions is regular enough. Hence, the notions themselves are nonlinear even when we apply them to linear PDE. In order to proceed we need some notation.

Definition 23. Let $\mathbb{W}$ be a finite dimensional metric vector space isometrically and densely contained into a compactification $\mathbb{K}$ of $\mathbb{W}$. Let also $T_a : \mathbb{W} \to \mathbb{W}$ denote the translation operation given by $T_a b := b - a$. Given a probability $\vartheta \in \mathscr{P}(\mathbb{K})$, we define $\vartheta \circ T_a \in \mathscr{P}(\mathbb{K})$ by duality via the formula

$$\langle \vartheta \circ T_a, \Phi \rangle := \int_{\mathbb{W}} \Phi(a + X)\, d\vartheta(X) + \int_{\mathbb{K}\setminus\mathbb{W}} \Phi(X)\, d\vartheta(X), \qquad \Phi \in C(\mathbb{K}).$$

Definition 23 requires translation of the part contained in the vector space while points "at infinity" are left intact.

Proposition 24 (Diffuse derivatives & $\mathcal{D}$-solutions vs linearity). *Let $u, v : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be measurable maps.*
a) If v is differentiable in measure on Ω with derivative $\mathcal{L}\mathrm{D}v$, (Def. 11), then $\mathcal{D}(u + v) = \mathcal{D}u \circ T_{\mathcal{L}\mathrm{D}v}$, a.e. on Ω. Here the diffuse Jets on both sides arise from the same infinitesimal sequence.
b) Consider the measurable maps $\boldsymbol{A}^q : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{Nn^q}_s \otimes \mathbb{R}^M$ and $f, g : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^M$ where $q = 1, ..., p$ and the linear systems $\boldsymbol{A} :: \mathrm{D}^{[p]}u = f$ and $\boldsymbol{A} :: \mathrm{D}^{[p]}v = g$. Here $\boldsymbol{A} = (\boldsymbol{A}^1, ..., \boldsymbol{A}^p)$. If u, v are $\mathcal{D}$-solutions, then $u + v$ is a $\mathcal{D}$-solution $\boldsymbol{A}(x) :: \mathrm{D}^{[p]}(u + v) = f + g$, when v is p-times differentiable in measure on Ω.

The notation "::" above is a convenient abbreviation of the multiple contraction

$$\sum_{\alpha_1, i_1} \mathbf{A}^1_{\mu;\alpha_1, i_1} \mathrm{D}_{i_1} u_{\alpha_1} + ... + \sum_{\alpha_p, i^p_1 ... i^p_p} \mathbf{A}^p_{\mu;\alpha_p, i^1_1, ..., i^p_p} \mathrm{D}^p_{i^p_1 ... i^p_p} u_{\alpha_p}.$$

The proof is based on the next general lemma.

Lemma 25. *Let $E \subseteq \mathbb{R}^n$ be measurable and $\mathbb{W}$ a finite dimensional metric vector space isometrically contained into a compactification $\mathbb{K}$ of $\mathbb{W}$. If $U^m, V^m : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{W}$ are measurable and such that $\delta_{U^m} \overset{*}{\rightharpoonup} \vartheta$ in $\mathscr{Y}(E, \mathbb{K})$ and $V^m \longrightarrow V$ a.e. on E, as $m \to \infty$. Then, we have $\delta_{U^m + V^m} \overset{*}{\rightharpoonup} \vartheta \circ T_V$ in $\mathscr{Y}(E, \mathbb{K})$ as $m \to \infty$.*

Proof of Lemma 25. Fix $\phi \in L^1(E)$, $\Phi \in C(\mathbb{K})$ and $\varepsilon > 0$. Since Φ is uniformly continuous, there is a *bounded* increasing modulus of continuity $\omega \in (C \cap L^\infty)[0, \infty)$ such that $|\Phi(X) - \Phi(Y)| \leq \omega(|X - Y|)$ for $X, Y \in \mathbb{K}$. Since $V^m \longrightarrow V$ a.e. on E, we obtain $V^m \longrightarrow V$ μ-a.e. on E where μ is the finite measure $\mu(A) := \|\phi\|_{L^1(A \cap E)}$, $A \subseteq \mathbb{R}^n$. It follows that $V^m \longrightarrow V$ in μ-measure as well. Hence,

$$\begin{aligned} \left| \int_E \phi \Big[\Phi(U^m + V^m) - \Phi(U^m + V) \Big] \right| &\leq \int_E |\phi| \omega\big(|V^m - V|\big) \\ &\leq \|\omega\|_{C(0,\infty)} \mu\big(\{|V^m - V| > \varepsilon\}\big) \; + \; \omega(\varepsilon)\mu(E). \end{aligned}$$

By letting $m \to \infty$ and then $\varepsilon \to 0$, the density of the linear span of products $\phi(x)\Phi(X)$ in $L^1\big(E, C(\mathbb{K})\big)$ and the definition of $\vartheta \circ T_V$ allow us to conclude. ☐

Proof of Proposition 24. If suffices to establish b) and only for $p = 1$. By assumption, we have that $\mathbf{A}^1(x) : \mathcal{L}\mathrm{D}v(x) = g(x)$ and also that for any $\Phi \in C^0_c(\mathbb{R}^{Nn})$,

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi(X) \Big[\mathbf{A}^1(x) : X - f(x) \Big] \, d[\mathcal{D}u(x)](X) \, = \, 0,$$

both being valid for a.e. on $x \in \Omega$. Here $\mathcal{D}u$ is any diffuse gradient. We fix any point x as above and replace Φ by $\Phi\big(\, \cdot + \mathcal{L}\mathrm{D}v(x)\big)$. Then, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi\big(X + \mathcal{L}\mathrm{D}v(x)\big) \Big[\mathbf{A}^1(x) : \big(X + \mathcal{L}\mathrm{D}v(x)\big) - f(x) - g(x) \Big] \, d[\mathcal{D}u(x)](X) \, = \, 0.$$

By the definition of $\mathcal{D}u \circ T_{\mathcal{L}\mathrm{D}v}$, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi(Y)\Big[\mathbf{A}^1(x) : Y - (f+g)(x)\Big]\, d\big[\mathcal{D}u(x) \circ T_{\mathcal{L}\mathrm{D}v(x)}\big](Y) \,=\, 0.$$

By utilising part a), the conclusion ensues. □

Example 26 (Nonlinearity of diffuse derivatives)**.** *Let $K \subseteq \mathbb{R}$ be a compact nowhere dense set of positive measure (e.g. $K = [0,1]\setminus(\cup_1^\infty(r_j - 3^{-j}, r_j + 3^{-j}))$ where $(r_j)_1^\infty$ is an enumeration of $\mathbb{Q}\cap[0,1]$). Then, for $u := \chi_K$ we have that $|\mathrm{D}^{1,h}u(x)| \to \infty$ as $h \to 0$ for $x \in K$ and $u' = 0$ on $\mathbb{R}\setminus K$. Hence, by Lemma 3 along any $h_m \to 0$ we have $\mathcal{D}u(x) = \delta_{\{\infty\}}$ for a.e. $x \in K$. However, for $v := -u$, we have $\mathcal{D}(u+v) = \delta_{\{0\}}$ a.e. on $\mathbb{R}$, while $\mathcal{D}u = \mathcal{D}v = \delta_{\{\infty\}}$ a.e. on K.*

Comparison with distributional solutions. Let us conclude this section with an *informal* discussion of the relation between distributional and $\mathcal{D}$-solutions. Let us first compare distributional to diffuse derivatives. First recall that the distributional gradient $\mathrm{D}u$ of $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ can be weakly* approximated by difference quotients: for any $\phi \in C^\infty_c(\mathbb{R}^n)$,

$$\langle \phi, \mathrm{D}u\rangle \,=\, \lim_{m\to\infty}\int_{\mathbb{R}^n} \phi\,\mathrm{D}^{1,h_m}u \,=\, \lim_{m\to\infty}\int_{\mathbb{R}^n}\phi\left(\int_{\overline{\mathbb{R}}^n} X\, d\big[\delta_{\mathrm{D}^{1,h_m}u}\big](X)\right).$$

If "bar_*" denotes the barycentre of the restriction of a measure on $\overline{\mathbb{R}}^n$ off $\{\infty\}$, the above can be rewritten as

$$\mathrm{bar}_*\left(\delta_{\mathrm{D}^{1,h_m}u}\right) \overset{*}{\rightharpoonup} \mathrm{D}u, \qquad \text{as } m\to\infty, \tag{2.13}$$

in the distributions $\mathscr{D}'\big(\mathbb{R}^n, \mathbb{R}^n\big)$. Along perhaps a further subsequence, we have

$$\delta_{\mathrm{D}^{1,h_m}u} \overset{*}{\rightharpoonup} \mathcal{D}u, \quad \text{in } \mathscr{Y}\big(\mathbb{R}^n, \overline{\mathbb{R}}^n\big), \qquad \text{as } m\to\infty. \tag{2.14}$$

By juxtaposing (2.13) with (2.14), our interpretation is that *the barycentre of the diffuse derivative (off $\{\infty\}$) is unique and equals the distributional derivative*: $\mathrm{bar}_*(\mathcal{D}u) = \mathrm{D}u$. Regarding the notions to solution, apparently $\mathcal{D}$-solutions are a more general theory than distributional solutions in the sense that they apply to more general PDEs and under weaker requirements. However, the two theories are not immediately comparable on their common domain of L^1_{loc} solutions of linear systems with smooth coefficients. On the one hand, Proposition 24 and Example 26 point out a property which is not generally true for diffuse derivatives but is always true for distributional derivatives. However, $\mathcal{D}$-solutions completely avoid the impossibility to multiply distributions. For example, if $\mathbf{A} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\mathbf{A}\cdot\mathrm{D}^{1,h_m}u \longrightarrow \begin{cases} \overset{*}{\not\rightharpoonup}\ \mathbf{A}\cdot\mathrm{D}u, & \text{in } \mathscr{D}'\big(\mathbb{R}^n,\mathbb{R}^n\big), \quad \text{[not well defined!]} \\ \overset{*}{\rightharpoonup}\ \mathbf{A}\cdot\mathcal{D}u, & \text{in } \mathscr{Y}\big(\mathbb{R}^n,\overline{\mathbb{R}}^n\big). \quad \text{[well defined!]} \end{cases}$$

Hence, although the product $\mathbf{A}(x)\cdot\mathrm{D}u(x) = \mathbf{A}\cdot\mathrm{bar}_*\left(\mathcal{D}u(x)\right)$ is ill-defined, diffuse derivatives make sense because they can be multiplied with measurable functions.

3. $\mathcal{D}$-solutions of the ∞-Laplacian and tangent systems

In this section we establish our first main result concerning $\mathcal{D}$-solutions. We treat the Dirichlet problem for the ∞-Laplace system (1.11) which is the fundamental equation of vectorial Calculus of Variations in the space L^∞ and arises from the functional (1.10).

Theorem 27 (Existence of ∞-Harmonic maps)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with $|\Omega| < \infty$, $n \geq 1$. For any $g \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, the Dirichlet problem*

$$\left\{ \begin{array}{rl} \Delta_\infty u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{array} \right. \tag{3.1}$$

has a $\mathcal{D}$-solution $u \in W^{1,\infty}_g(\Omega, \mathbb{R}^n)$ with respect to the standard frames *(Definition 13). In particular, for any $\mathcal{D}^2 u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{nn^2}_s\big)$ and $\Phi \in C_c\big(\mathbb{R}^{nn^2}_s\big)$, we have*

$$\int_{\overline{\mathbb{R}}^{nn^2}_s} \Phi(\boldsymbol{X}) \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes \mathrm{I} \Big) : \boldsymbol{X} d[\mathcal{D}^2 u](\boldsymbol{X}) = 0, \quad \text{a.e. on } \Omega.$$

Remark 28. Unfortunately, as we proved in [K2], in general it is impossible to obtain uniqueness of solutions to the equations of vectorial L^∞ problems **even within the class of smooth solutions**. Clearly, uniqueness in the vectorial case is not an issue of defining a "proper" notion of generalised solution, since even classical solutions in general are non-unique. Instead, *extra conditions* must be determined to select a "good" solution. On the other hand, uniqueness is standard in the scalar case (a celebrated theorem of Jensen, see e.g. [C], [K8]). Such phenomena are *not exclusive to the ∞-Laplacian*: for instance, the Dirichlet problem for the minimal surface system may have either non-existence or non-uniqueness in codimension greater than one (see [OL]), while for the minimal surface equation it is well posed.

In addition, the next corollary will also be established in the course of its proof.

Corollary 29 (Multiplicity & geometric properties of $\mathcal{D}$-solutions)**.** *In the setting of Theorem 27, if $n \geq 2$ then (3.1) actually has an infinite set of solutions. Moreover, for any $M > \big\|(\mathrm{D}g^\top \mathrm{D}g)^{1/2}\big\|_{L^\infty(\Omega)}$ there is a $\mathcal{D}$-solution $u = u(M)$ satisfying*

$$|\mathrm{D}u|^2 = nM^2, \quad \big|\det(\mathrm{D}u)\big| = M^n, \quad \text{a.e. on } \Omega. \tag{3.2}$$

Hence, the $\mathcal{D}$-solutions we construct are "critical points" with pre-assigned energy level, having also the geometric property of being full-rank solutions of the vectorial Eikonal equation.

3.1. **The idea of the proof.** Suppose that $u \in C^2(\Omega, \mathbb{R}^n)$ solves (1.11) and recall that $[\![\mathrm{D}u]\!]^\perp = \mathrm{Proj}_{(R(\mathrm{D}u))^\perp}$. By contracting derivatives, we rewrite the system as

$$\mathrm{D}u \, \mathrm{D}\Big(\frac{1}{2}|\mathrm{D}u|^2\Big) + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \Delta u = 0. \tag{3.3}$$

It follows that *smooth* solutions of the 1st order differential inclusion $\mathrm{D}u(x) \in \mathcal{K}_c$, $x \in \Omega$, where $c > 0$ is a parameter and

$$\mathcal{K}_c := \Big\{ X \in \mathbb{R}^{nn} : |X| = c, |\det(X)| > 0 \Big\},$$

actually are ∞-Harmonic mappings: indeed, if $\mathrm{D}u(\Omega) \subseteq \mathcal{K}_c$, then $|\mathrm{D}u|^2 \equiv c^2$ and $\det(\mathrm{D}u) \neq 0$ on Ω. In view of (3.3) we have that the system is satisfied because $|\mathrm{D}u| \equiv$ const and u is a submersion which gives $[\![\mathrm{D}u]\!]^\perp \equiv 0$. Hence, if we prove existence of a solution to the inclusion with the desired boundary data, this yields a solution to (3.1). However, the preceding arguments make sense only for classical or strong solutions. The starting point of the proof of Theorem 27 is to use the Dacorogna-Marcellini Baire Category method [DM] in order to construct Lipschitz solutions of the inclusion with the given boundary data. Then, by using the machinery of $\mathcal{D}$-solutions we make the previous ideas rigorous for Lipschitz maps, which is the natural regularity class. Note that our methodology is *not variational* and does not directly involve the functional (1.10).

3.2. **Proof of the main result.** The case $n = 1$ is trivial (see [K1]), so we may assume $n \geq 2$. A central ingredient in the proof of Theorem 27 is a result of independent interest, Theorem 30 below, which provides a method of constructing nonsmooth $\mathcal{D}$-solutions to nonlinear systems by "differentiating an equation".

Theorem 30 (Differentiating equations in the $\mathcal{D}$-sense)**.** *Let $\mathcal{F}$ be a C^1 map as in* (1.1) *and consider the p-th order system*

$$\mathcal{F}\Big(x, u(x), \mathrm{D}^{[p]}u(x)\Big) \,=\, 0, \quad x \in \Omega.$$

If $u \in W^{p,\infty}_{loc}(\Omega, \mathbb{R}^N)$ is a strong a.e. solution to the system, then u is a $\mathcal{D}$-solution to the "tangent system" on Ω with respect to the usual frames (Definition 14):

$$\mathcal{F}_x\big(\cdot, u, \mathrm{D}^{[p]}u\big) \,+\, \mathcal{F}_\eta\big(\cdot, u, \mathrm{D}^{[p]}u\big)\mathrm{D}u \,+\, \mathcal{F}_{\underline{\mathbf{X}}}\big(\cdot, u, \mathrm{D}^{[p]}u\big) :: \mathrm{D}^{[p+1]}u \;=\; 0.$$

For the notation "::" see Proposition 24. Theorem 30 is actually true for solutions which are merely $W^{p,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ or just p-times differentiable in measure (Definition 11), but then we have to assume certain growth bounds on the derivatives of $\mathcal{F}$. We invite the reader to note the *simplicity* with which we pass to limits in the proof below within the framework of $\mathcal{D}$-solutions.

Proof of Theorem 30. It suffices to prove only the case of $p = 1$ and with no explicit u dependence, the general case following analogously. Hence we suppose that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ solves $\mathcal{F}\big(x, \mathrm{D}u(x)\big) = 0$ for a.e. $x \in \Omega$ and we aim at showing

$$\mathcal{F}_x\big(x, \mathrm{D}u(x)\big) \,+\, \mathcal{F}_X\big(x, \mathrm{D}u(x)\big) : \mathrm{D}^2u(x) \,=\, 0, \quad x \in \Omega,$$

in the $\mathcal{D}$-sense (Definition 13). For a.e. point $x \in \Omega$ such that $\mathcal{F}\big(x, \mathrm{D}u(x)\big) = 0$ and $h \neq 0$ small enough, Taylor's theorem implies for each i the identity

$$\begin{aligned}
&\mathcal{F}_{x_i}\big(x, \mathrm{D}u(x)\big) \,+\, \mathcal{F}_X\big(x, \mathrm{D}u(x)\big) : \mathrm{D}^{1,h}_{e^i}\mathrm{D}u(x) \\
&\quad= \,-\,\mathrm{D}^{1,h}_{e^i}\mathrm{D}u(x) : \int_0^1 \Big\{\mathcal{F}_X\Big(x + \lambda h e^i, \mathrm{D}u(x) + \lambda\big[\mathrm{D}u(x+he^i) - \mathrm{D}u(x)\big]\Big) \\
&\qquad\qquad\qquad - \mathcal{F}_X\big(x, \mathrm{D}u(x)\big)\Big\}\, d\lambda \\
&\qquad - \int_0^1 \Big\{\mathcal{F}_{x_i}\Big(x + \lambda h e^i, \mathrm{D}u(x) + \lambda\big[\mathrm{D}u(x+he^i) - \mathrm{D}u(x)\big]\Big) \\
&\qquad\qquad - \mathcal{F}_{x_i}\big(x, \mathrm{D}u(x)\big)\Big\}\, d\lambda.
\end{aligned} \tag{3.4}$$

We fix an infinitesimal sequence $(h_m)_{m=1}^\infty \subseteq \mathbb{R} \setminus \{0\}$ and observe that by the weak* compactness of $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$, along a subsequence $h_{m_k} \to 0$ we have

$$\delta_{\mathrm{D}^{1,h_{m_k}}\mathrm{D}u} \overset{*}{\rightharpoonup} \mathcal{D}^2 u \;\text{ in }\; \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \quad \text{as } k \to \infty.$$

We now invoke that $|\mathrm{D}u| \in L^\infty(\Omega)$ to infer that since $\mathrm{D}u(\cdot + he^i) \longrightarrow \mathrm{D}u$ in $L^1(\Omega, \mathbb{R}^{Nn})$ as $h \to 0$, there is a further subsequence denoted again by $(h_{m_k})_{k=1}^\infty$ such that for a.e. $x \in \Omega$ we have $\mathrm{D}u(x + h_{m_k}e^i) \longrightarrow \mathrm{D}u(x)$ as $k \to \infty$. Next, we set

$$G_i^\infty(x, \mathbf{X}) \;:=\; \mathcal{F}_{x_i}\big(x, \mathrm{D}u(x)\big) \,+\, \sum_{\beta,j} \mathcal{F}_{X_{\beta j}}\big(x, \mathrm{D}u(x)\big)\, \mathbf{X}_{\beta j i}$$

and for $m \in \mathbb{N}$

$$
\begin{aligned}
G^m_i(x,\mathbf{X}) := \ & \mathcal{F}_{x_i}\big(x,\mathrm{D}u(x)\big) + \sum_{\beta,j}\mathcal{F}_{X_{\beta j}}\big(x,\mathrm{D}u(x)\big)\,\mathbf{X}_{\beta j i} \\
& + \sum_{\beta,j}\mathbf{X}_{\beta j i}\int_0^1 \Big\{\mathcal{F}_{X_{\beta j}}\Big(x+\lambda h_m e^i, \mathrm{D}u(x)+\lambda\big[\mathrm{D}u(x+h_m e^i)-\mathrm{D}u(x)\big]\Big) \\
& \qquad\qquad - \mathcal{F}_{X_{\beta j}}\big(x,\mathrm{D}u(x)\big)\Big\}\,d\lambda \\
& + \int_0^1 \Big\{\mathcal{F}_{x_i}\Big(x+\lambda h_m e^i, \mathrm{D}u(x)+\lambda\big[\mathrm{D}u(x+h_m e^i)-\mathrm{D}u(x)\big]\Big) \\
& \qquad - \mathcal{F}_{x_i}\big(x,\mathrm{D}u(x)\big)\Big\}\,d\lambda.
\end{aligned}
$$

By the C^1 regularity of $\mathcal{F}$ and that $\mathrm{D}u(\cdot+h_{m_k}e^i) \longrightarrow \mathrm{D}u$ a.e. on Ω as $k\to\infty$ (together with the Dominated convergence theorem and that $\mathrm{D}u \in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^{Nn})$), for a.e. $x\in\Omega$ we obtain that $G^{m_k}(x,\cdot) \longrightarrow G^\infty(x,\cdot)$ in $C\big(\mathbb{R}^{Nn^2}_s,\mathbb{R}^M\big)$, as $k\to\infty$. Moreover, in view of the definition of G^m, the identity (3.4) gives

$$
G^m\Big(x,\mathrm{D}^{1,h_m}\mathrm{D}u(x)\Big) = 0 \quad \text{a.e. on } \Omega,\ \ m\in\mathbb{N}.
$$

Hence, for any $\Phi \in C_c\big(\mathbb{R}^{Nn^2}_s\big)$ we have

$$
\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X})\,G^{m_k}(x,\mathbf{X})\,d\big[\delta_{\mathrm{D}^{1,h_{m_k}}\mathrm{D}u(x)}\big](\mathbf{X}) = 0 \quad \text{a.e. } x\in\Omega,
$$

for $k\in\mathbb{N}$. The convergence Lemma 16 now implies

$$
\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X})\,G^{\infty}(x,\mathbf{X})\,d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) = 0, \quad \text{a.e. } x\in\Omega,
$$

for any $\Phi \in C_c\big(\mathbb{R}^{Nn^2}_s\big)$ and any diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$. Hence, u is a $\mathcal{D}$-solution of $G^\infty\big(\cdot,\mathrm{D}^2u\big)=0$ and by the definition of G^∞, the result ensues. ☐

Proof of Theorem 27 (and Corollary 29). Assume we are given $\Omega \subseteq \mathbb{R}^n$ with finite measure and $g \in W^{1,\infty}(\Omega,\mathbb{R}^n)$. We begin with the next:

Claim 31. *If $M > \|(\mathrm{D}g^\top\mathrm{D}g)^{1/2}\|_{L^\infty(\Omega)}$, there exists $u \in W^{1,\infty}_g(\Omega,\mathbb{R}^n)$ such that $|\mathrm{D}u|^2 = nM^2$ and also $|\det(\mathrm{D}u)| = M^n$, both holding a.e. on Ω.*

Proof of Claim 31. Given a map $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^n$ in $W^{1,\infty}_g(\Omega,\mathbb{R}^n)$, let $\lambda_i(\mathrm{D}u)$ denote the ith singular value, that is the ith eigenvalue of $(\mathrm{D}u^\top\mathrm{D}u)^{1/2}$:

$$
\sigma\big((\mathrm{D}u^\top\mathrm{D}u)^{1/2}\big) = \big\{\lambda_1(\mathrm{D}u),\dots,\lambda_n(\mathrm{D}u)\big\}, \quad \lambda_i \le \lambda_{i+1}.
$$

Fix an $M>0$ as in statement and consider the Dirichlet problem:

$$
\left\{
\begin{array}{rll}
\lambda_i(\mathrm{D}v) = 1, & \text{a.e. in } \Omega, & i=1,...,n,\\
v = g/M, & \text{on } \partial\Omega.
\end{array}
\right. \tag{3.5}
$$

Then, we have the estimate

$$
\big\|\lambda_n(\mathrm{D}g)\big\|_{L^\infty(\Omega)} = \Big\|\max_{|e|=1}(\mathrm{D}g^\top\mathrm{D}g)^{1/2} : e\otimes e\Big\|_{L^\infty(\Omega)} \le \big\|(\mathrm{D}g^\top\mathrm{D}g)^{1/2}\big\|_{L^\infty(\Omega)}. \tag{3.6}
$$

In view of the results of [DM], the estimate (3.6) implies that the required compatibility condition is satisfied in regard to the problem (3.5). Hence there is a strong solution v to (3.5) such that $v - (g/M) \in W^{1,\infty}_0(\Omega,\mathbb{R}^n)$ for the given M and the boundary data g. Finally, since $\lambda_i(\mathrm{D}v) = 1$ a.e. on Ω, by setting $u := Mv$ we have

$$|\mathrm{D}u|^2 = M^2|\mathrm{D}v|^2 = M^2 \sum_{i=1\dots n} \lambda_i(\mathrm{D}v)^2 = nM^2, \quad \text{a.e. on } \Omega,$$

$$|\det(\mathrm{D}u)| = M^n|\det(\mathrm{D}v)| = M^n \prod_{i=1\dots n} \lambda_i(\mathrm{D}v) = M^n, \quad \text{a.e. on } \Omega,$$

and in addition, $u - g \in W^{1,\infty}_0(\Omega,\mathbb{R}^n)$. The proof of the claim is complete. $\square$

Now we may complete the proof. For the given boundary condition g, we fix an $M > 0$ as in the claim and consider one of its solutions $u \in W^{1,\infty}_g(\Omega,\mathbb{R}^N)$ which satisfies $|\mathrm{D}u|^2 - nM^2 = 0$, a.e. on Ω. We set $\mathcal{F}(X) := |X|^2 - nM^2$ for $X \in \mathbb{R}^{Nn}$ and apply Theorem 30 to infer that $u \in W^{1,\infty}_g(\Omega,\mathbb{R}^N)$ is a $\mathcal{D}$-solution to the tangent system $\mathcal{F}_X(\mathrm{D}u) : \mathrm{D}^2u = 0$; that is, for all i we have

$$\sum_{\beta,j} \mathrm{D}_j u_\beta(x)\, \mathrm{D}^2_{ij} u_\beta(x) = 0, \quad x \in \Omega, \text{ in the } \mathcal{D}\text{-sense.}$$

This means that when $\delta_{\mathrm{D}^{1,h_m}\mathrm{D}u} \overset{*}{\longrightarrow} \mathcal{D}^2u$ in $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{nn^2}_s\big)$ as $m \to \infty$, we have

$$\int_{\overline{\mathbb{R}}^{nn^2}_s} \sum_{\beta,j} \mathrm{D}_j u_\beta(x)\, \Phi(\mathbf{X})\, \mathbf{X}_{\beta ij}\, d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega,$$

for any fixed $\Phi \in C_c\big(\mathbb{R}^{nn^2}_s\big)$. We multiply the above equation by $\mathrm{D}_i u_\alpha(x)$ and sum with respect to i to obtain

$$\int_{\overline{\mathbb{R}}^{nn^2}_s} \sum_{\beta,j,i} \Phi(\mathbf{X})\, \mathrm{D}_i u_\alpha(x)\, \mathrm{D}_j u_\beta(x)\, \mathbf{X}_{\beta ij}\, d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega.$$

Finally, by Claim 31 we have $\det(\mathrm{D}u) \neq 0$ a.e. on Ω and as a result $\mathrm{D}u(x)$ has rank equal to n in $\mathbb{R}^{nn}$. Hence, the projection $[\![\mathrm{D}u(x)]\!]^\perp = \mathrm{Proj}_{R(\mathrm{D}u(x))^\perp}$ vanishes for a.e. $x \in \Omega$ and consequently we obtain

$$\int_{\overline{\mathbb{R}}^{nn^2}_s} \sum_{\beta,i} \Phi(\mathbf{X})\, \big|\mathrm{D}u(x)\big|^2 [\![\mathrm{D}u(x)]\!]^\perp_{\alpha\beta}\, \mathbf{X}_{\beta ii}\, d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C_c\big(\mathbb{R}^{nn^2}_s\big)$ and any diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{nn^2}_s\big)$. The last two equations imply that u is ∞-Harmonic in the $\mathcal{D}$-sense and the theorem follows. $\square$

4. $\mathcal{D}$-solutions of fully nonlinear degenerate elliptic systems

Fix $n, N \geq 1$, let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathcal{F} : \Omega \times \mathbb{R}^{Nn^2}_s \longrightarrow \mathbb{R}^N$ a Carathéodory map. In this section we establish our second main result, namely the *existence of a unique $\mathcal{D}$-solution* $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to the Dirichlet problem

$$\left\{ \begin{aligned} \mathcal{F}(\cdot, \mathrm{D}^2u) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \right. \tag{4.1}$$

together with a *partial regularity assertion of new type involving differentiability along rank-one directions* (instead of the usual partial regularity on a subset of the domain). We will assume that $f \in L^2(\Omega,\mathbb{R}^N)$ and $\mathcal{F}$ satisfies a degenerate ellipticity condition which in general does not guarantee that solutions are even *once* weakly differentiable. This extends previous results of the author in the class of strong solution for (4.1) ([K9, K11]) under a stronger ellipticity condition.

4.1. **The idea of the proof.** The solvability of (4.1) in the class of $\mathcal{D}$-solutions is based on the study of the linearised system with constant coefficients

$$\left\{ \begin{aligned} \mathbf{A} : \mathrm{D}^2 u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \right. \tag{4.2}$$

when $\mathbf{A}$ is a (perhaps degenerate) symmetric 4th order tensor and on a perturbation device provided by our ellipticity assumption for $\mathcal{F}$. The latter allows to solve (4.1) by solving (4.2) and using a fixed point argument in the guises of a classical theorem of Campanato ([C3]). In order to solve (4.2) in the $\mathcal{D}$-sense (and not just weakly) we impose a structural condition on $\mathbf{A}$ which allows to construct $\mathcal{D}$-solutions as maps having weakly twice differentiable projections along certain rank-one lines of $\mathbb{R}^{Nn}$. These are the "directions of ellipticity" of (4.2). We formalise this idea by introducing a "fibre" extension of the classical Sobolev spaces which consist of maps possessing only certain partial regularity along rank-one lines. Our fibre space counterparts are adapted to the degenerate nature of the problem and support feeble yet sufficient versions of weak compactness, trace operators and Poincaré inequalities established in [K10]. The proof is completed by characterising the fixed point as the unique $\mathcal{D}$-solution of the problem (4.1) in the fibre space.

4.2. **Fibre spaces, degenerate ellipticity and the main result.** Before stating our existence result we need some preparation. We will use the notation $\mathbf{A} \in \mathbb{R}^{Nn\times Nn}_s$ to symbolise symmetric linear maps $\mathbf{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^{Nn}$, i.e. 4th order tensors satisfying $\mathbf{A}_{\alpha i\beta j} = \mathbf{A}_{\beta j\alpha i}$ for all $\alpha,\beta = 1,...,N$, $i,j = 1,...,n$. The notation $N\big(\mathbf{A} : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}\big)$ and $N\big(\mathbf{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^{N}\big)$ will be used to symbolise the nullspaces of $\mathbf{A}$ when it acts as a linear map with domain and range like those indicated in the brackets, i.e. $Q \longmapsto \sum_{\alpha,\beta,i,j}\big(\mathbf{A}_{\alpha i\beta j}Q_{\beta j}\big)e^\alpha \otimes e^i$ and $\mathbf{X} \longmapsto \sum_{\alpha,\beta,i,j}\big(\mathbf{A}_{\alpha i\beta j}\mathbf{X}_{\beta ij}\big)e^\alpha$. We will also use similar notation for the respective ranges with "R" instead of "N". If $\mathbf{A}$ is rank-one non-negative, i.e. if the respective quadratic form is rank-one convex $\mathbf{A} : \eta\otimes a\otimes\eta\otimes a = \sum_{\alpha,\beta,i,j}\mathbf{A}_{\alpha i\beta j}\eta_\alpha a_i \eta_\beta a_j \geq 0$ $(\eta \in \mathbb{R}^N,\ a \in \mathbb{R}^n)$, we define

$$\begin{aligned} \Pi &:= \ R\Big(\mathbf{A} : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}\Big) && \subseteq \ \mathbb{R}^{Nn}, \\ \Sigma &:= \ \mathrm{span}\Big[\Big\{\eta \ \Big|\ \eta\otimes a \in \Pi\Big\}\Big] && \subseteq \ \mathbb{R}^{N}, \\ \Xi &:= \ \mathrm{span}\Big[\Big\{\eta\otimes(a\vee b)\ \Big|\ \eta\otimes a,\ \eta\otimes b \in \Pi\Big\}\Big] && \subseteq \ \mathbb{R}^{Nn^2}_s, \\ \nu &:= \min_{|\eta|=|a|=1,\,\eta\otimes a\in\Pi}\Big\{\mathbf{A} : \eta\otimes a\otimes\eta\otimes a\Big\} && > \ 0. \end{aligned} \tag{4.3}$$

We will call ν the *ellipticity constant of* $\boldsymbol{A}$, bearing in mind that strictly speaking $\mathbf{A}$ may not be elliptic and the respective infimum over $\mathbb{R}^{Nn}$ may vanish. We also recall that we will use the same letters Π, Ξ, Σ to symbolise the subspaces as well as the orthogonal projections on them.

The fibre Sobolev spaces. Given $\mathbf{A} \in \mathbb{R}^{Nn\times Nn}_s$ rank-one non-negative, let Σ, Π, Ξ be as in (4.3) and suppose Π *is spanned by rank-one directions.* A sufficient condition for this to happen is when $\mathbf{A}$ is decomposable (Definition 33 that follows). For simplicity, we treat only the L^2 second order case needed in this paper. We begin by identifying $W^{2,2}(\Omega,\mathbb{R}^N)$ with its isometric image $\tilde{W}^{2,2}(\Omega,\mathbb{R}^N)$ into a product of

L^2 spaces $\tilde W^{2,2}(\Omega,\mathbb{R}^N) \underset{\to}{\subsetneq} L^2\big(\Omega,\mathbb{R}^N\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s\big)$ via the map $u\longmapsto(u,\mathrm{D}u,\mathrm{D}^2u)$. We define the **fibre Sobolev space** $\mathscr{W}^{2,2}(\Omega,\Sigma)$ as the Hilbert space

$$\mathscr{W}^{2,2}(\Omega,\Sigma) \,:=\, \overline{\mathrm{Proj}_{L^2\left(\Omega,\Sigma\times\Pi\times\Xi\right)}\tilde W^{2,2}(\Omega,\mathbb{R}^N)}^{\|\cdot\|_{L^2(\Omega)}} \tag{4.4}$$

which we equip with the norm (written for $W^{2,2}$ maps)

$$\|u\|_{\mathscr{W}^{2,2}(\Omega,\Sigma)} \,:=\, \big\|\Sigma u\big\|_{L^2(\Omega)} + \big\|\Pi Du\big\|_{L^2(\Omega)} + \big\|\Xi\mathrm{D}^2u\big\|_{L^2(\Omega)}.$$

By the Mazur theorem, $\mathscr{W}^{2,2}(\Omega,\Sigma)$ can be characterised as

$$\mathscr{W}^{2,2}(\Omega,\Sigma) = \left\{\begin{array}{l}\big(u,G(u),\mathrm{G}^2(u)\big)\in L^2\big(\Omega,\Sigma\times\Pi\times\Xi\big)\;\big|\;\exists\,(u^m)_1^\infty\subseteq\\ W^{2,2}(\Omega,\mathbb{R}^N):\quad\text{we have weakly in } L^2 \text{ as } m\to\infty\\ \text{that }\;\big(\Sigma u^m,\Pi\mathrm{D}u^m,\Xi\mathrm{D}^2u^m\big)\longrightarrow\big(u,\mathrm{G}(u),\mathrm{G}^2(u)\big)\end{array}\right\}.$$

We will call $\mathrm{G}(u)\in L^2(\Omega,\Pi)$ the **fibre gradient** of u and $\mathrm{G}^2(u)\in L^2(\Omega,\Xi)$ the **fibre hessian** of u.

We now show that $\big(\mathrm{G}(u),\mathrm{G}^2(u)\big)$ *depend only on* $u\in L^2(\Omega,\Sigma)$ *and not on the approximating sequence.* Indeed, let $(u^m)_1^\infty$ and $(v^m)_1^\infty$ be sequences in $W^{2,2}(\Omega,\mathbb{R}^N)$ such that

$$\big(\Sigma u^m,\Pi\mathrm{D}u^m,\Xi\mathrm{D}^2u^m\big)\longrightarrow\big(u,\mathrm{G}(u),\mathrm{G}^2(u)\big),$$
$$\big(\Sigma v^m,\Pi\mathrm{D}v^m,\Xi\mathrm{D}^2v^m\big)\longrightarrow\big(v,\mathrm{G}(v),\mathrm{G}^2(v)\big),$$

weakly in $L^2\big(\Omega,\Sigma\times\Pi\times\Xi\big)$ as $m\to\infty$. We immediately have that $\Sigma u=\Sigma v$ a.e. on Ω and hence u,v represent the same element of $L^2(\Omega,\Sigma)$ because their projections on the subspace $\Sigma\subseteq\mathbb{R}^N$ coincide, whilst by definition $\Sigma^\perp u\equiv\Sigma^\perp v\equiv 0$. Similarly, since Σ,Π,Ξ are spanned by directions of the form η, $\eta\otimes a$ and $\eta\otimes(a\vee b)$ respectively, for any $\phi\in C^\infty_c(\Omega)$, $\eta\otimes a\in\Pi$ and $\eta\in\Sigma$ we have

$$\int_\Omega\phi\big(G(u)-G(v)\big):\eta\otimes a \,=\, \lim_{m\to\infty}\int_\Omega\phi\big(\Pi\mathrm{D}u^m-\Pi\mathrm{D}v^m\big):\eta\otimes a$$
$$=\lim_{m\to\infty}\int_\Omega\phi\mathrm{D}_a\left[\eta\cdot\big(u^m-v^m\big)\right] \,=\, -\lim_{m\to\infty}\int_\Omega\mathrm{D}_a\phi\,\left[\eta\cdot\Sigma\big(u^m-v^m\big)\right] \,=\,0$$

and hence $G(u),G(v)$ coincide as elements of $L^2(\Omega,\Pi)$ because their projections on the subspace $\Pi\subseteq\mathbb{R}^{Nn}$ coincide. The remaining case is analogous.

Further, by using the standard properties of equivalence between strong and weak L^2 directional derivatives, we have that $\mathrm{G}(u),\mathrm{G}^2(u)$ can be characterised as "fibre" derivatives of u: for any directions $\eta\in\Sigma$, $\eta\otimes a\in\Pi$ and $\eta\otimes(a\vee b)\in\Xi$, we have $\mathrm{G}(u):(\eta\otimes a)=D_a(\eta\cdot u)$ and also

$$\mathrm{G}^2(u):\big(\eta\otimes(a\vee b)\big) \,=\, D^2_{ab}(\eta\cdot u) \,=\, D_b\big(\mathrm{G}(u):(\eta\otimes a)\big),$$

a.e. on Ω, where D_a, D^2_{ab} are the usual directional derivatives. In general, the fibre spaces are strictly larger than their "non-degenerate" counterparts and there are elements of them which are not even $W^{1,1}_{\mathrm{loc}}$. For instance, take $\mathbf{A}=\eta\otimes a\otimes\eta\otimes a$, $|a|=1$. Then, for any $f\in W^{2,2}(\mathbb{R})$, $g\in L^2(\mathbb{R}^n)$ and $\zeta\in C^\infty_c(\mathbb{R}^n)$, the map $u(x):=\zeta(x)\big[f(a\cdot x)+g\big([I-a\otimes a]x\big)\big]\eta$ is an element of $\mathscr{W}^{2,2}(\Omega,\Sigma)$ arising from this $\mathbf{A}$, but $D_b(\eta\cdot u)$ may not exist in L^2 for any $b\perp a$. Similarly to the second order case, we may also define

$$\mathscr{W}^{1,2}_0(\Omega,\Sigma) \,:=\, \overline{\mathrm{Proj}_{L^2(\Omega,\Sigma\times\Pi)}\tilde W^{1,2}_0(\Omega,\mathbb{R}^N)}^{\|\cdot\|_{L^2(\Omega)}}, \tag{4.5}$$

equipped with the obvious respective norm $\|\cdot\|_{\mathscr{W}^{1,2}(\Omega)}$. Further functional properties of the fibre spaces (traces, Poincaré inequality) needed for the proof of Theorem 34 will be discussed after its statement. The fibre space $\big(\mathscr{W}^{2,2}\cap\mathscr{W}^{1,2}_0\big)(\Omega,\Sigma)$ is the appropriate setup within which we obtain compactness and uniqueness of $\mathcal{D}$-solutions for the Dirichlet problems (4.1)-(4.2), by utilising the hypotheses introduced in the next paragraph.

Degenerate ellipticity and decomposability. Now we introduce our ellipticity hypothesis for (4.1) and a condition for tensors $\mathbf{A}\in\mathbb{R}^{Nn\times Nn}_s$ guaranteeing their range Π is spanned by rank-one directions.

Definition 32 (Degenerate ellipticity)**.** The Carathéodory map $\mathcal{F}:\Omega\times\mathbb{R}^{Nn^2}_s\longrightarrow\mathbb{R}^N$ is called degenerate elliptic when there exists a rank-one non-negative $\mathbf{A}\in\mathbb{R}^{Nn\times Nn}_s$, constants $B,C\geq 0$ with $B+C<1$ and a positive function A satisfying $A,1/A\in L^\infty(\Omega)$ such that

$$\Big|\mathbf{A}:\mathbf{Z}\,-\,A(x)\Big(\mathcal{F}(x,\mathbf{X}+\mathbf{Z})-\mathcal{F}(x,\mathbf{X})\Big)\Big|\,\leq\,B\nu\,|\Xi\mathbf{Z}|\;+\;C\,|\mathbf{A}:\mathbf{Z}|,$$

for a.e. $x\in\Omega$ and all $\mathbf{X},\mathbf{Z}\in\mathbb{R}^{Nn^2}_s$. We moreover require $\mathcal{F}$ to be valued in the subspace $\Sigma\subseteq\mathbb{R}^N$, i.e. $\mathcal{F}(x,\mathbf{X})\in\Sigma$, for a.e. $x\in\Omega$ and all $\mathbf{X}\in\mathbb{R}^{Nn^2}_s$.

Definition 32 is an extension to the degenerate realm of the strict ellipticity assumption introduced in [K9]. In the elliptic case we have $\Sigma=\mathbb{R}^N$, $\Pi=\mathbb{R}^{Nn}$ and $\Xi=\mathbb{R}^{Nn^2}_s$ and then (4.1) is solvable in the class of strong solutions (for more details see [K9]). If $\mathbf{A}_{\alpha i\beta j}=\delta_{\alpha\beta}\delta_{ij}$ and $A(x)=\text{const}$, we reduce to the classical notion introduced by Campanato ([C1]-[C3]). It is easy to exhibit non-trivial examples of Carathéodory maps satisfying Definition 32, see Remark 35IV) that follows. It is quite restrictive, but classical examples ([LU]) show that even if $N=1$, extra assumption are needed for the linear uniformly elliptic equation $\mathbf{A}:\mathrm{D}^2u=f$ if $\mathbf{A}$ is discontinuous. Below is the structural hypothesis for tensors.

Definition 33 (Decomposability)**.** We call $\mathbf{A}\in\mathbb{R}^{Nn\times Nn}_s$ *decomposable* when it can be written as $\mathbf{A}_{\alpha i\beta j}=B^1_{\alpha\beta}A^1_{ij}+\cdots+B^N_{\alpha\beta}A^N_{ij}$ and:

i) The matrices $\{B^1,...,B^N\}\subseteq\mathbb{R}^{N^2}_s$ are non-negative with ranges $\Sigma^1,...,\Sigma^N$ mutually orthogonal in $\mathbb{R}^N$.

ii) The matrices $\{A^1,...,A^N\}\subseteq\mathbb{R}^{n^2}_s$ are non-negative and if $\lambda^\gamma_{i_0}$ is the smallest positive eigenvalue of A^γ, the eigenspaces $N\big(A^\gamma-\lambda^\gamma_{i_0}I\big)$ have non-trivial intersection.

We discuss certain implications of these hypotheses and some examples after the main result which we give right next.

$\mathcal{D}$-solutions for fully nonlinear degenerate elliptic systems. Below is the principal result of this section followed by some relevant comments.

Theorem 34 (Existence-Uniqueness-Partial Regularity)**.** *Let $\Omega\subseteq\mathbb{R}^n$ be a strictly convex bounded domain with C^2 boundary and $\mathcal{F}:\Omega\times\mathbb{R}^{Nn^2}_s\longrightarrow\mathbb{R}^N$ a map which satisfies Definition 32 with respect to a decomposable $\mathbf{A}$ (Definition 33). Let also Ξ,Π,Σ be given by (4.3) and suppose that $|\mathcal{F}(\cdot,0)|\in L^2(\Omega)$. Then, the problem*

$$\left\{\begin{aligned}\mathcal{F}(\cdot,\mathrm{D}^2u)\,&=\,f, &&\text{in }\Omega,\\ u\,&=\,0, &&\text{on }\partial\Omega,\end{aligned}\right.$$

has, for any $f \in L^2(\Omega, \Sigma)$, a unique $\mathcal{D}$-solution $u : \mathbb{R}^n \supseteq \overline{\Omega} \longrightarrow \mathbb{R}^N$ in the fibre space $(\mathscr{W}^{1,2}_0 \cap \mathscr{W}^{2,2})(\Omega, \Sigma)$ (see (4.4), (4.5)) with respect to certain orthonormal frames (see (2.2)) depending only on $\mathcal{F}$ (Definition 14). In particular, $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ is a vanishing Lebesgue point of u, whilst for any $\mathcal{D}^2 u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\boldsymbol{X}) \Big(\mathcal{F}(x, \boldsymbol{X}) - f(x)\Big)\, d[\mathcal{D}^2 u(x)](\boldsymbol{X}) \,=\, 0, \quad a.e.\ x \in \Omega,$$

for any $\Phi \in C_c\big(\mathbb{R}^{Nn^2}_s\big)$.

Remark 35. I) [**Compatibility**] f must be valued into Σ because this is a compatibility condition due to the degeneracy of the problem. For example, the 2×2 system $\Delta u_1 = f_1$, $0 = f_2$ has no solution whatsoever in any sense unless $f_2 \equiv 0$.

II) [**Partial regularity**] The solution we obtain in Theorem 34 possess differentiable projections along certain rank-one lines, but in general this can not be improved further. In particular, *the solution is not partially regular in the standard sense* of being more regular on a subset of the domain with full measure. For, choose any $f \in C(\overline{D})$ not weakly differentiable with respect to x_1 for any x_2 over the unit disc of $\mathbb{R}^2$. Then, the problem

$$D^2_{22} u = 0 \text{ in } D \ \text{ and } \ u = 0 \text{ on } \partial \mathrm{D}$$

has the unique explicit $\mathcal{D}$-solution (which is not in $W^{1,1}_{\text{loc}}(\Omega)$)

$$u(x_1, x_2) \,=\, -v(x_1, x_2) \ + \int_{-\infty}^{x_2}\int_{-\infty}^{t_2} f(x_1, s_2)\, ds_2\, dt_2,$$

where for $(x_1, x_2) \in \overline{D}$, we set $w(x_1, x_2) := \int_{-\infty}^{x_2}\int_{-\infty}^{t_2} f(x_1, s_2)\, ds_2\, dt_2$ and also

$$\begin{aligned} v(x_1, x_2) := \frac{x_2}{2\sqrt{1 - x_1^2}} &\left[w\Big(x_1, \sqrt{1 - x_1^2}\Big) - w\Big(x_1, -\sqrt{1 - x_1^2}\Big)\right] \\ &+ \frac{1}{2}\left[w\Big(x_1, \sqrt{1 - x_1^2}\Big) + w\Big(x_1, -\sqrt{1 - x_1^2}\Big)\right]. \end{aligned}$$

III) [**Decomposability**] Definition 33 trivialises when either $N = 1$ or $n = 1$ since any non-negative matrix $A \in \mathbb{R}^{n^2}_s$ or $B \in \mathbb{R}^{N^2}_s$ satisfies it. When $\max\{N, n\} \geq 2$, it is non-trivial, but in view of its constructive nature it is trivial to exhibit $\mathbf{A}$'s satisfying it. Also, any decomposable $\mathbf{A}$ must be non-negative: if $Q \in \mathbb{R}^{Nn}$,

$$\mathbf{A} : Q \otimes Q = \sum_{\gamma} \big((B^\gamma)^{1/2} Q (A^\gamma)^{1/2}\big)\big((B^\gamma)^{1/2} Q (A^\gamma)^{1/2}\big) \,\geq\, 0.$$

IV) [**Examples of nonlinearities**] Fix $\mathbf{A} \in \mathbb{R}^{Nn \times Nn}_s$ and an $f \in C^{0,1}\big(\mathbb{R}^{Nn^2}_s, \mathbb{R}^N\big)$ with Lipschitz constant $\mathrm{Lip}(f)$. Then, for any positive A with $A, 1/A \in L^\infty(\Omega)$,

$$\mathcal{F}(x, \mathbf{X}) := \big(A(x)\big)^{-1}\Big[(1 + \gamma)\mathbf{A} : \mathbf{X} + \Sigma f(\Xi \mathbf{X})\Big]$$

satisfies Definition 32 when $\nu|\gamma| + \mathrm{Lip}(f) \,<\, \nu$. Linear examples satisfying Definition 32 are given by any $\mathbf{A} : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{Nn \times Nn}_s$ measurable such that $\big|\big(\mathbf{A} - A(x)\mathbf{A}(x)\big) : \mathbf{Z}\big| \,\leq\, B\nu\,|\Xi \mathbf{Z}|$ for some $0 < B < 1$ and A positive such that $A, 1/A \in L^\infty(\Omega)$.

V) [**Partial monotonicity**] Under the assumptions of Theorem 34, $\mathcal{F}$ satisfies

$$(4.6)\qquad \left\{ \begin{array}{l} \text{For a.e. } x \in \Omega,\ \mathcal{F}(x,\cdot) \text{ is constant along the subspace } \Xi^{\perp}, \text{ i.e.} \\ \qquad\qquad \mathcal{F}(x,\mathbf{X}) = \mathcal{F}(x,\Xi\mathbf{X}), \quad \mathbf{X} \in \mathbb{R}^{Nn^2}_s. \end{array} \right.$$

Here Ξ is as in (4.3). Condition (4.6) is *strictly weaker* than the decoupling condition $\mathcal{F}_\alpha(\mathbf{X}) = \mathcal{F}_\alpha(\mathbf{X}_\alpha)$ assumed in vector-valued viscosity solutions. To see it, we first note that if $\mathbf{A}$ satisfies Definition 33, then $\Xi^{\perp} \subseteq N\big(\mathbf{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\big)$ (this is established in the proof, see (4.7) below). By using that $\mathbf{A}:\mathbf{X} = \mathbf{A}:(\Xi\mathbf{X})$, we have $\mathbf{A}:\mathbf{Z} = 0$ and $\Xi\mathbf{Z} = 0$ when $\mathbf{Z} \in \Xi^{\perp}$. Hence, Definition 32 gives $\big|A(x)\big(\mathcal{F}(x,\mathbf{X}+\mathbf{Z}) - \mathcal{F}(x,\mathbf{X})\big)\big| \leq 0$ for all $\mathbf{Z} \in \Xi^{\perp}$ and $\mathbf{X} \in \mathbb{R}^{Nn^2}_s$.

Next we gather some properties of the fibre spaces essentially proved in [K10] but without the formalism of the fibre spaces.

Remark 36 (Basic properties of the fibre Sobolev space counterparts, cf. [K10])**.**
(I) [**Poincaré inequality**] For any $\Omega \Subset \mathbb{R}^n$, unit vectors a, η and $u \in W^{1,2}_0(\Omega,\mathbb{R}^N)$, we have

$$\|\eta\cdot u\|_{L^2(\Omega)} \leq \operatorname{diam}(\Omega)\,\big\|D_a(\eta\cdot u)\big\|_{L^2(\Omega)}.$$

(II) [**Norm equivalence**] The seminorm $\|G^2(\cdot)\|_{L^2(\Omega)}$ on the fibre space $(\mathscr{W}^{1,2}_0 \cap \mathscr{W}^{2,2})(\Omega,\Sigma)$ (see (4.4), (4.5)) is equivalent to its natural norm

$$\|\cdot\|_{\mathscr{W}^{2,2}(\Omega)} = \|\cdot\|_{L^2(\Omega)} + \|G(\cdot)\|_{L^2(\Omega)} + \|G^2(\cdot)\|_{L^2(\Omega)}.$$

(III) [**Trace operator**] If $\Omega \Subset \mathbb{R}^n$ is strictly convex and $a \in \mathbb{R}^n \setminus \{0\}$, then there is a closed set $E \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(E) = 0$ such that for any $\Gamma \Subset \partial\Omega \setminus E$, we have

$$\|v\|_{L^2(\Gamma,\mathcal{H}^{n-1})} \leq C\Big(\|v\|_{L^2(\Omega)} + \big\|D_a v\big\|_{L^2(\Omega)}\Big),$$

for some universal $C = C(\Gamma) > 0$ and all $v \in C^1(\overline{\Omega})$. Hence, there is a well-defined trace operator $T : \mathscr{W}^{1,2}(\Omega,\mathbb{R}^N) \to L^2_{\text{loc}}(\partial\Omega \setminus E, \mathcal{H}^{n-1};\mathbb{R}^N)$.

For the proof of Theorem 34 we need an important estimate established next.

4.3. A priori degenerate estimates.

Herein we establish an a priori estimate for strong solutions in $(W^{2,2}\cap W^{1,2}_0)(\Omega,\mathbb{R}^N)$ of a regularisation of the linear system $\mathbf{A} : \mathrm{D}^2 u = f$ when $\mathbf{A}$ is decomposable. This is a generalisation of the elliptic estimate of [K9] (the latter extending the Miranda-Talenti inequality) to the *degenerate* realm.

Theorem 37 (Degenerate hessian estimate)**.** *Suppose $\Omega \Subset \mathbb{R}^n$ is a convex C^2 domain and $\boldsymbol{A} \in \mathbb{R}^{Nn\times Nn}_s$ satisfies Definition 33, $n, N \geq 1$. If Ξ, ν are as in (4.3), for any $u \in (W^{2,2}\cap W^{1,2}_0)(\Omega,\mathbb{R}^N)$ and $\varepsilon \geq 0$ we have the estimate*

$$\big\|\Xi\mathrm{D}^2 u\big\|_{L^2(\Omega)} \leq \frac{1}{\nu}\,\big\|\boldsymbol{A}^{(\varepsilon)} : \mathrm{D}^2 u\big\|_{L^2(\Omega)}$$

and also the property

$$(4.7)\qquad \Xi \supseteq N\Big(\boldsymbol{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\Big)^{\perp}.$$

The tensor $\boldsymbol{A}^{(\varepsilon)}$ *is the rank-one (strictly) positive regularisation of* $\boldsymbol{A}$ *given by* $\boldsymbol{A}^{(\varepsilon)}_{\alpha i\beta j} := \sum_{\gamma=0}^{N} B^{(\varepsilon)\gamma}_{\alpha\beta} A^{(\varepsilon)\gamma}_{ij}$*, where* B^γ*,* A^γ *are as appearing in Definition 33 and*

$$B^{(\varepsilon)\gamma} := \begin{cases} B^\gamma, & \gamma = 1, ..., N, \\ \varepsilon I - \varepsilon\big(B^1 + \cdots + B^N\big), & \gamma = 0, \end{cases}$$
$$A^{(\varepsilon)\gamma} := \begin{cases} A^\gamma + \varepsilon I, & \gamma = 1, ..., N, \\ \varepsilon I, & \gamma = 0. \end{cases}$$

Note that in the vectorial case $N \geq 2$ of Theorem 37, the "correct" approximation is *not* the vanishing viscosity one, although it reduces to such when $N = 1$.

Proof of Theorem 37. The first step is to prove a scalar version of the theorem.

Claim 38. *Let* $\Omega \Subset \mathbb{R}^n$ *be* C^2 *and convex and let also* $A \geq 0$ *in* $\mathbb{R}^{n^2}_s$*. Then, there exists a subspace* $H \subseteq \mathbb{R}^{n^2}_s$ *such that* $H \supseteq N\Big(A : \mathbb{R}^{n^2}_s \to \mathbb{R}\Big)^{\perp}$ *and for any* $u \in (W^{2,2} \cap W^{1,2}_0)(\Omega)$ *and any* $\varepsilon \geq 0$ *we have the estimate*

$$\big\|H\mathrm{D}^2 u\big\|_{L^2(\Omega)} \leq \frac{1}{\nu(A)}\big\|A : \mathrm{D}^2 u + \varepsilon\Delta u\big\|_{L^2(\Omega)}$$

where $\nu(A) := \min_{|a|=1,\, a\in T}\{A : a \otimes a\}$ *and* $T := R\big(A : \mathbb{R}^n \to \mathbb{R}^n\big)$.

Proof of Claim 38. By the Spectral theorem, we can find a diagonal matrix Λ with entries $0 \leq \lambda_1 \leq ... \leq \lambda_n$ and $O \in O(n)$ such that $A = O\Lambda^{1/2}(O\Lambda^{1/2})^\top$ and

$$\Lambda = \left[\begin{array}{c|ccc} 0 & & 0 & \\ \hline & \lambda_{i_0} & & 0 \\ 0 & & \ddots & \\ & 0 & & \lambda_n \end{array}\right].$$

Evidently, $\big\{\lambda_1, ..., \lambda_n\big\} = \big\{0, ..., 0, \lambda_{i_0}, ..., \lambda_n\big\}$ are the eigenvalues of A and λ_{i_0} is the smallest positive eigenvalue. We also fix $\varepsilon \geq 0$ and set

$$\Theta := \big(\Lambda + \varepsilon I\big)^{1/2}, \quad \Gamma := O\Theta. \tag{4.8}$$

Then, since A equals $O\Lambda O^\top$ and Θ is symmetric, we have

$$A + \varepsilon I = O\Lambda O^\top + O(\varepsilon I)O^\top = O\Theta(O\Theta)^\top = \Gamma\Gamma^\top \tag{4.9}$$

and also $\nu(A) = \lambda_{i_0}$ ($\nu(A)$ is defined in the statement). We define

$$\begin{aligned} H^0 &:= \left\{X \in \mathbb{R}^{n^2}_s \ : \ X = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & (X_{ij})^{j=i_0,...,n}_{i=i_0,...,n} \end{array}\right]\right\}, \\ H &:= \left\{X \in \mathbb{R}^{n^2}_s \ : \ O^\top X O \in H^0\right\} \end{aligned} \tag{4.10}$$

and claim the following algebraic inequality:

$$\big|\Theta X\Theta\big| \geq \nu(A)\big|H^0 X\big|, \quad X \in \mathbb{R}^{n^2}_s. \tag{4.11}$$

Indeed, since $\Theta_{ij}=0$ when $i\neq j$ and $\Theta_{ii}=\sqrt{\lambda_i+\varepsilon}$, in view of (4.10) we have

$$\begin{aligned}\big|\Theta X\Theta\big|^2 &= \sum_{i,j,k,l,p,q=1}^{n}\big(\Theta_{ik}X_{kl}\Theta_{lj}\big)\big(\Theta_{ip}X_{pq}\Theta_{qj}\big) = \\ &= \sum_{i,j=1}^{n}\big(\Theta_{ii}X_{ij}\Theta_{jj}\big)^2 \ \geq \ \sum_{i,j=i_0}^{n}\big(\lambda_i+\varepsilon\big)\,(X_{ij})^2\big(\lambda_j+\varepsilon\big) \ \geq \\ &\geq \ (\lambda_{i_0})^2\sum_{i,j=i_0}^{n}(X_{ij})^2 \ = \ \nu(A)^2\big|H^0X\big|^2.\end{aligned}$$

Hence, (4.11) has been established. In order to conclude, the goal is to reduce to the classical Miranda-Talenti inequality (see [M, T, K9]) which says that for $U\Subset\mathbb{R}^n$ convex C^2 domain and any $v\in(W^{2,2}\cap W^{1,2}_0)(U)$, we have

$$\big\|D^2v\big\|_{L^2(U)} \leq \big\|\Delta v\big\|_{L^2(U)}. \tag{4.12}$$

It suffices to consider $\varepsilon>0$ since the case $\varepsilon=0$ follows by letting $\varepsilon\to 0$. Given any $u\in C^2(\overline{\Omega})\cap C^1_0(\Omega)$, we set $U:=\Gamma^{-1}\Omega$ and $v(x):=u(\Gamma x)$, $x\in U$. Then, $\mathrm{D}^2_{ij}v(x)=\sum_{p,q=1}^{n}D^2_{pq}u(\Gamma x)\Gamma_{pi}\Gamma_{qj}$ and hence, by (4.8) and (4.9) we obtain

$$\begin{aligned} D^2v(x) \ &= \ \Gamma^\top\mathrm{D}^2u(\Gamma x)\Gamma \ = \ \Theta\Big(O^\top D^2u(\Gamma x)O\Big)\Theta, \\ \Delta v(x) \ &= \ \mathrm{D}^2u(\Gamma x):\Gamma\Gamma^\top \ = \ \mathrm{D}^2u(\Gamma x):(A+\varepsilon I).\end{aligned} \tag{4.13}$$

Now note that since Ω is a C^2 bounded convex domain, U is a C^2 bounded convex domain as well as image of such a set under a linear invertible mapping. We now apply (4.12) to v over $U\subseteq\mathbb{R}^n$ and in view of (4.13) we obtain

$$\begin{aligned}\int_U\big|\mathrm{D}^2u(\Gamma x):(A+\varepsilon I)\big|^2dx &\geq \int_U\Big|\Theta\Big(O^\top D^2u(\Gamma x)O\Big)\Theta\Big|^2dx \\ &\overset{(4.11)}{\geq} \nu(A)^2\int_U\Big|H^0\Big(O^\top D^2u(\Gamma x)O\Big)\Big|^2dx.\end{aligned}$$

By the change of variables $y:=\Gamma x$ and by using that O is orthogonal, we get

$$\big\|\mathrm{D}^2u:(A+\varepsilon I)\big\|_{L^2(\Omega)} \geq \ \nu(A)\Big\|O\Big(H^0\left(O^\top D^2uO\right)\Big)O^\top\Big\|_{L^2(\Omega)}. \tag{4.14}$$

Now we claim that the orthogonal projection on the subspace $H\subseteq\mathbb{R}^{n^2}_s$ is given by

$$HX = O\Big(H^0\left(O^\top XO\right)\Big)O^\top. \tag{4.15}$$

Once (4.15) has been established, the desired estimate follows from (4.14), (4.15) and a standard density argument. Indeed, if K denotes the linear operator defined by the right hand side of (4.15), for any $X\in\mathbb{R}^{n^2}_s$ we have

$$\begin{aligned}K(KX) &= O\Big(H^0\Big(O^\top O\Big(H^0\left(O^\top XO\right)\Big)O^\top O\Big)\Big)O^\top = \\ &= O\Big(H^0H^0\big(O^\top XO\big)\Big)O^\top = O\Big(H^0\big(O^\top XO\big)\Big)O^\top = KX.\end{aligned}$$

Hence, $K^2 = K$. Moreover, K is symmetric as a map $\mathbb{R}^{n^2}_s \longrightarrow \mathbb{R}^{n^2}_s$: by using that H^0 is symmetric, we have

$$(K X) : Y \,=\, \Big(O\Big(H^0\left(O^\top X O\right)\Big)O^\top\Big) : Y \,=\, H^0\left(O^\top X O\right) : \left(O^\top Y O\right) =$$
$$= \left(O^\top X O\right) : H^0\left(O^\top Y O\right) \,=\, X : \Big(O\Big(H^0\left(O^\top Y O\right)\Big)O^\top\Big) \,=\, X : (KY),$$

for $X, Y \in \mathbb{R}^{n^2}_s$. Hence, (4.15) follows. It remains to demonstrate the claimed property of H. To this end, fix $X \perp H$. Then, the projection of X on H vanishes and as a result of (4.15) we obtain $H^0(O^\top X O) = 0$. By recalling that $A = O\Lambda O^\top$, we have $A : X = \Lambda : (O^\top X O)$ and since $\Lambda \in H^0$, we conclude that $A : X = 0$. Hence, we proved $H^\perp \subseteq N(A : \mathbb{R}^{n^2}_s \to \mathbb{R})$, as desired. $\square$

Next we characterise the space $H \subseteq \mathbb{R}^{n^2}_s$ of Claim 38 in terms of the range of A.

Claim 39. *In the setting of Claim 38, we have the identity*

$$H \,=\, \mathrm{span}\Big[\Big\{a \vee b \,\Big|\, a, b \in R\big(A : \mathbb{R}^n \to \mathbb{R}^n\big)\Big\}\Big] \,=\, T \vee T.$$

Proof of Claim 39. We begin by observing that in view of (4.10), we have $H = O\,H^0 O^\top$ where $O \in O(n)$. Since $H^0 = \mathrm{span}\big[\big\{e^i \vee e^j \,\big|\, i, j = i_0, ..., n\big\}\big]$, H has a basis of the form $Oe^i \vee Oe^j$, $i, j = i_0, ..., n$. We recall now that $A = O\Lambda O^\top$ where Λ is a diagonal matrix with entries the eigenvalues $\{0, ..., 0, \lambda_{i_0}, ..., \lambda_n\}$ of A. We define the vectors $a^i := Oe^i = \big(O_{1i}, ..., O_{ni}\big)^\top$, $i = 1, ..., n$. Then, $\{a^1, ..., a^n\}$ is an orthonormal frame of $\mathbb{R}^n$ corresponding to the columns of the matrix A and is a set of eigenvectors of A. Since $\{a^{i_0}, ..., a^n\}$ correspond to the nonzero eigenvalues $\{\lambda_{i_0}, ..., \lambda_n\}$, the nullspace $N\big(A : \mathbb{R}^n \to \mathbb{R}^n\big)$ is spanned by $\{a^1, ..., a^{i_0-1}\}$ and hence $R\big(A : \mathbb{R}^n \to \mathbb{R}^n\big) = \mathrm{span}\big[\big\{a^{i_0}, ..., a^n\big\}\big]$. Since H has a basis of the form $\{a^i \vee a^j : i, j = i_0, ..., n\}$, the claim follows. $\square$

Now we begin working towards the vector case $N \geq 2$. Let us first verify that $\mathbf{A}^{(\varepsilon)}$ is rank-one positive. Indeed, if $0 < \varepsilon < 1$, $\eta \in \mathbb{R}^N$, $a \in \mathbb{R}^n$, we have

$$\mathbf{A}^{(\varepsilon)} : \eta \otimes a \otimes \eta \otimes a \,\geq\, \min_{\delta=0,...,N}\Big(A^{(\varepsilon)\delta} : a \otimes a\Big)\left[\sum_{\gamma=0}^{N} B^{(\varepsilon)\gamma} : \eta \otimes \eta\right] \,\geq$$
$$\geq\, \varepsilon\,|a|^2\left[\sum_{\gamma=1}^{N} B^\gamma + \varepsilon\left(I - \sum_{\delta=1}^{N} B^\delta\right)\right] : \eta \otimes \eta \,\geq\, \varepsilon^2\,|\eta|^2|a|^2,$$

as claimed. The next step is to characterise the range Π of decomposable tensors $\mathbf{A} \in \mathbb{R}^{Nn\times Nn}_s$ in terms of the matrices B^γ, A^γ composing $\mathbf{A}$.

Claim 40. *Let $\Pi \subseteq \mathbb{R}^{Nn}$ be the range of $\boldsymbol{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^{Nn}$ (see (4.3)). Then, $\Pi = \oplus_\gamma\left(\Sigma^\gamma \otimes T^\gamma\right)$, where $\Sigma^\gamma \subseteq \mathbb{R}^N$ and $T^\gamma \subseteq \mathbb{R}^n$ are given by*

$$Si^\gamma \,=\, R\left(B^\gamma : \mathbb{R}^N \to \mathbb{R}^N\right), \quad T^\gamma \,=\, R\left(A^\gamma : \mathbb{R}^n \to \mathbb{R}^n\right) \tag{4.16}$$

Proof of Claim 40. We first observe that by Definition 33, $\Sigma^\gamma \perp \Sigma^\delta$ if $\gamma \neq \delta$ and this implies that $\Sigma^\gamma \otimes T^\gamma \perp \Sigma^\delta \otimes T^\delta$ if $\gamma \neq \delta$. Let now $Q \in \mathbb{R}^{Nn}$. Then, $\mathbf{A} : Q$ is given in index form by $\sum_{\gamma,\beta,j} B^\gamma_{\alpha\beta} Q_{\beta j} A^\gamma_{ji}$ which by (4.16) shows that $\Pi \subseteq \oplus_\gamma\big(\Sigma^\gamma \otimes T^\gamma\big)$. Conversely, let $R \in \oplus_\gamma\big(\Sigma^\gamma \otimes T^\gamma\big)$. Then, R can be written as $R = \sum_{\gamma,\kappa}\left(B^\gamma \eta^{\kappa\gamma}\right) \otimes \left(A^\gamma a^{\kappa\gamma}\right)$ for some $\eta^{\kappa\gamma} \in \Sigma^\gamma$, $a^{\kappa\gamma} \in T^\gamma$. We note that when

$\gamma \neq \delta$, we have $\big(B^\delta \otimes A^\delta\big)\big(\sum_\kappa \eta^{\kappa\gamma} \otimes a^{\kappa\gamma}\big) =$ because $\eta^{\kappa\gamma} \perp \Sigma^\delta$ if $\gamma \neq \delta$. By defining $Q := \sum_{\gamma,\kappa} \eta^{\kappa\gamma} \otimes a^{\kappa\gamma}$ it is immediate to verify that $\mathbf{A} : Q = R$. This establishes that $\Pi \supseteq \oplus_\gamma \big(\Sigma^\gamma \otimes T^\gamma\big)$, therefore completing the proof. ☐

Next we prove an upper bound on $\nu(\mathbf{A})$ in terms of B^γ, A^γ.

Claim 41. *Let ν be given (4.3) and Σ^γ, T^γ by (4.16). Then, we have the estimate*

$$\nu \leq \left(\min_\gamma \min_{\eta \in \Sigma^\gamma, \, |\eta|=1} \big\{ B^\gamma : \eta \otimes \eta \big\} \right) \left(\min_\delta \min_{a \in T^\delta, \, |a|=1} \big\{ A^\delta : a \otimes a \big\} \right).$$

Proof of Claim 41. We begin by noting that on top of the decomposability we may further assume that all the matrices A^γ have the same smallest positive eigenvalue $\lambda^\gamma_{i_0}$ equal to 1 for all $\gamma = 1, ..., N$ which is realised at a common eigenvector $\bar{a} \in \mathbb{R}^n$. Indeed, existence of $\bar{a}$ follows from Definition 33 since the eigenspaces $N\big(A^\gamma - \lambda^\gamma_{i_0} I\big)$ intersect for all γ at least along a common line in $\mathbb{R}^n$. Further, by replacing $\{B^1, ..., B^N\}$, $\{A^1, ..., A^N\}$ by the rescaled families $\{\tilde{B}^1, ..., \tilde{B}^N\}$, $\{\tilde{A}^1, ..., \tilde{A}^N\}$ where $\tilde{B}^\gamma := \lambda^\gamma_{i_0} B^\gamma$, $\tilde{A}^\gamma := (1/\lambda^\gamma_{i_0}) A^\gamma$, we have that the new families have the same properties as the original and in addition all the new A^γ matrices have the same minimum positive eigenvalue normalised to 1. Hence, we may assume

$$\exists \ \bar{a} \in \partial\mathbb{B}^n_1 \bigcap_{\gamma=1}^{N} T^\gamma \quad : \qquad \lambda^\gamma_{i_0} = \min_{a \in T^\gamma, \, |a|=1} \big\{ A^\gamma : a \otimes a \big\} = A^\gamma : \bar{a} \otimes \bar{a} = 1, \tag{4.17}$$

for $\gamma = 1, ..., N$. By using (4.17), Claim 40 and that $\cup_\gamma \big(\Sigma^\gamma \otimes T^\gamma\big) \subseteq \oplus_\gamma \big(\Sigma^\gamma \otimes T^\gamma\big)$

$$\begin{aligned} \nu &= \min_{|\eta|=|a|=1, \, \eta \otimes a \in \Pi} \sum_\delta \Big(B^\delta : \eta \otimes \eta\Big)\Big(A^\delta : a \otimes a\Big) \\ &\leq \min_{|\eta|=|a|=1, \, \eta \otimes a \in \cup_\gamma (\Sigma^\gamma \otimes T^\gamma)} \sum_\delta \Big(B^\delta : \eta \otimes \eta\Big)\Big(A^\delta : a \otimes a\Big), \end{aligned}$$

and hence

$$\begin{aligned} \nu &\leq \min_\gamma \left(\min_{|\eta|=|a|=1, \, \eta \otimes a \in \Sigma^\gamma \otimes T^\gamma} \sum_\delta \Big(B^\delta : \eta \otimes \eta\Big)\Big(A^\delta : a \otimes a\Big) \right) \\ &\leq \min_\gamma \left(\min_{|\eta|=|a|=1, \, \eta \otimes a \in \Sigma^\gamma \otimes T^\gamma} \sum_\delta \Big(B^\delta : \eta \otimes \eta\Big)\Big(A^\delta : a \otimes a\Big) \right) \\ &\leq \min_\gamma \left(\min_{|\eta|=1, \, \eta \in \Sigma^\gamma} \sum_\delta \Big(B^\delta : \eta \otimes \eta\Big)\Big(A^\delta : \bar{a} \otimes \bar{a}\Big) \right) \\ &= \min_\gamma \min_{|\eta|=1, \, \eta \in \Sigma^\gamma} \sum_\delta \Big(B^\delta : \eta \otimes \eta\Big). \end{aligned}$$

Since $B^\delta : \eta \otimes \eta = 0$ if $\eta \in \Sigma^\gamma$ for $\gamma \neq \delta$, by using (4.17) again we conclude that $\nu \leq \min_\gamma \min_{\eta \in \Sigma^\gamma, \, |\eta|=1} \big\{B^\gamma : \eta \otimes \eta\big\}$, as desired. ☐

Now we complete the proof by using the previous claims. We define

$$\Xi := \bigoplus_\gamma \Big(\Sigma^\gamma \otimes T^\gamma \vee T^\gamma \Big) \quad \subseteq \ \mathbb{R}^{Nn^2}_s, \tag{4.18}$$

and for brevity we set $\Xi^\gamma := T^\gamma \vee T^\gamma$ where Σ^γ, T^γ are as in (4.16). Fix $u \in C^2(\overline{\Omega}, \mathbb{R}^N) \cap C^1_0(\Omega, \mathbb{R}^N)$. Then, for $\gamma, \alpha = 1, ..., N$, by Claims 38, 39 applied to the scalar function $(\Sigma^\gamma u)_\alpha \in C^2(\overline{\Omega}) \cap C^1_0(\Omega)$, we have

$$\int_\Omega \Big|\Xi^\gamma D^2(\Sigma^\gamma u)_\alpha\Big|^2 \le \int_\Omega \Big|A^{(\varepsilon)\gamma} : D^2(\Sigma^\gamma u)_\alpha\Big|^2,$$

where we have used that $A^{(\varepsilon)\gamma} = A^\gamma + \varepsilon I$ (by the definition of $\mathbf{A}^{(\varepsilon)}$) and we have employed the normalisation of (4.17) which forces $\lambda^\gamma_{i_0} = \nu(A^\gamma) = 1$. By summing in α, γ, the above estimate and (4.18) give

$$\int_\Omega \big|\Xi\, D^2 u\big|^2 = \int_\Omega \sum_\gamma \Big|\Sigma^\gamma \otimes \Xi^\gamma : \mathrm{D}^2 u\Big|^2 \le \int_\Omega \sum_\gamma \Big|\Sigma^\gamma\big(D^2 u : A^{(\varepsilon)\gamma}\big)\Big|^2. \tag{4.19}$$

We also set $C^{(\varepsilon)\gamma} := \Sigma^\gamma\big(D^2 u : A^{(\varepsilon)\gamma}\big)$ for $\gamma = 1, ..., N$. Then, (4.19) says

$$\int_\Omega \big|\Xi\, D^2 u\big|^2 \le \int_\Omega \sum_{\gamma=1}^N \big|C^{(\varepsilon)\gamma}\big|^2. \tag{4.20}$$

By the definition of $\mathbf{A}^{(\varepsilon)}$, we have that $B^{(\varepsilon)\gamma} \perp B^{(\varepsilon)\delta}$ for $\gamma \neq \delta$ in $\{0, 1, ..., N\}$. By using this fact, we calculate

$$\begin{aligned} \big|\mathbf{A}^{(\varepsilon)} : \mathrm{D}^2 u\big|^2 &= \left(\sum_{\gamma=0}^N B^{(\varepsilon)\gamma}\big(D^2 u : A^{(\varepsilon)\gamma}\big)\right) \cdot \left(\sum_{\delta=0}^N B^{(\varepsilon)\delta}\big(D^2 u : A^{(\varepsilon)\delta}\big)\right) \\ &= \sum_{\gamma=0}^N \Big(B^{(\varepsilon)\gamma}\big(D^2 u : A^{(\varepsilon)\gamma}\big)\Big) \cdot \Big(B^{(\varepsilon)\gamma}\big(D^2 u : A^{(\varepsilon)\gamma}\big)\Big) \end{aligned}$$

and hence

$$\big|\mathbf{A}^{(\varepsilon)} : \mathrm{D}^2 u\big|^2 = \Big|B^{(\varepsilon)0}\big(D^2 u : A^{(\varepsilon)0}\big)\Big|^2 + \sum_{\gamma=1}^N \Big|B^{(\varepsilon)\gamma}\big(D^2 u : A^{(\varepsilon)\gamma}\big)\Big|^2 \ge$$

$$\ge \sum_{\gamma=1}^N \Big|B^\gamma\big(D^2 u : A^{(\varepsilon)\gamma}\big)\Big|^2 = \sum_{\gamma=1}^N \big|B^\gamma C^{(\varepsilon)\gamma}\big|^2 \ge \sum_{\gamma=1}^N \max_{|\eta|=1}\Big(B^\gamma : \big(C^{(\varepsilon)\gamma} \otimes \eta\big)\Big)^2 \ge$$

$$\ge \sum_{\gamma=1}^N \Big(B^\gamma : \Big(\mathrm{sgn}(C^{(\varepsilon)\gamma}) \otimes \mathrm{sgn}(C^{(\varepsilon)\gamma})\Big)\Big)^2 \big|C^{(\varepsilon)\gamma}\big|^2.$$

As a result, we obtain

$$\big|\mathbf{A}^{(\varepsilon)} : \mathrm{D}^2 u\big|^2 \ge \left(\min_{\delta=1,\dots,N} \min_{|\eta|=1,\, \eta \in \Sigma^\delta} \big\{B^\delta : \eta \otimes \eta\big\}\right)^2 \sum_{\gamma=1}^N \big|C^{(\varepsilon)\gamma}\big|^2. \tag{4.21}$$

By using Claim 41 (and also the normalisation condition (4.17)), (4.21) gives

$$\int_\Omega \big|\mathbf{A}^{(\varepsilon)} : \mathrm{D}^2 u\big|^2 \ge \nu^2 \int_\Omega \sum_{\delta=1}^N \big|C^{(\varepsilon)\delta}\big|^2. \tag{4.22}$$

Hence, by (4.22) and (4.20) we obtain the desired estimate for smooth u, the general case following by a density argument. We complete the proof by showing that Ξ satisfies (4.7). If $\mathbf{X} \perp \Xi$, by (4.18) $\mathbf{X}$ is normal to $\Sigma^\gamma \otimes H^\gamma$ for any $\gamma = 1, ..., N$, where $H^\gamma := T^\gamma \vee T^\gamma$. Hence the projection of $\mathbf{X}$ on $\Sigma^\gamma \otimes H^\gamma$ vanishes: $(\Sigma^\gamma \otimes H^\gamma)\mathbf{X} = 0$. By Claim 38 we have $A^\gamma : X = A^\gamma : (H^\gamma X)$ if $X \in \mathbb{R}^{n^2}$. Hence, $B^\gamma(\mathbf{X} : A^\gamma) = 0$

for $\gamma = 1, ..., N$ and by summing in γ we obtain $\mathbf{A} : \mathbf{X} = 0$. Thus, we have shown $\Xi^{\perp} \subseteq N\big(\mathbf{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\big)$ as desired. □

4.4. **Proof of the main result.** Now we establish our second main result by utilising the a priori estimate of Subsection 4.3.

Proof of Theorem 34. The fist step is to prove existence of a map in the fibre space $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ solving in a certain sense the linear problem.

Claim 42. *In the setting of Theorem 34 and under the same assumptions, for any $f \in L^2(\Omega, \Sigma)$ there exists a unique $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ such that $\boldsymbol{A} : \mathrm{G}^2(u) = f$ a.e. on Ω, where $\mathrm{G}^2(u)$ is the fibre hessian of u.*

Proof of Claim 42. The proof is based on the approximation by strictly elliptic systems and relies on the stable estimate of Theorem 37. Let $\mathbf{A}^{(\varepsilon)}$ be the approximation of $\mathbf{A}$ of Theorem 37 and consider for a fixed $f \in L^2(\Omega, \Sigma)$ the system $\mathbf{A}^{(\varepsilon)} : \mathrm{D}^2 u^\varepsilon = f$, a.e. on Ω. By standard lower semicontinuity and regularity results (see e.g. [D, GM]), the problem has for any $\varepsilon > 0$ a unique strong a.e. solution $u^\varepsilon \in (W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$. By Theorem 37 and Remark 36, we have the estimate

$$\big\|\Sigma u^\varepsilon\big\|_{L^2(\Omega)} + \big\|\Pi D u^\varepsilon\big\|_{L^2(\Omega)} + \big\|\Xi \mathrm{D}^2 u^\varepsilon\big\|_{L^2(\Omega)} \leq \frac{C}{\nu}\big\|f\big\|_{L^2(\Omega)}$$

for some universal $C > 0$. By the definition of $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ ((4.4),(4.5)), there exists u such that $\big(\Sigma u^\varepsilon, \Pi D u^\varepsilon, \Xi D^2 u^\varepsilon\big) \longrightarrow \big(u, \mathrm{G}(u), \mathrm{G}^2(u)\big)$, along a sequence $\varepsilon_k \to 0$ in L^2. Now we pass to the weak limit in the equations. By the form of the approximation $\mathbf{A}^{(\varepsilon)}$ and Definition 33, we have

$$\sum_{\gamma=1}^{N} B^{(\varepsilon)\gamma}\Big(\mathrm{D}^2 u^\varepsilon : A^{(\varepsilon)\gamma}\Big) = f - B^{(\varepsilon)0}\big(\mathrm{D}^2 u^\varepsilon : A^{(\varepsilon)0}\big),$$

a.e. on Ω. By using that $B^{(\varepsilon)\gamma} = B^\gamma$ for $\gamma = 1, ..., N$ and that $B^{(\varepsilon)0} \perp B^1 + \cdots + B^N$, we may project the system above on the range of $B^1 + \cdots + B^N$ which we denote by Σ. Then, since $\Sigma f = f$ and $A^{(\varepsilon)\gamma} = A^\gamma + \varepsilon I$, we obtain

$$\sum_{\gamma=1}^{N} B^\gamma\Big(\varepsilon \Delta u^\varepsilon \ + \ \mathrm{D}^2 u^\varepsilon : A^\gamma\Big) = f,$$

a.e. on Ω. Moreover, by (4.7) (and in view of Remark 35), we deduce that for any $\phi \in C^\infty_c(\Omega, \mathbb{R}^N)$, integration by parts gives

$$\int_\Omega \Big(\mathbf{A} : \big(\Xi \mathrm{D}^2 u^\varepsilon\big) - f\Big) \cdot \phi = -\varepsilon \int_\Omega \sum_{\gamma=1}^{N} B^\gamma(\Sigma u^\varepsilon) \cdot \Delta\phi.$$

By letting $\varepsilon_k \to 0$, we obtain $\mathbf{A} : \mathrm{G}^2(u) = f$, a.e. on Ω. We finally show uniqueness. Let $v, w \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ be two solutions of the system. Then, there are sequences $(v^m)_1^\infty, (w^m)_1^\infty \subseteq (W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$ such that $v^m - w^m \longrightarrow v - w$ with respect to $\|\cdot\|_{\mathscr{W}^{2,2}(\Omega)}$ as $m \to \infty$. By assumption we have $\mathbf{A} : G^2(v - w) = 0$ a.e. on Ω, and hence $\mathbf{A} : D^2(v^m - w^m) =: f^m$ a.e. on Ω and $f^m \to 0$ in $L^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$. Hence, by Theorem 37 and Remark 36,

$$\|f^m\|_{L^2(\Omega)} \geq \nu\big\|\Xi : D^2(v^m - w^m)\big\|_{L^2(\Omega)} \geq C\big\|\Sigma(v^m - w^m)\big\|_{L^2(\Omega)}$$

and by letting $m \to \infty$ we see that $v \equiv w$, hence uniqueness ensues. □

An essential ingredient in order to pass to the nonlinear problem is the next result of Campanato ([C3], [K9]) which we recall for the convenience of the reader.

Lemma 43 (Campanato's bijectivity of near operators). *Let $\mathfrak{X} \neq \emptyset$ be a set, $(X, \|\cdot\|)$ a Banach space and $\mathscr{F}, \mathscr{A} : \mathfrak{X} \longrightarrow X$ maps such that*

$$\Big\| \mathscr{F}(u) - \mathscr{F}(v) - \big(\mathscr{A}(u) - \mathscr{A}(v)\big)\Big\| \leq K \big\| \mathscr{A}(u) - \mathscr{A}(v) \big\|$$

for some $K \in (0,1)$ and all $u, v \in \mathfrak{X}$. Then, if $\mathscr{A}$ is bijective, $\mathscr{F}$ is bijective as well.

Now we employ Lemma 43 in order to show existence of a map in the fibre space $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ solving in a certain sense the nonlinear problem.

Claim 44. *In the setting of Theorem 34 and under the same assumptions, for any $f \in L^2(\Omega, \Sigma)$ there exists a unique $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ such that $\mathcal{F}\big(\cdot, \mathrm{G}^2(u)\big) = f$ a.e. on Ω where $\mathrm{G}^2(u)$ is the fibre hessian of u.*

Proof of Claim 44. For any fixed $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$, we have that $\mathbf{A} : \mathrm{G}^2(u)$ is in $L^2(\Omega, \Sigma)$ because $\mathrm{G}^2(u) \in L^2(\Omega, \Xi)$ and also $\mathbf{A} : \mathbf{X}$ lies is in $\Sigma \subseteq \mathbb{R}^N$ for any $\mathbf{X} \in \Xi \subseteq \mathbb{R}^{Nn^2}_s$. Moreover, by Definition 32 we have

$$\big|\mathcal{F}\big(\cdot, \mathrm{G}^2(u)\big)\big| \leq \left(\frac{(C+1)|\mathbf{A}| + B\nu}{\operatorname{ess\,inf}_{x \in \Omega}[A(x)]} \right) |\mathrm{G}^2(u)| + \big|\mathcal{F}\big(\cdot, 0\big)\big|,$$

a.e. on Ω. Hence, $\mathcal{F}\big(\cdot, \mathrm{G}^2(u)\big)$ is in $L^2(\Omega, \Sigma)$ as well. The previous considerations imply that the maps

$$\begin{aligned}
\mathscr{A} \ &: \ (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma) \longrightarrow L^2(\Omega, \Sigma), \quad \mathscr{A}(u) := \mathbf{A} : \mathrm{G}^2(u), \\
\mathscr{F} \ &: \ (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma) \longrightarrow L^2(\Omega, \Sigma), \quad \mathscr{F}(u) := \mathcal{F}\big(\cdot, \mathrm{G}^2(u)\big),
\end{aligned}$$

are well defined. By Claim 42, $\mathscr{A}$ is bijective. We complete the claim by showing that $\mathscr{F}$ is near $\mathscr{A}$ in the sense of Lemma 43. For any $u, v \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$, by Definition 32 and Theorem 37 we have

$$\begin{aligned}
&\Big\| A(\cdot)\Big(\mathcal{F}\big(\cdot, \mathrm{G}^2(u)\big) - \mathcal{F}\big(\cdot, \mathrm{G}^2(v)\big)\Big) - \mathbf{A} : \big(\mathrm{G}^2(u) - \mathrm{G}^2(v)\big) \Big\|_{L^2(\Omega)} \\
&\qquad \leq B\nu \big\| \mathrm{G}^2(u) - \mathrm{G}^2(v) \big\|_{L^2(\Omega)} + C \big\| \mathbf{A} : \big(\mathrm{G}^2(u) - \mathrm{G}^2(v)\big) \big\|_{L^2(\Omega)} \\
&\qquad \leq (B + C) \big\| \mathbf{A} : \big(\mathrm{G}^2(u) - \mathrm{G}^2(v)\big) \big\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, $\hat{\mathscr{F}}(u) := A(\cdot)\mathcal{F}\big(\cdot, \mathrm{G}^2(u)\big)$ is bijective and since $A, 1/A \in L^\infty(\Omega)$, the same is true for $\mathscr{F}$. The claim ensues. □

The next claim completes the proof of Theorem 34.

Claim 45. *In the setting of Claim 44 and under the same assumptions, there exists an orthonormal frame $\{E^1, ..., E^N\} \subseteq \mathbb{R}^N$ and for each $\alpha = 1, ..., N$ there is an orthonormal frame $\{E^{(\alpha)1}, ..., E^{(\alpha)n}\} \subseteq \mathbb{R}^n$ (both depending only on $\mathcal{F}$) such that, the map u corresponding to $f \in L^2(\Omega, \Sigma)$ is the unique $\mathcal{D}$-solution of $\mathcal{F}(\cdot, \mathrm{D}^2) = f$ in the fibre space $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$.*

Proof of Claim 45. Step 1 (The frames). By (4.3) and (4.16) there is a frame $\{E^\alpha|\alpha\}$ of $\mathbb{R}^N$ and for each α there is a frame $\{E^{(\alpha)i}|i\}$ of $\mathbb{R}^n$ such that each of the mutually orthogonal subspaces $\Sigma^\gamma \subseteq \mathbb{R}^N$ is spanned by a subset of vectors E^α and for the same index γ, T^γ is spanned by $\{E^{(\alpha)i_0},...,E^{(\alpha)n}\}$ which is a set of eigenvectors of A^γ. By (4.3) and (4.18) there are also induced frames of $\mathbb{R}^{Nn}$ and $\mathbb{R}^{Nn^2}_s$ of matrices as in (2.2). These frames are such that a subset of the $E^{\alpha ij}$'s spans the subspace Ξ and the rest are orthogonal to Ξ.

Step 2 (Sufficiency). Let now $u \in (\mathscr{W}^{2,2}\cap \mathscr{W}^{1,2}_0)(\Omega,\Sigma)$ be the map of Claim 44 which satisfies $\mathcal{F}(\cdot,\mathrm{G}^2(u)) = f$ a.e. on Ω. Let also us fix any infinitesimal sequence $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^2}$ with respect to the frames of Step 1 (see Definition 4) and let $\mathcal{D}^2u$ be any diffuse hessian of u arising from this sequence $\delta_{\mathrm{D}^{2,h_{\underline{m}}}u} \overset{*}{\longrightarrow} \mathcal{D}^2 u$ in $\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$ as $\underline{m}\to\infty$, perhaps along subsequences. By the characterisation of the fibre hessian $\mathrm{G}^2(u)\in L^2(\Omega,\Xi)$ in terms of directional derivatives of projections (Subsection 4.2),

$$\mathrm{G}^2(u) = \sum_{\alpha,i,j\,:\,E^{\alpha ij}\in\Xi}\Big(\mathrm{G}^2(u): E^{\alpha ij}\Big)E^{\alpha ij}, \quad \text{a.e. on } \Omega, \tag{4.23}$$

because the projection of $\mathrm{G}^2(u)$ along $E^{\alpha ij}$ is non-zero only for those $E^{\alpha ij}$ spanning Ξ. Since $\mathcal{F}$ is a Carathéodory map and $\mathcal{F}\big(x,\mathrm{G}^2(u)(x)\big) = f(x)$ for a.e. $x\in\Omega$, by (4.23) and in view of (2.4) we get

$$\mathcal{F}\left(x, \sum_{\alpha,i,j\,:\,E^{\alpha ij}\in\Xi}\left[\mathrm{D}^{2,h_{m_1^2}h_{m_2^2}}_{E^{(\alpha)i}E^{(\alpha)j}}\big(E^\alpha\cdot u\big)\right](x)E^{\alpha ij}\right)\longrightarrow f(x),$$

for a.e. $x\in\Omega$ as $\underline{m}\to\infty$. By Remark 35V), the above is equivalent to

$$\mathcal{F}\Big(x,\mathrm{D}^{2,h_{\underline{m}}}u(x)\Big) = \mathcal{F}\left(x, \sum_{\alpha,i,j}\left[\mathrm{D}^{2,h_{m_1^2}h_{m_2^2}}_{E^{(\alpha)i}E^{(\alpha)j}}\big(E^\alpha\cdot u\big)\right](x)E^{\alpha ij}\right)\longrightarrow f(x),$$

for a.e. $x\in\Omega$, as $\underline{m}\to\infty$. We set $f^{\underline{m}}(x) := \mathcal{F}\big(x,\mathrm{D}^{2,h_{\underline{m}}}u(x)\big) - f(x)$ and note that we have $f^{\underline{m}}\longrightarrow 0$, a.e. on Ω as $\underline{m}\to\infty$. By the above, for any $\Phi\in C_c\big(\mathbb{R}^{Nn^2}_s\big)$,

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s}\Phi(\mathbf{X})\Big[\mathcal{F}(x,\mathbf{X}) - \big(f(x)+f^{\underline{m}}(x)\big)\Big]\,d\big[\delta_{\mathrm{D}^{2,s_{\underline{m}}}u(x)}\big](\mathbf{X}) = 0, \quad \text{a.e. } x\in\Omega.$$

Since $f^{\underline{m}}\to 0$ a.e. on Ω as $\underline{m}\to\infty$, we apply the Convergence Lemma 16 to obtain

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s}\Phi(\mathbf{X})\big[\mathcal{F}(x,\mathbf{X}) - f(x)\big]\,d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) = 0, \quad \text{a.e. } x\in\Omega,$$

for any $\Phi\in C_c\big(\mathbb{R}^{Nn^2}_s\big)$. Hence, the map u of Claim 44 is a $\mathcal{D}$-solution of (4.1).

Step 3 (Necessity). We now finish the proof by showing that any $\mathcal{D}$-solution w of (4.1) with respect to the frames of Step 1 which lies in the fibre space $(\mathscr{W}^{2,2}\cap \mathscr{W}^{1,2}_0)(\Omega,\Sigma)$ actually coincides with the map u of Claim 44. By Theorem 21, we have that the $\mathcal{D}$-solution w can be characterised by the property that for any $R>0$, the cut off associated to $\mathcal{F}$ (see Definition 20) satisfies $\mathcal{F}\big(\cdot,\big[\mathrm{D}^{2,h_{\underline{m}}}w\big]^R\big)\longrightarrow f$, a.e. on Ω as $\underline{m}\to\infty$. By using Remark 35V), we have for any $R>0$ that $\mathcal{F}\big(\cdot,\big[\Xi\mathrm{D}^{2,h_{\underline{m}}}w\big]^R\big)\longrightarrow f$, a.e. on Ω as $\underline{m}\to\infty$. Since w is in $(\mathscr{W}^{2,2}\cap \mathscr{W}^{1,2}_0)(\Omega,\Sigma)$, by the properties of the fibre space we get $\Xi\mathrm{D}^{2,h_{\underline{m}}}w\longrightarrow G^2(w)$ in L^2 and hence a.e. on Ω along perhaps subsequences. By passing to the limit as $\underline{m}\to\infty$ and then as $R\to\infty$, we obtain that $\mathcal{F}(\cdot,G^2(w)) = f$, a.e. on Ω. Hence, $w\equiv u$. ∎

By recalling Remark 36 regarding the boundary trace values of maps in the fibre space, we conclude that the proof of Theorem 34 is now complete. □

Remark 46 (Regularity of $\mathcal{D}$-solutions)**.** In a sense, Claim 45 says that all diffuse hessians of the $\mathcal{D}$-solution u when restricted to the subspace of non-degeneracies have the "functional" representation $\mathrm{G}^2(u)$ *inside the coefficients*. Indeed, by decomposing $\mathbb{R}^{Nn^2}_s = \Xi \oplus \Xi^\perp$, the restriction of any $\mathcal{D}^2 u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$ to Ξ is given by the fibre hessian: $\mathcal{D}^2 u(x) \llcorner \Xi = \delta_{G^2 u(x)}$, a.e. $x \in \Omega$. Although such a simple representation is not possible in general (compare e.g. with Theorems 27, 30), it is expected that weaker versions of such results should be true (see Proposition 12).

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