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# On torsion of class groups of CM tori.

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## Abstract

Let  $T$  be an algebraic torus over  $\mathbb{Q}$  such that  $T(\mathbb{R})$  is compact. Assuming the Generalised Riemann Hypothesis, we give a lower bound for the size of the class group of  $T$  modulo its  $n$ -torsion in terms of a small power of the discriminant of the splitting field of  $T$ . As a corollary, we obtain an upper bound on the  $n$ -torsion in that class group. This generalises known results on the structure of class groups of CM fields.

## 1 Introduction.

This paper is motivated by Zhang’s “ $\epsilon$ -conjecture”, found in [11], proposing that the size of  $n$ -torsion in the class groups of CM fields of fixed degree grows slower than any positive power of the discriminant:

**Conjecture 1.1 ( $\epsilon$ -conjecture)** *Fix a totally real number field  $F$  and a positive integer  $n$ . Then, for any real  $\epsilon > 0$ , there exists a constant  $C(\epsilon)$  such that, for any quadratic CM extension  $L$  and any order  $\mathcal{O}$  of  $L$  containing the ring of integers of  $F$ , the  $n$ -torsion of the class group of  $\mathcal{O}$  has the following bound:*

$$\#\text{Pic}(\mathcal{O})[n] \leq C(\epsilon) \text{disc}(\mathcal{O})^\epsilon.$$

Recent results on torsion of the class groups of number fields, due to Ellenberg and Venkatesh, can be found in [3].

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Understanding  $n$ -torsion in the class groups of CM tori arises as a natural problem in the study of Shimura varieties. In particular, it is part of several strategies for proving the André-Oort conjecture. The Galois action on special points of Shimura varieties is given by reciprocity morphisms of CM tori. To give a lower bound for the size of the orbits, one needs to bound the size of the images of the induced maps on class groups. Lower bounds for the size of class groups modulo  $n$ -torsion yield estimates on these quantities (see [6] and [7] for further details). Our results are primarily of this form.

Note that the Mumford-Tate group of a CM point is a CM torus. We work in the slightly more general setting of algebraic tori over  $\mathbb{Q}$  whose real points are compact. Their splitting fields are CM fields. Let  $\mathbb{A}_f$  denote the finite adeles over  $\mathbb{Q}$ . For an arbitrary algebraic torus  $M$  over  $\mathbb{Q}$ , we denote by  $K_M^m$  the maximal compact open subgroup of  $M(\mathbb{A}_f)$ . For any such torus we denote by  $h_M$  its class group i.e.

$$h_M = M(\mathbb{Q}) \backslash M(\mathbb{A}_f) / K_M^m.$$

Given any  $n \in \mathbb{N}$  we denote by  $h_M[n]$  the  $n$ -torsion. For an arbitrary number field  $F$ , we denote by  $\Delta_F$  the absolute value of the discriminant of  $F$ . Assuming the Generalised Riemann Hypothesis for CM fields we obtain the following bound.

**Theorem 1.2** *Assume the Generalised Riemann Hypothesis for CM fields. Let  $T$  be an algebraic torus over  $\mathbb{Q}$  of dimension  $d$ , with splitting field  $L$ , such that  $T(\mathbb{R})$  is compact. Then we have*

$$|h_T/h_T[n]| \gg_{\epsilon,d} \Delta_L^{\frac{c}{2^n} + \epsilon},$$

for all  $\epsilon > 0$ , where  $c$  is a positive constant depending only on  $d$ .

By splitting field we refer to the smallest field over which  $T$  becomes a product of copies of  $\mathbb{G}_m$ . Our method relies on the fact that  $T$  is isogenous to a product  $T_1 \times \cdots \times T_s$  of simple tori. We fix surjective maps with connected kernels from  $R := \text{Res}_{L/\mathbb{Q}} \mathbb{G}_{m,L}$  to each of the  $T_i$  and take  $r$  to be the product of these surjections followed by an isogeny to  $T$ . For a prime  $p$ , split in  $L$ ,  $R(\mathbb{Q}_p)$  is isomorphic to the cocharacter group of  $R$  tensored with  $\mathbb{Q}_p^*$ . We use GRH to find small split primes and take an arbitrary product of powers of uniformisers lying over these primes in  $R(\mathbb{A}_f)$ . After projection to the factors  $R(\mathbb{Q}_p)$ , for each such prime  $p$ , these elements are permuted by

$\text{Gal}(L/\mathbb{Q})$  over a basis for the cocharacter group. We embed these elements diagonally into the product of the  $R(\mathbb{A}_f)$  and assume that this lies in the kernel of the map induced by  $r$  on class groups i.e. the image under  $r$  is an element  $\pi k \in T(\mathbb{Q})K_T^m$ . The element  $\pi$  gives us elements  $\pi_i$  in the  $T_i(\mathbb{Q})$ . We show that  $L$  is generated over  $\mathbb{Q}$  by the images of the  $\pi_i$  under bases for the character groups of the  $T_i$ . We make them integral, scaling them by the primes  $p$  raised to the absolute values of the exponents of the uniformisers. We may take a basis for  $L$  over  $\mathbb{Q}$  in terms of these elements, whose  $\mathbb{Z}$ -span is an order in  $\mathcal{O}_L$ , the ring of integers, which relates  $\Delta_L$  to the absolute values of images of the  $\pi_i$  and their Galois conjugates. However, these absolute values are controlled by the primes found under GRH, a uniform bound on the character bases, and the exponents of the uniformisers. Since these primes are ‘small’, compared to  $\Delta_L$ , we are able to bound the exponents from below. A group theoretic argument converts this into a lower bound for the size of the class group modulo  $n$ -torsion.

Our method relies crucially on the assumption that  $T(\mathbb{R})$  is compact. However, note that, whilst the class group of a CM field  $L$  is the class group of the torus  $R_L := \text{Res}_{L/\mathbb{Q}}\mathbb{G}_m$ , whose real points are not compact, this torus lies in an exact sequence

$$1 \rightarrow M \rightarrow R_L \rightarrow \mathbb{G}_m \rightarrow 1,$$

where  $M$  is a torus over  $\mathbb{Q}$  whose real points are compact. Its  $\mathbb{Q}$ -points are precisely the elements of  $L$  of norm 1. This exact sequence of tori induces a morphism of class groups

$$h_M \rightarrow h_{R_L},$$

whose kernel has order bounded in terms of  $d$  only by [6], Theorem 5.1. We have another induced morphism from  $h_{R_L}$  to  $h_{\mathbb{G}_m} = 1$ . Again by Theorem 5.1. of [6], we deduce that

$$\frac{|h_{R_L}|}{|h_T|} \ll_{n_L, \epsilon} \Delta_L^\epsilon,$$

for any  $\epsilon > 0$ , where  $n_L = [L : \mathbb{Q}]$ . Since we have a morphism

$$h_T[n] \rightarrow h_{R_L}[n],$$

with uniformly bounded kernel, for any  $n \in \mathbb{N}$ , Theorem 1.2 applies to CM fields and, indeed, to any extension of a suitable torus by a product of copies of  $\mathbb{G}_m$ .

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## 2 Corollary on $n$ -torsion.

Before proceeding to the proof of Theorem 1.2 we give an implied upper bound on the size of  $n$ -torsion. All that is required is a simple upper bound on the class group of  $T$ . We have the following theorem.

**Theorem 2.1** *Let  $T$  be an algebraic torus over  $\mathbb{Q}$ , with dimension  $d$  and splitting field  $L$ . Then we have*

$$|h_T| \ll_{\epsilon, d} \Delta_L^{\frac{s}{2} + \epsilon},$$

for all  $\epsilon > 0$ , where  $s$  is the number of simple subtori of  $T$ .

**Proof.** Proposition 2.1. of [4] ensures that we can put  $T$  into the exact sequence

$$1 \rightarrow T \rightarrow R_L^s \rightarrow M \rightarrow 1,$$

where  $M$  is a  $\mathbb{Q}$ -torus. Theorem 5.1. of [6] ensures that the induced map on class groups

$$h_T \rightarrow h_{R^s}$$

has kernel of uniformly bounded order and the class group  $h_{R^s}$  is simply the  $s$ -fold direct product of the class group  $Cl(L)$  of  $L$ . By (1) of [2] we have

$$|Cl(L)| \ll_{\epsilon, n_L} \Delta_L^{\frac{1}{2} + \epsilon},$$

where  $n_L$  is the degree of  $L$  over  $\mathbb{Q}$ . We will later demonstrate that  $n_L$  is bounded in terms of  $d$  only, which completes the proof.  $\square$

The combination of Theorems 2.1 and 1.2 yield the following bound on the size of  $n$ -torsion in the class group.

**Theorem 2.2** *Assume the Generalised Riemann Hypothesis for CM fields. Let  $T$  be an algebraic torus over  $\mathbb{Q}$  of dimension  $d$ , with splitting field  $L$ , such that  $T(\mathbb{R})$  is compact. Then we have*

$$|h_T[n]| \ll_{\epsilon, d} \Delta_L^{\frac{s}{2} - \frac{c}{2n} + \epsilon},$$

for all  $\epsilon > 0$ , where  $c$  is a positive constant depending only on  $d$  and  $s$  is the number of simple subtori of  $T$ .

### 3 A group theoretic argument.

The proof of theorem 1.2 will combine the ideas of two papers, [1] and [10]. Following [1], for an arbitrary Abelian group  $G$  and  $l \in \mathbb{N}$ , let  $\mathcal{M}_G(l)$  be the smallest integer  $A$  such that for any  $l$  elements  $g_1, \dots, g_l \in G$ , not necessarily distinct, there exist  $a_1, \dots, a_l \in \mathbb{Z}$ , not all zero, with  $\sum_{i=1}^l |a_i| \leq A$ , such that

$$g_1^{a_1} \cdots g_l^{a_l} = 1.$$

In what follows we will demonstrate the following:

$$\mathcal{M}_{h_T}(l) > \frac{c \log \Delta_L}{\log(l) + \log \log \Delta_L}, \quad (1)$$

for any  $l \in \mathbb{N}$ , provided  $\Delta_L$  is greater than a constant depending only on  $d$ . Here we prove that inequality (1) implies Theorem 1.2.

**Proof.** We follow the proof of Lemma 5.1. of [1]. Let  $G$  be a finite Abelian group, set  $l = |G|$ , and take  $g_1, \dots, g_l \in G$ . If  $g_i = 1$  for some  $i \in \{1, \dots, l\}$  then we clearly have a nontrivial relation between the  $g_i$  and so  $A = 1$  with  $A$  defined as it was above. Otherwise, an element of  $G$  appears twice amongst our  $g_i$  and there exist  $i$  and  $j$  such that  $i \neq j$  and  $g_i g_j^{-1} = 1$ . Either way, we have a nontrivial relation with  $A \leq 2$ . Hence, we have shown that

$$\mathcal{M}_G(|G|) \leq 2.$$

Henceforth, let  $G = h_T/h_T[n]$ . Then, by Lemma 5.1. (iii) of [1], we have

$$\mathcal{M}_{h_T}(|G|) \leq n \mathcal{M}_G(|G|)$$

and, therefore, by the preceding argument,

$$\mathcal{M}_{h_T}(|G|) \leq 2n.$$

Substituting  $\mathcal{M}_{h_T}(|G|)$  into (1), we obtain the desired result

$$l = |h_T/h_T[n]| > \frac{\Delta_L^{\frac{c}{2n}}}{\log \Delta_L},$$

provided  $\Delta_L > 2$ . □

The remainder of this paper is devoted to the proof of (1).

## 4 Covering $T$ .

For an arbitrary algebraic torus  $M$  over  $\mathbb{Q}$ , we denote by  $X^*(M)$  its character group i.e. the free  $\mathbb{Z}$ -module  $\text{Hom}(M_{\overline{\mathbb{Q}}}, \mathbb{G}_{m, \overline{\mathbb{Q}}})$  with the natural Galois action. The corresponding representation

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL(X^*(M)),$$

has kernel  $\text{Gal}(\overline{\mathbb{Q}}/F)$ , for some finite, Galois extension  $F$ , which we refer to as the *splitting field* of  $M$ .

We denote by  $X_*(M)$  the group of cocharacters of  $M$ , by which we refer to the  $\mathbb{Z}$ -module  $\text{Hom}(\mathbb{G}_{m, \overline{\mathbb{Q}}}, M_{\overline{\mathbb{Q}}})$ , again assuming a  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action in the natural way, also factoring through  $\text{Gal}(F/\mathbb{Q})$ . There is a natural bilinear map

$$X_*(M) \times X^*(M) \rightarrow \mathbb{Z},$$

identifying  $X_*(M)$  with the dual  $\mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ -module  $\text{Hom}(X^*(M), \mathbb{Z})$  of  $X^*(M)$ .

Recall that we have a semisimple category whose objects are algebraic tori and whose morphisms are the usual homomorphisms of algebraic tori (which form an abelian group) tensored with  $\mathbb{Q}$ . Therefore, our given  $T$  is isogenous to a product of tori  $T_1 \times \cdots \times T_s$ , where the  $T_i$  are simple i.e. they contain no proper subtori. Each  $T_i$  splits over a Galois extension  $L_i$  and the compositum of these fields is  $L$ . Two algebraic tori are isogenous precisely when their character groups are isomorphic when tensored with  $\mathbb{Q}$ .

Consider the torus  $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m, F}$ , derived from the multiplicative group  $\mathbb{G}_{m, F}$  over a number field  $F$  by restriction of scalars to  $\mathbb{Q}$ . Tori of this form,

along with their finite direct products, are often called *quasisplit*. They are characterised by the property that their character groups are permutation modules with respect to their Galois action. For example, the character group of  $R_L$  is  $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$ . In this regard, these tori are the easiest to study. We will make use of their tractability via the following special case of a general result (Proposition 2.2.) found in [4].

**Lemma 4.1** *Let  $M$  be a simple algebraic torus over  $\mathbb{Q}$ , split over a Galois field  $F$ . Then  $M$  can be covered by the quasisplit torus*

$$R_F := \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}.$$

*That is,  $M$  can be put into an exact sequence*

$$1 \rightarrow N \rightarrow R_F \rightarrow M \rightarrow 1,$$

*where  $N$  is a  $\mathbb{Q}$ -torus.*

**Proof.** The fact that  $M$  may be covered by some finite product  $R_F^l$  is Proposition 2.2. of [4]. However, since the Hom functor commutes with products, an element of

$$\text{Hom}(R_F^l, M)$$

is a product of morphisms into  $M$ , each of which has an image constituting a subgroup of  $M$ . Since  $M$  is simple, all but one of these images must be trivial i.e. we may assume  $l = 1$ .  $\square$

In light of this, each  $T_i$  may be covered by a copy of  $R_L$ . Any such morphism of tori is equivalent to an injection of character groups

$$\xi : X^*(T_i) \hookrightarrow X^*(R_L).$$

We have proved the existence of such embeddings, but have not specified one precisely. We identify  $X^*(R_L)$  with  $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$  and choose a basis by enumerating elements of the Galois group. We denote this basis

$$\{\psi_1, \dots, \psi_{n_L}\}.$$

We define an inner product on  $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$  by letting

$$\langle \psi_i, \psi_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$



and extending  $\mathbb{Z}$ -bilinearly.

Consider one of the  $T_i$ . For a fixed embedding

$$\xi : X^*(T_i) \hookrightarrow X^*(R_L),$$

and a chosen basis

$$\{\chi_1, \dots, \chi_{d_i}\},$$

where  $d_i$  will denote the dimension of  $T_i$ , let  $m_\xi$  denote

$$\max\{|\langle \chi_j, \psi_k \rangle|\} \in \mathbb{N},$$

for  $j = 1, \dots, d_i$  and  $k = 1, \dots, n_L$ . For each  $T_i$  we choose an embedding of  $X^*(T_i)$  into  $X^*(R_L)$  and a basis such that  $m_\xi$  is minimal i.e. the coordinates of this basis, with respect to the canonical basis of  $X^*(R_L)$ , have the smallest upper bound amongst all possible choices.

Thus, we have a collection of surjective maps of tori

$$R_L \rightarrow T_i.$$

We consider their direct product, yielding another surjection

$$R_L^s \rightarrow T_1 \times \dots \times T_s.$$

Now, since  $T$  is isogenous to  $T_1 \times \dots \times T_s$ , we may choose a surjection

$$\lambda : T_1 \times \dots \times T_s \rightarrow T,$$

with kernel of smallest degree, which we will denote by  $n_\lambda$ .

We denote the composition of our product map with  $\lambda$  as

$$r : R_L^s \rightarrow T,$$

another surjective morphism.

**Lemma 4.2** *Consider the morphism*

$$f : R_L^s \rightarrow T_1 \times \dots \times T_s,$$

*the direct product of the previously defined surjections of  $R$  on to the  $T_i$ , each followed by raising to the power  $n_\lambda$ . Then there exists a unique morphism,*

$$g : T \rightarrow T_1 \times \dots \times T_s,$$

*such that  $f = g \circ r$ .*

**Proof.** Let  $S$  be the kernel of  $r$ . By the universal property of quotients, any morphism from  $R_L^s$  vanishing on  $S$  factors uniquely through  $r$ .  $\square$

## 5 Uniform boundedness.

Firstly, we recall a standard property of integral matrix groups due to Minkowski.

**Theorem 5.1** *For any  $d \in \mathbb{N}$ , the number of isomorphism classes of finite groups contained in  $GL_d(\mathbb{Z})$  is finite.*

We have fixed an algebraic torus  $T$  over  $\mathbb{Q}$  of dimension  $d$  with splitting field  $L$ . In other words, for any choice of basis, we have a faithful representation

$$\rho : Gal(L/\mathbb{Q}) \hookrightarrow GL_d(\mathbb{Z}).$$

Thus, by the previous theorem,  $n_L = |Gal(L/\mathbb{Q})|$  is bounded in terms of the dimension of  $T$  only.

Secondly, we refer to a standard result from the theory of integral representations of finite groups [6].

**Theorem 5.2** *Let  $H$  be a finite group and let  $d \in \mathbb{N}$ . Then the number of isomorphism classes of integral representations of  $H$  of dimension  $d$  is finite.*

Recall, then, that we chose a surjection

$$\lambda : T_1 \times \cdots \times T_s \rightarrow T,$$

with kernel of smallest degree  $n_\lambda$ . Tori over  $\mathbb{Q}$  with splitting field  $L$  induce  $d$ -dimensional representations of the Galois group of  $L$ . By Theorem 5.1, there are only finitely many choices for this group. Therefore, by Theorem 5.2, only finitely many isomorphism classes of such tori exist. Therefore,  $n_\lambda$  is bounded in terms of  $d$  only.

Recall, also, that we constructed

$$f : R_L^s \rightarrow T_1 \times \cdots \times T_s,$$

via the composition of our original direct product of surjections and raising to the power  $n_\lambda$ . This corresponds to embeddings

$$X^*(T_i) \hookrightarrow X^*(R_L),$$

for each  $i$ . We have chosen the canonical basis for  $X^*(R_L)$  and we choose the bases for the  $X^*(T_i)$  to be the bases chosen in the previous section, all multiplied by  $n_\lambda$ . The previously stated results yield the following:

**Lemma 5.3** *The coordinates of these bases for the  $X^*(T_i)$  with respect to the canonical basis of  $X^*(R_L)$  are bounded in terms of  $d$  only.*

## 6 Uniformisers.

We have identified  $X^*(R_L)$  with  $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$  and chosen the canonical basis  $\{\psi_1, \dots, \psi_{n_L}\}$  by enumerating the elements of the Galois group. The inner product on  $X^*(R_L)$  satisfies the invariance property

$$\langle \sigma\psi, \psi' \rangle = \langle \psi, \sigma^{-1}\psi' \rangle,$$

for any  $\sigma \in \text{Gal}(L/\mathbb{Q})$  and  $\psi, \psi' \in X^*(R_L)$ . Now,  $X^*(R_L)$  is naturally isomorphic to its dual  $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$ -module  $\text{Hom}(X^*(R_L), \mathbb{Z})$ , sending  $\psi_i$  to  $\langle \psi_i, - \rangle$ , which we denote  $\varphi_i$ , and extending  $\mathbb{Z}$ -linearly. Via our perfect pairing

$$X_*(R_L) \times X^*(R_L) \rightarrow \mathbb{Z},$$

we have an isomorphism of  $X_*(R_L)$  with the dual of  $X^*(R_L)$ . We identify  $\varphi_i$  with its image in  $X_*(R_L)$ . Thus, we obtain a basis

$$\{\varphi_1, \dots, \varphi_{n_L}\}$$

of  $X_*(R_L)$ , which is that obtained by enumerating the elements of  $\text{Gal}(L/\mathbb{Q})$ .

Let  $p$  be a rational prime, completely split in  $L$ . The basis

$$\{\chi_1, \dots, \chi_{n_L}\}$$

induces an isomorphism of  $R_L(\mathbb{Q}_p)$  with

$$\prod \mathbb{Q}_p^* \approx X_*(R_L) \otimes \mathbb{Q}_p^*.$$

Let  $P$  be the element of  $R_L(\mathbb{Q}_p)$  such that  $\chi_1(P) = p$  and  $\chi_i(P) = 1$  for  $i = 2, \dots, n_L$ . In fact,  $P$  is a uniformiser corresponding to a place lying above  $p$  and the Galois orbit of its image under the above isomorphism corresponds to a complete set of uniformisers at places lying above  $p$ .

We have a morphism from  $X_*(R_L) \otimes \mathbb{Q}_p^*$  to  $X_*(R_L)$ , applying the valuation map

$$v_p : \mathbb{Q}_p^* \rightarrow \mathbb{Z}$$

to each factor. Under this morphism,  $P$  is sent to the basis element  $\varphi_1$  and the Galois orbit of  $P$  yields the complete set of basis elements.

Now let  $l$  be a natural number and let  $p_1, \dots, p_l$  be rational primes completely split in  $L$ . For each  $p_i$ , let  $P_i$  be the element of  $R_L(\mathbb{Q}_{p_i})$  associated to  $p_i$  via the above construction. We embed each  $R_L(\mathbb{Q}_{p_i})$  into

$$R_L(\mathbb{A}_f) = (\mathbb{A}_f \otimes L)^*$$

in the natural way. For integers  $a_1, \dots, a_l$ , we consider the element

$$I = P_1^{a_1} \dots P_l^{a_l}$$

belonging to  $R_L(\mathbb{A}_f)$ .

Recall that we have a surjective map of  $\mathbb{Q}$ -tori,

$$r : R_L^s \rightarrow T.$$

Following [7], we have an induced map on the corresponding class groups, which we denote

$$r_h : h_{R_L^s} \rightarrow h_T.$$

Let  $H = h_{R_L^s} / \ker r_h$ . We will show that

$$\mathcal{M}_H(l) > \frac{c \log \Delta_L}{\log(l) + \log \log \Delta_L},$$

for any  $l \in \mathbb{N}$ , provided  $\Delta_L$  is greater than a uniform constant. Since  $H$  injects into  $h_T$ , it is an easy observation that  $\mathcal{M}_{h_T}(l) \geq \mathcal{M}_H(l)$ , for  $l \in \mathbb{N}$ , thus yielding (1).

Henceforth, let  $I_i$  denote the embedding of  $I$  into the  $i^{\text{th}}$  factor of  $R_L^s(\mathbb{A}_f)$ . We denote by  $\underline{I}$  the product of the  $I_i$  i.e.  $I$  embedded diagonally into  $R_L^s(\mathbb{A}_f)$ . Recall that, by Lemma 4.2, we have the following commutative diagram.

$$\begin{array}{ccccc} R_L^s(\mathbb{A}_f) & \twoheadrightarrow & T_1(\mathbb{A}_f) \times \dots \times T_s(\mathbb{A}_f) & \twoheadrightarrow & T(\mathbb{A}_f) \\ & \searrow f & & & \swarrow g \\ & & T_1(\mathbb{A}_f) \times \dots \times T_s(\mathbb{A}_f) & & \end{array}$$

with  $r$  being the composite homomorphism from  $R_L^s(\mathbb{A}_f)$  to  $T(\mathbb{A}_f)$ .

Assume that  $\underline{I}$  belongs to the kernel of  $r_h$  i.e.

$$r(\underline{I}) = \pi k \in T(\mathbb{Q})K_T^m,$$

where this product is unique up to an element of  $T(\mathbb{Q}) \cap K_T^m$ , which, by Theorem 2.5 of [10], is a finite group of order bounded in terms of the dimension of  $T$  only. Now,  $f(\underline{I}) = g(\pi k)$ , but  $f$  is a product of morphisms from  $R_L(\mathbb{A}_f)$  into the  $T_i(\mathbb{A}_f)$ , so we write  $f(\underline{I})$  as

$$(f_1(I), \dots, f_s(I)).$$

As  $g$  is a morphism into  $T_1(\mathbb{A}_f) \times \cdots \times T_s(\mathbb{A}_f)$ , we write  $g(\pi k)$  as

$$(g_1(\pi k), \dots, g_s(\pi k)).$$

Thus, we have

$$f_i(I) = \pi_i k_i \in T_i(\mathbb{Q})K_{T_i}^m,$$

for  $i = 1, \dots, s$ , where  $\pi_i = g_i(\pi)$  and  $k_i = g_i(k)$ .

Now, recall the bases for the  $X^*(T_i)$  under the embeddings into  $X^*(R_L)$  induced by the  $f_i$ . We denote their elements as  $\chi_{i,j}$ , where  $i = 1, \dots, s$  and  $j = 1, \dots, d_i$ . Furthermore, let

$$\pi_{i,j} = \chi_{i,j}(\pi_i).$$

The following lemma is a generalisation of Lemma 2.13 of [10] to the case of a torus covered by an arbitrary finite product of quasisplit tori.

**Lemma 6.1** *The field  $L'$  generated over  $\mathbb{Q}$  by the  $\pi_{i,j}$  is  $L$ .*

**Proof.** Clearly  $L' \subseteq L$ , so let  $\sigma \in \text{Gal}(L/\mathbb{Q})$  and assume that  $\sigma$  acts trivially on  $L'$ . We need to show that such a  $\sigma$  is itself trivial. We have a faithful representation of  $\text{Gal}(L/\mathbb{Q})$  on the product of the  $X_*(T_i)$  and, thus, it is equivalent to show that  $\sigma$  acts trivially on each  $X_*(T_i) \otimes \mathbb{Q}$ .

Recall our surjective maps of tori from  $R_L$  to  $T_i$ , which we denoted  $f_i$ . After tensoring our cocharacter modules with  $\mathbb{Q}$  we retrieve short exact sequences

$$0 \rightarrow \mathbb{Q} \otimes \Delta_i \rightarrow \mathbb{Q} \otimes \Gamma \rightarrow \mathbb{Q} \otimes X_*(T_i) \rightarrow 0,$$

where  $\Gamma = \mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$  and the  $\Delta_i$  are  $\Gamma$ -submodules of  $X_*(R_L) \cong \Gamma$ . They correspond to the kernels of our surjections, which we denote  $N_i$ . We will show that  $\sigma$  acts trivially on the

$$\mathbb{Q} \otimes (X_*(R_L)/X_*(N_i)).$$

We will consider in turn the elements  $I_i$ . Since  $f$  is a product of the maps  $f_i$ , we will consider the  $I_i$  as belonging to  $R_L(\mathbb{A}_f)$  mapping into  $T_i(\mathbb{A}_f)$ . The elements under scrutiny here are the  $\pi_i \in T_i(\mathbb{Q})$ , which are diagonally embedded in  $T_i(\mathbb{A}_f)$ . Therefore, we relabel  $p_1, P_1$  and  $a_1$  as  $p, P$  and  $a$ , respectively, and project from  $R_L(\mathbb{A}_f)$  to  $R_L(\mathbb{Q}_p)$  i.e. we turn our attention from  $I_i$  to its image  $P^a \in R(\mathbb{Q}_p)$ , mapping under  $f_i$  to  $\pi_i k_{i,p}$ , where  $k_{i,p}$  is the  $p$ -component of  $k_i$ .

Since  $p$  splits each  $T_i$ , we also have isomorphisms

$$T_i(\mathbb{Q}_p) \approx X_*(T_i) \otimes \mathbb{Q}_p^*,$$

sending  $x \in T_i(\mathbb{Q}_p)$  to

$$(\chi_{i,1}(x), \dots, \chi_{i,d_i}(x)),$$

where we tacitly assume a choice

$$\{\mu_{i,1}, \dots, \mu_{i,d_i}\}$$

of the natural dual basis to our character basis already chosen; as described earlier for  $R_L$ . The valuation  $v_p : \mathbb{Q}_p^* \rightarrow \mathbb{Z}$  evaluates each factor, inducing an isomorphism between  $T_i(\mathbb{Q}_p)/K_{T_i,p}^m$  and  $X_*(T_i)$ , where  $K_{T_i,p}^m$  is the maximal compact open subgroup of  $T_i(\mathbb{Q}_p)$ .

We have the following commutative diagrams

$$\begin{array}{ccc} R_L(\mathbb{Q}_p) & \longrightarrow & X_*(R_L) \\ \downarrow & & \downarrow \\ T_i(\mathbb{Q}_p)/K_{T_i,p}^m & \longrightarrow & X_*(T_i), \end{array}$$

where the bottom arrow is the isomorphism just mentioned and the top arrow is the corresponding quotient version for  $R_L(\mathbb{Q}_p)$  described earlier. The righthand arrow is the map of cocharacters induced by  $f_i$  and the left arrow is  $f_i$  composed with factoring out by  $K_{T_i,p}^m$ .

The element  $P^a$  is mapped to the class of  $\pi_i$  in  $T_i(\mathbb{Q}_p)/K_{T_i,p}^m$ , which is mapped to

$$(v_p(\chi_{i,1}(\pi_i)), \dots, v_p(\chi_{i,d_i}(\pi_i))) \in X_*(T_i).$$

On the other hand,  $P^a$  is mapped to

$$(v_p(\psi_1(P^a)), \dots, v_p(\psi_{n_L}(P^a))) \in X_*(R_L),$$

which is  $a\varphi_1$ . In this form, the action of a  $\tau \in \text{Gal}(L/\mathbb{Q})$  is clear, sending this image to

$$(v_p((\tau^{-1}\psi_1)(P^a)), \dots, v_p((\tau^{-1}\psi_{n_L})(P^a))).$$

Note that, since  $\text{Gal}(L/\mathbb{Q})$  permutes the characters, the Galois orbit of the image of  $P^a$  comprises precisely the elements  $a\varphi_i$ , which constitute a basis for  $X_*(R_L) \otimes \mathbb{Q}$ .

We claim that the image of this orbit in  $X_*(T_i)$  consists of the elements

$$(v_p((\tau^{-1}\chi_{i,1})(\pi_i)), \dots, v_p((\tau^{-1}\chi_{i,d_i})(\pi_i))) \in X_*(T_i). \quad (2)$$

To see this, let  $\tau\mu_{i,j}$  be denoted by

$$\sum_{k=1}^{d_i} n_{j,k}^{i,\tau} \mu_{i,k}.$$

Thus, the image of

$$\sum_{j=1}^{d_i} \chi_{i,j}(\pi_i) \mu_{i,j} \in X_*(T_i) \otimes \mathbb{Q}_p^*$$

under  $\tau \in \text{Gal}(L/\mathbb{Q})$  will be

$$\sum_{k=1}^{d_i} \sum_{j=1}^{d_i} n_{j,k}^{i,\tau} \chi_{i,j}(\pi_i) \mu_{i,k}.$$

The  $k^{\text{th}}$  coefficient here is equal to  $(\tau^{-1}\chi_{i,k})(\pi_i)$  if we have  $n_{k,j}^{i,\tau^{-1}} = n_{j,k}^{i,\tau}$ , for all  $j, k = 1, \dots, d_i$ , but this is simply the Galois invariance of the inner product we previously placed on  $X^*(R_L)$ .

Now, since  $\sigma$  fixes each  $\pi_{i,j}$  and the characters  $\chi_{i,j}$  are the basis of a  $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$ -module,  $\sigma$  clearly fixes each of the elements of  $X_*(T_i)$  depicted in (2). Thus, by our exact sequence,  $\sigma$  fixes the Galois orbit of the image of  $P^a$  in

$$\mathbb{Q} \otimes (X_*(R_L)/X_*(S_i)).$$

Since these elements span the above space, the claim follows. □

**Remark 6.2** *It is worth noting that any element in  $R_L^s(\mathbb{A}_f)$  with a nonzero valuation at a place lying above a split prime in each of the  $s$  factors produces generators for  $L$  via this argument.*

## 7 Small split primes.

The remainder of the proof follows the concluding pages of [7].

We have  $I = P_1^{a_1} \cdots P_l^{a_l} \in R_L(\mathbb{A}_f)$  embedded diagonally into  $R_L^s(\mathbb{A}_f)$ . We denote this element  $\underline{I}$ . The image of  $\underline{I}$  in  $T_1(\mathbb{A}_f) \times \cdots \times T_l(\mathbb{A}_f)$  under  $f$  is

$$f(\underline{I}) = (\pi_1 k_1, \dots, \pi_l k_l) \in T_1(\mathbb{Q})K_{T_1}^m \times \cdots \times T_l(\mathbb{Q})K_{T_l}^m.$$

We consider the images  $\pi_{i,j}$  of the  $\pi_i$  under the elements  $\chi_{i,j}$  of the character bases. Due to Lemma 5.3, we have a bound  $B$ , say, on the coordinates of these basis elements with respect to the canonical basis of  $X^*(R_L)$ . This bound is dependent on  $d$  only. We replace the  $\pi_{i,j}$  by  $(p_1^{|a_1|} \cdots p_l^{|a_l|})^B \pi_{i,j}$ , which belong to  $\mathcal{O}_L$ , the ring of integers of  $L$ .

By virtue of Lemma 6.1, we may form a primitive element

$$\alpha = \sum_{i,j} a_{i,j} \pi_{i,j}$$

for the field  $L$ , where the  $a_{i,j}$  are integers with absolute value bounded by some constant  $E$  depending only on  $d$ . We take the  $\mathbb{Q}$ -basis  $\{1, \alpha, \alpha^2, \dots, \alpha^{n_L-1}\}$  for  $L$ . Consequently,  $\mathbb{Z}[1, \alpha, \alpha^2, \dots, \alpha^{n_L-1}]$  is an order in  $\mathcal{O}_L$ . We denote the absolute value of its discriminant as  $\Delta'_L$ . Therefore,

$$\Delta'_L \geq \Delta_L.$$

On the other hand,  $\Delta'_L$  is the square of the determinant of the matrix

$$(\tau_i(\alpha^j))_{i,j},$$

where the  $\tau_i$  range over the elements of  $\text{Gal}(L/\mathbb{Q})$  and  $0 \leq j \leq n_L - 1$ . An inequality of Hadamard then states that, given an upper bound  $C$  on the values of

$$|\tau_i(\alpha^j)|,$$

then

$$\Delta'_L \leq n_L^{n_L} C^{2n_L} \leq c_1 C^{n_L},$$

where  $c_1 > 0$  is a constant bounded in terms of  $d$  only.

The splitting field  $L$  of  $T$  is a Galois CM-field. For a character  $\chi$ , we denote by  $\bar{\chi}$  the image of  $\chi$  under the automorphism of  $L$  induced by complex conjugation on  $\mathbb{C}$ . It is at this point that we use the fact that  $T(\mathbb{R})$  is compact and is, therefore, a product of circles (see [8], Section 10.1). This implies that  $\chi_{i,j} \bar{\chi}_{i,j}$  is the trivial character for every  $i$  and  $j$  (writing the group law multiplicatively).



Thus, for each  $\tau \in \text{Gal}(L/\mathbb{Q})$ ,

$$|\tau(\pi_{i,j})| = (p_1^{|a_1|} \cdots p_l^{|a_l|})^B$$

and, therefore, by the preceding discussion,

$$|\tau_i(\alpha^j)| \leq (dE(p_1^{|a_1|} \cdots p_l^{|a_l|}))^{B(n_L-1)}.$$

Hence, our calculation yields

$$c_1^{-1} \Delta_L \leq (dE(p_1^{|a_1|} \cdots p_l^{|a_l|}))^{2Bn_L(n_L-1)}.$$

It remains to find the  $p_i$  for any given  $l$ . We require the following corollary of a special case of the effective Chebotarev Density Theorem, a proof of which can be found in [1].

**Theorem 7.1** *Let  $F$  be any number field. Assume the Generalised Riemann Hypothesis holds for the Dedekind zeta function of  $F$ . Let  $\pi_F(x)$  denote the number of rational primes  $p$  completely split in  $F$  such that  $p \leq x$ . There exist positive, absolute constants  $c_2$  and  $c_3$  that are effectively calculable such that, for all  $x \geq c_2(\log \Delta_F)^2(\log \log \Delta_F)^4$ ,*

$$\pi_F(x) \geq c_3 \frac{x}{\log x}.$$

We assume the Generalised Riemann Hypothesis for  $CM$  fields. Let  $l$  be any natural number. We require at least  $l$  primes completely split in  $L$ , so let

$$x = c_4 l \log l + c_1 (\log \Delta_L)^2 (\log \log \Delta_L)^4 > 1,$$

where  $c_4$  is a positive, absolute constant, such that

$$c_3 \frac{x}{\log x} \geq l.$$

It is uniform since

$$\frac{x}{\log x} \geq \frac{c_4 l \log l}{\log c_4 + 2 \log l},$$

and so our requirement is satisfied when, for example,  $\frac{c_3 c_4}{\log c_4 + 2} \geq 1$ .

Thus, by Theorem 7.1, we are able to find  $l$  primes  $p_1, \dots, p_l$  completely split in  $l$  such that  $p_i \leq x$ . Subsequently, we return to our inequality

$$c_1^{-1} \Delta_L \leq (dE(p_1^{|a_1|} \cdots p_l^{|a_l|}))^{2Bn_L(n_L-1)},$$

choosing for the  $p_i$  those just found. As before, we let

$$A = \sum_{i=1}^l |a_i|,$$

yielding

$$\log(F^{-1}\Delta_L) \leq 2ABn_L(n_L - 1) \log x,$$

provided  $\Delta_L > F$ , where  $F = c_1(dE)^{2Bn_L(n_L-1)}$ . Now, there exists a positive, absolute constant  $c_5$  such that

$$\log x \leq c_5(\log l + \log \log \Delta_L),$$

provided  $\Delta_L$  exceeds a positive, absolute constant. Combining these two inequalities yields a lower bound for  $A$ , which implies (1).

□

**Remark 7.2** *The constant  $c$  given in the statement of (1) will be  $\frac{c_6}{2Bn_L(n_L-1)}$ , where  $c_6$  can be taken to be  $1 - \epsilon$ , for any  $0 < \epsilon < 1$ , with each value of  $\epsilon$  setting a uniform lower bound on the size of  $\Delta_L$ .*

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