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Degrees of strongly special subvarieties and the André–Oort conjecture

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Abstract. In this paper we give a new proof of the André–Oort conjecture under the generalised Riemann hypothesis. In fact, we generalise the strategy pioneered by Edixhoven, and implemented by Klingler and Yafaev, to all special subvarieties. Thus, we remove ergodic theory from the proof of Klingler, Ullmo and Yafaev and replace it with tools from algebraic geometry. Our key ingredient is a lower bound for the degrees of strongly special subvarieties coming from Prasad's volume formula for S-arithmetic quotients of semisimple groups.

1. Introduction

Definition 1.1. Given a set Σ of special subvarieties of a Shimura variety S, we denote by $\mathbf{\Sigma}$ the subset $\bigcup_{V\in\Sigma} V$ of S.

This paper is concerned with the following conjecture:

Conjecture 1.2 (André–Oort). Let S be a Shimura variety and let Σ be a set of special points in S. Every irreducible component of the Zariski closure of $\mathbf{\Sigma}$ in S is a special subvariety.

For the definition of Shimura varieties, special points and special subvarieties we refer the reader to the introduction of [9]. The current paper is complementary to the aforementioned article in the following sense: Klingler and Yafaev consider the above conjecture with Σ replaced by a set of special subvarieties, rather than just points. Via extra machinery developed by Ullmo and Yafaev [19], the authors prove the conjecture, assuming the generalised Riemann hypothesis (GRH), by repeatedly replacing the elements of Σ with higher dimensional special subvarieties. They rely on a lower bound, obtained by Ullmo and Yafaev, on the degree of the Galois orbit of a special subvariety. As one ranges through the elements of Σ, this bound is either bounded from above or tends to infinity. In the case that it tends to infinity, the authors are able to proceed using their generalisation of a method pioneered by Edixhoven that compares Galois orbits and Hecke correspondences. Though technical, the proof relies on a simple geometric idea.

In the case that the lower bound is bounded for all elements in Σ, Klingler and Yafaev appeal to a result by Ullmo and Yafaev [19], generalising the equidistribution of strongly special subvarieties demonstrated by Clozel and Ullmo [4]. Our motivation was to remove this element of the proof, thus eliminating the dependency on the extremely deep and complicated theorems of Ratner. In this paper, we achieve this aim, thus reproving the following theorem of Clozel and Ullmo:

Theorem 1.3. *Let Z be a subvariety of a Shimura variety S. There exists a finite set $\{V_1, \dots, V_k\}$ of positive dimensional, strongly special subvarieties $V_i \subset Z$ such that, if $V \subset Z$ is a positive dimensional, strongly special subvariety, then $V \subset V_i$ for some $i \in \{1, \dots, k\}$.*

Therefore, under the GRH, we are able to prove the André–Oort conjecture solely via the geometric strategy of Edixhoven. In fact, the case dealt with here is less technical and does not depend on the GRH. We employ similar tools from algebraic geometry and the theory of reductive groups over local fields. The main ingredient is the following lower bound for the degrees of strongly special subvarieties. We refer the reader to Sections 2, 3, 4 and 6 for the relevant definitions and explanations.

Theorem 1.4. *Let (G, X) be a Shimura datum such that $G = G^{\mathrm{ad}}$ and fix a connected component X^+ of X. Fix a faithful representation*

$$\rho : G \hookrightarrow \mathrm{GL}_n$$

and let K be a neat compact open subgroup of $G(\mathbb{A}_f)$ such that K is the product of compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. There exist positive constants c and δ such that, if V is a strongly special subvariety of $S_K(G, X)$, defined by (H, X_H), then

$$\deg_{\mathcal{L}_K} V > c \cdot \Pi(H, K_H)^{\delta}.$$

This bound replaces the lower bound on the degrees of Galois orbits used in [9]. Otherwise, the strategy is largely similar, though somewhat simplified in this case since we will not need an analogue of [9, Lemma 9.2.3]. Given a strongly special subvariety V, contained in an irreducible subvariety Z, one obtains a lower bound for the degree of V in terms of a product of 'bad' primes (see Theorem 1.4). One then obtains a 'good' prime p, small compared to the degree of V, such that there exists a 'suitable' Hecke correspondence T at p satisfying $V \subset T(V)$. Thus, V is contained in the intersection $Z \cap T(Z)$. However, if the dimension of Z is only one greater than that of V, comparing their degrees leads one to realise that the intersection $Z \cap T(Z)$ cannot be proper. Therefore, since Z is irreducible, it must be contained in $T(Z)$. In this case, a geometric argument implies that there exists a strongly special subvariety $V' \subset Z$ such that $V \subsetneq V'$. On the other hand, if the intersection $Z \cap T(Z)$ is proper, one chooses an irreducible component containing V and repeats the above procedure.

This result represents the full generalisation of a strategy tested in [5] for removing ergodic theory from the proof of the André–Oort conjecture. However, we also hope that the bounds presented here will lead to useful developments in the wider world of the Zilber–Pink conjectures.

2. Generalities

Unless stated otherwise, all varieties (except for linear algebraic groups) will be defined over $\mathbb{C}$ and identified with their set of $\mathbb{C}$-points. We will denote by $\mathbb{A}_f$ the ring of finite (rational) adeles and by $\widehat{\mathbb{Z}}$ the product of $\mathbb{Z}_p$ over all primes p.

For any algebraic group G, we will denote by G^{ad} the quotient of G by its centre. If G is defined over $\mathbb{Q}_p$ and ρ is a faithful representation, we will consider G as a subgroup of $\mathrm{GL}_{n,\mathbb{Q}_p}$. For such a subgroup, we will denote by $G_{\mathbb{Z}_p}$ the Zariski closure of G in $\mathrm{GL}_{n,\mathbb{Z}_p}$. We will say that G is *unramified* if it is quasi-split and splits over an unramified extension of $\mathbb{Q}_p$. If G and ρ are defined over $\mathbb{Q}$, then the previous definitions make sense for $G_{\mathbb{Q}_p}$ and $\rho_{\mathbb{Q}_p}$ for any prime p. A subgroup $K_p \subset G(\mathbb{Q}_p)$ is called *hyperspecial* if there exists a smooth reductive group scheme $\mathbf{G}$ over $\mathbb{Z}_p$ such that $\mathbf{G}_{\mathbb{Q}_p} = G_{\mathbb{Q}_p}$ and $\mathbf{G}(\mathbb{Z}_p) = K_p$ (see [2, Section 4.6]). By a reductive group scheme, we mean a group scheme with reductive fibres.

Given an algebraic torus T over a field k and a representation

$$\rho : T \hookrightarrow \mathrm{GL}_{n,k},$$

let l/k be a Galois extension such that T_l splits. One obtains a decomposition $l^n = \bigoplus_\chi V_\chi$, summing over characters $\chi : T_l \to \mathbb{G}_{m,l}$ of T, where V_χ is the l-subspace on which T_l acts via χ. We refer to those characters χ such that $V_\chi \neq \{0\}$ as the characters *intervening* in l^n. The characters of T form a free $\mathbb{Z}$-module $X^*(T)$ equipped with an action of $\mathrm{Gal}(l/k)$. After choosing a basis for $X^*(T)$, one may refer to the *coordinates* of a character $\chi \in X^*(T)$.

Let X be a complete irreducible variety and let $\mathcal{L}$ be a line bundle on X with topological first Chern class

$$c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}).$$

Given an irreducible subvariety V of X, we define the degree of V, with respect to $\mathcal{L}$, as in [9, Section 5.1], by

$$\deg_{\mathcal{L}} V := c_1(\mathcal{L})^{\dim V} \cap [V] \in H_0(X, \mathbb{Z}) = \mathbb{Z},$$

where $[V] \in H_{2\dim V}(X, \mathbb{Z})$ denotes the fundamental class of V and $\cap$ denotes the cap product between $H^{2\dim V}(X, \mathbb{Z})$ and $H_{2\dim V}(X, \mathbb{Z})$. We will also put

$$\int_V c_1(\mathcal{L})^{\dim V} := \deg_{\mathcal{L}} V.$$

When the variety X is a disjoint union of irreducible components X_i, the function $\deg_{\mathcal{L}}$ is defined as the sum $\sum_i \deg_{\mathcal{L}|X_i}$.

3. Reductions

Consider a Shimura datum (G, X), a connected component X^+ of X, and a compact open subgroup K of $G(\mathbb{A}_f)$. We will write $\mathrm{Sh}_K(G, X)$ for the corresponding Shimura variety and denote by $S_K(G, X)$ the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$. Recall that the André–Oort conjecture is equivalent for all choices of K.

We will write $\overline{\mathrm{Sh}_K(G, X)}$ for the Baily–Borel compactification of $\mathrm{Sh}_K(G, X)$ (as defined in [9, Proposition 5.3.1]) and $\mathcal{L}_K$ for the corresponding ample line bundle (as defined in [9, Proposition 5.3.2])). For an irreducible subvariety V of $\mathrm{Sh}_K(G, X)$ we will denote by $\overline{V}$ the Zariski closure of V in $\overline{\mathrm{Sh}_K(G, X)}$. We will write $\deg_{\mathcal{L}_K} V$ for $\deg_{\mathcal{L}_K} \overline{V}$.

We denote by X_{ad} the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of morphisms from $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ to $G^{\mathrm{ad}}_{\mathbb{R}}$ that contains the image of X. Then $(G^{\mathrm{ad}}, X_{\mathrm{ad}})$ is a Shimura datum, referred to as the *adjoint Shimura datum*. The image K_{ad} of K in $G^{\mathrm{ad}}(\mathbb{A}_f)$ is a compact open subgroup and we have an induced morphism

$$\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K_{\mathrm{ad}}}(G^{\mathrm{ad}}, X_{\mathrm{ad}}).$$

By [8, Proposition 2.2], V is special if and only if its image V_{ad} is special. Hence, for the purposes of the André–Oort conjecture, we may assume that $G = G^{\mathrm{ad}}$.

Let $\alpha \in G(\mathbb{A}_f)$ and let T_α be the associated Hecke correspondence on $\mathrm{Sh}_K(G,X)$. By the definition of a special subvariety, V is special if and only if one (or, equivalently, all) of the irreducible components of $T_\alpha(V)$ is (are) special. In particular, in order to prove the André–Oort conjecture, it suffices to consider sets of special subvarieties Σ such that the Zariski closure of $\boldsymbol{\Sigma}$ in $\mathrm{Sh}_K(G,X)$ is irreducible and contained in $S_K(G,X)$.

In this paper, we will often have an inclusion of Shimura data $(G_1,X_1) \subset (G_2,X_2)$ and a compact open subgroup $K_1 := K_2 \cap G_1(\mathbb{A}_f)$ of $G_1(\mathbb{A}_f)$, where K_2 is a compact open subgroup of $G_2(\mathbb{A}_f)$. Thus, we obtain a morphism

$$\phi : \mathrm{Sh}_{K_1}(G_1,X_1) \to \mathrm{Sh}_{K_2}(G_2,X_2)$$

and, by [19, Lemma 2.2], if K_2 is neat, ϕ is generically injective. In this case, we will use the same symbol for a subvariety of $\mathrm{Sh}_{K_1}(G_1,X_1)$ and its image in $\mathrm{Sh}_{K_2}(G_2,X_2)$.

4. Choosing a measure

Consider a special subvariety V of $S_K(G,X)$. By [19, Lemma 2.1], there exists a Shimura subdatum (H,X_H) of (G,X) and a connected component X_H^+ of X_H contained in X^+ such that H is the generic Mumford–Tate group on X_H and V is the image of $X_H^+ \times \{1\}$ in $\mathrm{Sh}_K(G,X)$. We will denote by K_H the intersection $K \cap H(\mathbb{A}_f)$ and by Γ_H the intersection $H(\mathbb{Q})_+ \cap K_H$, where $H(\mathbb{Q})_+$ is the stabiliser of X_H^+ in $H(\mathbb{Q})$. Thus, V is the image of $\Gamma_H \backslash X_H^+$ in $\mathrm{Sh}_K(G,X)$. We refer to (H,X_H) as the Shimura datum *defining* V and we say that V is *strongly special* if the image of H in G^{ad} is semisimple.

The space X_H^+ is isomorphic to $H^{\mathrm{ad}}(\mathbb{R})^+/K_\infty$, where K_∞ is a maximal compact subgroup of $H^{\mathrm{ad}}(\mathbb{R})^+$. Let $\mathfrak{h}$ denote the Lie algebra of $H^{\mathrm{ad}}(\mathbb{R})^+$ and let $\mathfrak{h}^*$ denote the dual of $\mathfrak{h}$. Any real, non-zero, left-invariant differential form ω of maximal degree r on $H^{\mathrm{ad}}(\mathbb{R})^+$ corresponds to an element of $\bigwedge^r \mathfrak{h}^*$.

Since $\mathfrak{h}$ admits a Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K_∞ and $\mathfrak{p}$ is the tangent space of X_H^+ at the point K_∞, we can write $\omega = \omega_{\mathfrak{k}} \wedge \omega_{\mathfrak{p}}$, where $\omega_{\mathfrak{k}}$ and $\omega_{\mathfrak{p}}$ correspond to real multilinear forms on $\mathfrak{k}$ and $\mathfrak{p}$, respectively. In this paper, we will always choose $\omega_{\mathfrak{k}}$ so that, with respect to the measure it determines, the volume of K_∞ is one.

Consider the unique (up to isomorphism) $\mathbb{R}$-anisotropic form H^c of H^{ad}, i.e. the real algebraic group H^c isomorphic to H^{ad} over $\mathbb{C}$ such that $H^c(\mathbb{R})$ is compact. Then $H^c(\mathbb{R})$ is a connected, maximal, compact subgroup of $H^c(\mathbb{C})$ containing a copy of K_∞ and the quotient $\check{X}_H := H^c(\mathbb{R})/K_\infty$ is called the *compact dual* of X_H^+. It contains X_H^+ as an open subset.

Considering multilinear forms on the complexification $\mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \otimes \mathbb{C}$, the form ω extends $\mathbb{C}$-linearly to a complex, left-invariant differential form $\omega_{\mathbb{C}}$ on $H^{\mathrm{ad}}(\mathbb{C})$. As in [13, Proportionality Theorem 3.2], the Lie algebra of $H^c(\mathbb{R})$ inside $\mathfrak{h}_{\mathbb{C}}$ is equal to $\mathfrak{k} \oplus i\mathfrak{p}$. We will always choose $\omega_{\mathfrak{p}}$ so that, with respect to the measure determined by $\omega_{\mathbb{C}}$, the volume of $H^c(\mathbb{R})$ or, equivalently, any maximal compact subgroup of $H^{\mathrm{ad}}(\mathbb{C})$, is one. Therefore, the volume of $\check{X}_H$ is also one.

We will denote by μ the Haar measure on $H^{\mathrm{ad}}(\mathbb{R})^+$ determined by ω. We will also denote by μ the volume measure on X_H^+ determined by $\omega_{\mathfrak{p}}$. When we consider the volume measures induced on arithmetic quotients of either $H^{\mathrm{ad}}(\mathbb{R})^+$ or X_H^+, we will again use μ.

5. Degrees of strongly special subvarieties

In order to prove Theorem 1.4, we will prove the following theorem, relating the degree of a special subvariety to its volume:

Theorem 5.1. *Let (G, X) be a Shimura datum, let X^+ be a connected component of X, and let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. There exists a constant c_1 such that, if V is a special subvariety of $S_K(G, X)$, defined by (H, X_H), then*

$$\deg_{\mathcal{L}_K} V > c_1 \cdot \mu(\Gamma_H \setminus X_H^+).$$

In this paper, a *constant* will be taken to mean a positive real number.

Proof. By [9, Corollary 5.3.10],

$$\deg_{\mathcal{L}_K} V \geq \deg_{\mathcal{L}_{K_H}} V$$

and, for the remainder of this section, V will refer to the connected component $\Gamma_H \setminus X_H^+$ of $\mathrm{Sh}_{K_H}(H, X_H)$.

Consider a smooth compactification $\overline{V}^{\mathrm{sm}}$ of V, thus providing a canonical birational map

$$\pi : \overline{V}^{\mathrm{sm}} \to \overline{V},$$

as in the proof of [13, Proposition 3.4 (b)]. By [9, Proposition 5.3.2 (1)], the exterior product $\Omega^{\dim X_H}$ of the cotangent bundle Ω on X_H descends to $\mathrm{Sh}_{K_H}(H, X_H)$ and extends uniquely to an ample line bundle on $\overline{V}$ (this is the restriction of $\mathcal{L}_{K_H}$). By [13, Proposition 3.4 (b)], the pullback $\pi^*\mathcal{L}_{K_H}$ of $\mathcal{L}_{K_H}$ to $\overline{V}^{\mathrm{sm}}$ is the unique extension $\overline{E}$ of [13, Main Theorem 3.1]. Of course, $\Omega^{\dim X_H}$ is the restriction of the exterior product $\check{\Omega}^{\dim X_H}$ of the cotangent bundle $\check{\Omega}$ on the compact dual $\check{X}_H$. By [13, Proportionality Theorem 3.2], we have

$$\deg_{\pi^*\mathcal{L}_{K_H}} \overline{V}^{\mathrm{sm}} = (-1)^{\dim X_H} \cdot \mu(\Gamma_H \setminus X_H^+) \cdot \int_{\check{X}_H} c_1(\check{\Omega}^{\dim X_H})^{\dim X_H}.$$

However, since π is birational, the projection formula (see [9, Section 5.1]) implies that

$$\deg_{\pi^*\mathcal{L}_{K_H}} \overline{V}^{\mathrm{sm}} = \deg_{\mathcal{L}_{K_H}} V.$$

Furthermore, up to isomorphism, the number of Hermitian symmetric spaces corresponding to Shimura subdata of (G, X) is finite. Therefore,

$$(-1)^{\dim X_H} \cdot \int_{\check{X}_H^+} c_1(\check{\Omega}^{\dim X_H})^{\dim X_H},$$

may assume only finitely many positive values. □

6. Volumes of strongly special subvarieties

In this section, we prove a lower bound for the volume of a strongly special subvariety, concluding the proof of Theorem 1.4. First, however, suppose that G is a reductive group over $\mathbb{Q}$ and L is a finite Galois extension over which G is split. Since almost all places of L are unramified over $\mathbb{Q}$, it follows that $G_{\mathbb{Q}_p}$ is split over an unramified extension for almost all primes p. Furthermore, by [17, Lemma 4.9 (ii)], $G_{\mathbb{Q}_p}$ is quasi-split for almost all primes p. Therefore, we let $\Sigma(G)$ denote the finite set of primes p such that $G_{\mathbb{Q}_p}$ is not unramified.

Suppose that K is a compact open subgroup of $G(\mathbb{A}_f)$, equal to a product of compact open subgroups $K_p \subset G(\mathbb{Q}_p)$; fix a faithful representation $G \hookrightarrow \mathrm{GL}_n$. By [9, Section 4.1.5], $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ for almost all primes p. Thus, by [18, Section 3.9.1], K_p is hyperspecial for almost all p. Therefore, we let $\Sigma(K)$ denote the finite set of primes p such that K_p is not hyperspecial. Finally, we let $\Sigma(G,K)$ denote the set of primes belonging to either $\Sigma(G)$ or $\Sigma(K)$ and we let $\Pi(G,K)$ denote their product.

Theorem 6.1. *Let (G,X) be a Shimura datum such that $G = G^{\mathrm{ad}}$ and let X^+ be a connected component of X. Fix a faithful representation*

$$\rho : G \hookrightarrow \mathrm{GL}_n$$

and let K be a neat compact open subgroup of $G(\mathbb{A}_f)$, equal to a product of compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. There exist constants c_2 and δ such that, if V is a strongly special subvariety of $S_K(G,X)$, defined by (H, X_H), then

$$\mu(\Gamma_H \backslash X_H^+) > c_2 \cdot \Pi(H, K_H)^{\delta}.$$

In this section, we will use the term *uniform* to mean depending only on (G,X), K and ρ.

Note that, since K_H is neat, Γ_H injects into $H^{\mathrm{ad}}(\mathbb{R})^+$ and so acts freely on X_H^+. Let $\mathrm{ad} : H \to H^{\mathrm{ad}}$ denote the natural map. Since K_∞ has volume one with respect to the measure determined by $\omega_{\mathfrak{k}}$, we have

$$\mu(\Gamma_H \backslash X_H^+) = \mu(\mathrm{ad}(\Gamma_H) \backslash H^{\mathrm{ad}}(\mathbb{R})^+).$$

Since H is semisimple, we have a central isogeny $\pi : \widetilde{H} \to H$, where $\widetilde{H}$ is simply connected and whose centre we denote $Z_{\widetilde{H}}$. We denote by $Z \subset Z_{\widetilde{H}}$ the kernel of π. Note that, by [10, Proposition 1.4.5], the maximal split tori (resp. parabolic subgroups) of $\widetilde{H}$ are in bijection via this morphism with the maximal split tori (resp. parabolic subgroups) of H. Therefore, $\Sigma(\widetilde{H}) = \Sigma(H)$.

Since π is finite, and therefore proper,

$$\widetilde{K}_H := \pi_{\mathbb{A}_f}^{-1}(K_H)$$

is a compact open subgroup of $\widetilde{H}(\mathbb{A}_f)$ and we let

$$\widetilde{\Gamma}_H := \widetilde{H}(\mathbb{Q}) \cap \widetilde{K}_H,$$

which is equal to $\pi^{-1}(\Gamma_H)$. Since K_H is necessarily a product of compact open subgroups $K_{H,p} \subset H(\mathbb{Q}_p)$, $\widetilde{K}_H$ is also a product of compact open subgroups $\widetilde{K}_{H,p} \subset \widetilde{H}(\mathbb{Q}_p)$. Let $K^m_{\widetilde{H}}$ be a maximal compact open subgroup of $\widetilde{H}(\mathbb{A}_f)$ containing $\widetilde{K}_H$ and let

$$\Gamma^m_{\widetilde{H}} := \widetilde{H}(\mathbb{Q}) \cap K^m_{\widetilde{H}}.$$

Again $K^m_{\widetilde{H}}$ is a product of maximal compact open subgroups $K^m_{\widetilde{H},p} \subset \widetilde{H}(\mathbb{Q}_p)$.

By [11, Theorem 5.2], $\widetilde{H}(\mathbb{R})$ is connected and so acts on $H^{\mathrm{ad}}(\mathbb{R})^+$ through $\mathrm{ad} \circ \pi$. Therefore, we have two finite projections

$$\mathrm{ad}(\Gamma_H) \backslash H^{\mathrm{ad}}(\mathbb{R})^+ \leftarrow \mathrm{ad} \circ \pi(\widetilde{\Gamma}_H) \backslash H^{\mathrm{ad}}(\mathbb{R})^+ \to \mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}}) \backslash H^{\mathrm{ad}}(\mathbb{R})^+$$

yielding the equality

$$\mu(\mathrm{ad}(\Gamma_H) \backslash H^{\mathrm{ad}}(\mathbb{R})^+) = \frac{[\mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}}) : \mathrm{ad} \circ \pi(\widetilde{\Gamma}_H)]}{[\mathrm{ad}(\Gamma_H) : \mathrm{ad} \circ \pi(\widetilde{\Gamma}_H)]} \cdot \mu(\mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}}) \backslash H^{\mathrm{ad}}(\mathbb{R})^+).$$

Lemma 6.2. *There exists a uniform constant c_3 such that*

$$[\mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}}) : \mathrm{ad} \circ \pi(\widetilde{\Gamma}_H)] > c_3[K^m_{\widetilde{H}} : \widetilde{K}_H].$$

Proof. Consider the surjective map

$$\Gamma^m_{\widetilde{H}} \to \mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}})/\mathrm{ad} \circ \pi(\widetilde{\Gamma}_H).$$

Since $\widetilde{\Gamma}_H = \pi^{-1}(\Gamma_H)$, the kernel is equal to

$$(\mathrm{ad} \circ \pi)^{-1}(\mathrm{ad} \circ \pi(\pi^{-1}(\Gamma_H))) \cap \Gamma^m_{\widetilde{H}},$$

which is readily seen to be $Z_{\widetilde{H}}(\mathbb{Q}) \cdot \pi^{-1}(\Gamma_H)$. Therefore, we turn our attention to $[\Gamma^m_{\widetilde{H}} : \widetilde{\Gamma}_H]$. Write

$$K^m_{\widetilde{H}} = \coprod_{i=1}^{n} k_i \widetilde{K}_H,$$

where $k_i \in K^m_{\widetilde{H}}$. By strong approximation (as in [11, Theorem 4.16]), applied to $\widetilde{H}$, each k_i can be written as $q_i k_i'$, where $q_i \in \widetilde{H}(\mathbb{Q})$ and $k_i' \in \widetilde{K}_H$. Therefore, in the above, we may replace k_i with q_i. Intersecting both sides with $\widetilde{H}(\mathbb{Q})$ we obtain

$$\Gamma^m_{\widetilde{H}} = \coprod_{i=1}^{n} q_i \widetilde{\Gamma}_H,$$

and so $q_i \in \Gamma^m_{\widetilde{H}}$. □

Lemma 6.3. *There exist uniform constants c_4 and C such that*

$$[\mathrm{ad}(\Gamma_H) : \mathrm{ad} \circ \pi(\widetilde{\Gamma}_H)] < c_4 C^{|\Sigma(H, K_H)|}.$$

Proof. Consider the surjective map

$$\Gamma_H \to \mathrm{ad}(\Gamma_H)/\mathrm{ad} \circ \pi(\widetilde{\Gamma}_H).$$

The kernel is equal to

$$\mathrm{ad}^{-1}(\mathrm{ad} \circ \pi(\widetilde{\Gamma}_H)) \cap \Gamma_H,$$

which is readily seen to be

$$(\pi(\widetilde{\Gamma}_H) \cdot Z_H(\mathbb{Q})) \cap \Gamma_H = \pi(\widetilde{\Gamma}_H) \cdot (Z_H(\mathbb{Q}) \cap \Gamma_H),$$

where Z_H is the centre of H, whose order is uniformly bounded by the proof of [19, Lemma 2.4]. Therefore, we turn our attention to $[\Gamma_H : \pi(\widetilde{\Gamma}_H)]$.

Recall that Galois cohomology yields an exact sequence

$$\widetilde{H}(\mathbb{Q}) \to H(\mathbb{Q}) \to H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), Z(\overline{\mathbb{Q}})).$$

Therefore, $\pi(\widetilde{\Gamma}_H) \setminus \Gamma_H$ embeds as a subgroup of the Abelian group $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), Z(\overline{\mathbb{Q}}))$. On the other hand, $\pi(\widetilde{\Gamma}_H) \setminus \Gamma_H$ embeds into

$$\pi(\widetilde{K}_H) \setminus K_H = \prod_p \pi(\widetilde{K}_{H,p}) \setminus K_{H,p}$$

and, again, Galois cohomology tells us that, for all primes p,

$$\pi(\widetilde{K}_{H,p}) \setminus K_{H,p} \hookrightarrow H^1(\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p), Z(\overline{\mathbb{Q}_p})).$$

However, now consider a prime p such that $H_{\mathbb{Q}_p}$ is unramified and $K_{H,p}$ is hyperspecial. Since $\widetilde{H}_{\mathbb{Q}_p}$ is also unramified, it follows that $\widetilde{H}(\mathbb{Q}_p)$ also possesses hyperspecial subgroups by [18, Section 3.8.2]. Therefore, by [18, Section 3.8.1], there exist smooth reductive group schemes $\widetilde{\mathbf{H}}$ and $\mathbf{H}$ over $\mathbb{Z}_p$, the generic fibres of which are $\widetilde{H}_{\mathbb{Q}_p}$ and $H_{\mathbb{Q}_p}$, such that $K_{H,p} = \mathbf{H}(\mathbb{Z}_p)$ and $\widetilde{\mathbf{H}}(\mathbb{Z}_p)$ is a hyperspecial subgroup of $\widetilde{H}(\mathbb{Q}_p)$. By [20, Lemma 2.3.1], the central isogeny $\pi_{\mathbb{Q}_p}$ extends uniquely to a central isogeny $\pi_{\mathbb{Z}_p} : \widetilde{\mathbf{H}} \to \mathbf{H}$. Therefore, the kernel $\mathbf{Z}$ of $\pi_{\mathbb{Z}_p}$ is a finite group scheme of multiplicative type such that $\mathbf{Z}_{\mathbb{Q}_p} = Z_{\mathbb{Q}_p}$. Over a finite Galois extension F of $\mathbb{Q}$, Z_F is isomorphic to a product of roots of unity, whose orders we denote $n_1, \dots, n_r$. Therefore, by [7, Exposé X, Lemme 4.1], if p is coprime to the n_i, $\mathbf{Z}_{\mathbb{F}_p}$ is smooth.

Therefore, by [15, Lemma 6.5], for any prime $p \notin \Sigma(H, K_H)$ and coprime to the n_i, we have an exact sequence

$$\widetilde{\mathbf{H}}(\mathbb{Z}_p) \to \mathbf{H}(\mathbb{Z}_p) \to H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p), \mathbf{Z}(\mathbb{Z}_p^{\mathrm{un}})),$$

where $\mathbb{Q}_p^{\mathrm{un}}$ is the maximal unramified extension of $\mathbb{Q}_p$ and $\mathbb{Z}_p^{\mathrm{un}}$ is its ring of integers. However, since $\widetilde{K}_{H,p}$ clearly contains $\widetilde{\mathbf{H}}(\mathbb{Z}_p)$ and, by [18, Section 3.8.2], $\widetilde{\mathbf{H}}(\mathbb{Z}_p)$ is maximal among compact subgroups of $\widetilde{H}(\mathbb{Q}_p)$, we have $\widetilde{\mathbf{H}}(\mathbb{Z}_p) = \widetilde{K}_{H,p}$.

By [6, Theorem 5.1], the degree $[F : \mathbb{Q}]$ is bounded in terms of the dimension of any maximal torus of $\widetilde{H}$ containing Z, which is itself bounded by a uniform constant. By the proof of [19, Lemma 2.4], the order of Z is also bounded by a uniform constant and so the same can be said of $|H^1(\mathrm{Gal}(F/\mathbb{Q}), Z(F))|$. Therefore, we may consider the image of the quotient $\pi(\widetilde{\Gamma}_H) \setminus \Gamma_H$ in $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/F), Z(\overline{\mathbb{Q}}))$, whose image in $H^1(\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_\upsilon), Z(\overline{\mathbb{Q}_p}))$ is contained in $H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/F_\upsilon), \mathbf{Z}(\mathbb{Z}_p^{\mathrm{un}}))$ for all places υ of F lying above a prime $p \notin \Sigma(H, K_H)$ coprime to the n_i.

We identify the three previous cohomology groups with the groups

$$\prod_{i=1}^r (F^*)^{n_i} \setminus F^*, \quad \prod_{i=1}^r (F_\upsilon^*)^{n_i} \setminus F_\upsilon^* \quad \text{and} \quad \prod_{i=1}^r (\mathcal{O}_{F_\upsilon}^*)^{n_i} \setminus \mathcal{O}_{F_\upsilon}^*$$

and choose a uniformiser $\xi_\upsilon \in F$ at each place υ lying above a prime $p \in \Sigma(H, K_H)$ or dividing one of the n_i. Therefore, the image of

$$\pi(\widetilde{\Gamma}_H) \setminus \Gamma_H \to \prod_{i=1}^r (F^*)^{n_i} \setminus F^*$$

is contained in the subgroup generated by $\mathcal{O}_F^*$ and the ξ_υ. Now, $\mathcal{O}_F^*$ is a finitely generated Abelian group whose rank and torsion subgroup are uniformly bounded and, since the n_i are bounded by the order of Z, we are done. □

We now appeal to Prasad's formula.

Lemma 6.4. *There exist uniform constants c_5 and δ_1 such that*

$$\mu(\mathrm{ad}\circ\pi(\Gamma^m_{\widetilde{H}}) \setminus H^{\mathrm{ad}}(\mathbb{R})^+) > c_5 \cdot \Pi(\widetilde{H}, K^m_{\widetilde{H}})^{\delta_1}.$$

Proof. Let

$$\widetilde{\omega} := \frac{1}{|Z_{\widetilde{H}}|}\omega^*,$$

where ω^* is the pullback of ω to $\widetilde{H}(\mathbb{R})$. Denote by $\widetilde{\mu}$ the measure determined by $\widetilde{\omega}$ on $\widetilde{H}(\mathbb{R})$ and its arithmetic quotients. By [11, Proposition 5.1], $\widetilde{H}(\mathbb{R}) \to H^{\mathrm{ad}}(\mathbb{R})^+$ is surjective. On the other hand, the kernel of the map

$$\Gamma^m_{\widetilde{H}} \setminus \widetilde{H}(\mathbb{R}) \to \mathrm{ad}\circ\pi(\Gamma^m_{\widetilde{H}}) \setminus H^{\mathrm{ad}}(\mathbb{R})^+$$

is $\Gamma^m_{\widetilde{H}} \cap Z_{\widetilde{H}}(\mathbb{R}) \setminus Z_{\widetilde{H}}(\mathbb{R})$. It follows from the proof of [19, Lemma 2.4] that

$$\frac{\mu(\mathrm{ad}\circ\pi(\Gamma^m_{\widetilde{H}}) \setminus H^{\mathrm{ad}}(\mathbb{R})^+)}{\widetilde{\mu}(\Gamma^m_{\widetilde{H}} \setminus \widetilde{H}(\mathbb{R}))}$$

is greater than a uniform constant.

Since $\widetilde{H}$ is simply connected, it is a direct product $H_1 \times \cdots \times H_s$ of quasi-simple, simply connected subgroups. We can write $\widetilde{\omega} = \omega_1 \wedge \cdots \wedge \omega_s$, where ω_i is a real, non-zero, left-invariant differential form of maximal degree on $H_i(\mathbb{R})$. Since $\mathrm{ad}\circ\pi$ is surjective, of degree $|Z'|$ and proper, the preimage of a maximal compact subgroup of $H^{\mathrm{ad}}(\mathbb{C})$ is a maximal compact subgroup of $\widetilde{H}(\mathbb{C})$, whose volume with respect to the measure determined by $\widetilde{\omega}_{\mathbb{C}}$ is one. The ω_i are, therefore, determined up to multiplication by a non-zero multiplicative constant. We choose this constant so that the volume of any maximal compact subgroup of $H_i(\mathbb{C})$ is one. We denote the measures determined on the $H_i(\mathbb{R})$ by μ_i. Since $K^m_{\widetilde{H}}$ is maximal, it is a product of maximal compact open subgroups

$$K^m_{H_i} \subset H_i(\mathbb{A}_f),$$

each a product of maximal compact open subgroups $K^m_{H_i,p} \subset H_i(\mathbb{Q}_p)$. Hence,

$$\widetilde{\mu}(\Gamma^m_{\widetilde{H}} \setminus \widetilde{H}(\mathbb{R})) = \prod_{i=1}^{s} \mu_i(\Gamma^m_{H_i} \setminus H_i(\mathbb{R})),$$

where $\Gamma^m_{H_i} := H_i(\mathbb{Q}) \cap K^m_{H_i}$.

By [21, Section 3.3], each H_i is of the form $\mathrm{Res}_{K_i/\mathbb{Q}} H'_i$, where K_i is a totally real number field and H'_i is a simply connected, absolutely quasi-simple group. Now,

$$H_i(\mathbb{Q}_p) = \mathrm{Res}_{K_i/\mathbb{Q}} H'_i(\mathbb{Q}_p) = H'_i(K_i \otimes_{\mathbb{Q}} \mathbb{Q}_p) = \prod_{v|p} H'_i(K_{i,v}),$$

where the product runs over the places v of K_i lying above p and $K_{i,v}$ is the completion of K_i with respect to the valuation determined by v. Thus, $K^m_{H_i,p}$ is a product of maximal compact open subgroups $K^m_{H'_i,v} \subset H'_i(K_{i,v})$.

Since $H_i(\mathbb{R}) = \prod_{v|\infty} H_i'(K_{i,v})$, we can write

$$\omega_i = \wedge_{v|\infty} \omega_{i,v},$$

where $\omega_{i,v}$ is a real, non-zero, left-invariant differential form of maximal degree on $H_i'(K_{i,v})$. We choose the $\omega_{i,v}$ so that the volume of any maximal compact subgroup of $H_i'(\mathbb{C})$ is one. Note that, by [16, Section 3.5], for each archimedean place v of K_i, the Haar measure $\mu_{i,v}$ determined by $\omega_{i,v}$ on $H_i'(\mathbb{R})$ is precisely that defined in [16, Section 3.6]. Therefore, by [16, Theorem 3.7],

$$\mu_i(\Gamma^m_{H_i} \setminus H_i(\mathbb{R})) = D_{K_i}^{\frac{1}{2}\dim H_i'} \cdot |N_{K_i/\mathbb{Q}}(\Delta_{L_i/K_i})|^{\frac{s_i}{2}} \cdot \left(\prod_{j=1}^{r_i} \frac{m_{i,j}!}{(2\pi)^{m_{i,j}+1}}\right)^{[K_i:\mathbb{Q}]} \cdot \tau_{K_i}(H_i') \cdot \xi_i,$$

where

- D_{K_i} is the absolute value of $\operatorname{disc}(K_i)$,
- L_i is the splitting field of the quasi-split inner form $\mathcal{H}_i$ of H_i',
- $N_{K_i/\mathbb{Q}}$ is the norm on K_i,
- Δ_{L_i/K_i} is the relative discriminant of L_i over K_i,
- s_i is the integer defined in [16, Section 0.4],
- r_i is the absolute rank of $\mathcal{H}_i$,
- the $m_{i,j}$ are the exponents of the simple, simply connected, compact real-analytic Lie group of the same type as $\mathcal{H}_i$,
- $\tau_{K_i}(H_i') = 1$ is the Tamagawa number of H_i' (see [16, Section 3.3]),
- ξ_i is the product, over all finite places v of K_i, of local factors $\xi_{i,v}$.

Note first that $\dim H_i'$, s_i, r_i, the $m_{i,j}$ and $[K_i : \mathbb{Q}]$ are all positive integers, with the possible exception of s_i when $L_i = K_i$, in which case it becomes irrelevant. It is also worth noting that they are all uniformly bounded.

Recall that Δ_{L_i/K_i} is an ideal in $\mathcal{O}_{K_i}$ with the property that the prime ideals dividing it are precisely those that ramify in $\mathcal{O}_{L_i}$, i.e. those places v of K_i such that $\mathcal{H}_{i,K_{i,v}}$ does not split over an unramified extension of $K_{i,v}$. Its norm $N_{K_i/\mathbb{Q}}(\Delta_{L_i/K_i})$ is divisible by precisely those primes p such that there exists a place v lying above p and dividing Δ_{L_i/K_i}.

By [16, Section 2.10], $\xi_{i,v} > 1$ for all non-archimedean places v of K_i. Furthermore, if

- $H'_{i,K_{i,v}}$ is not quasi-split,
- $K^m_{H_i',v}$ is not special,

or

- $H'_{i,K_{i,v}}$ splits over an unramified extension of $K_{i,v}$ and $K^m_{H_i',v}$ is not hyperspecial,

then

$$\xi_{i,v} \geq q_{i,v}^{r_{i,v}+1} \cdot (q_{i,v}+1)^{-1},$$

where $q_{i,v}$ is the cardinality of the residue field $k_{i,v}$ of $K_{i,v}$ and $r_{i,v} \geq 1$ is the rank of $\mathcal{H}_{i,K_{i,v}}$ over the maximal unramified extension of $K_{i,v}$. Therefore, let Σ_i be the set of primes p such that, for some place v of K_i lying above p, either of the following holds:

- $H'_{i,K_{i,v}}$ is not quasi-split,
- $H'_{i,K_{i,v}}$ does not split over an unramified extension of $K_{i,v}$,
- $K^m_{H'_i,v}$ is not a hyperspecial subgroup of $H'_i(K_{i,v})$.

Then there exist uniform constants c_6 and δ_2 such that

$$\mu_i(\Gamma^m_{H_i} \setminus H_i(\mathbb{R})) > c_6 D_{K_i}^{\frac{1}{2}\dim H'_i} \cdot \prod_{p\in\Sigma_i} p^{\delta_2}.$$

However, recall that

$$H_{i,\mathbb{Q}_p} = \prod_{v|p} \mathrm{Res}_{K_{i,v}/\mathbb{Q}_p} H'_{i,K_{i,v}}.$$

Therefore, by [1, Section 6.19], the set $\Sigma(H_i)$ is contained in the union of Σ_i and the set of primes p dividing D_{K_i}. On the other hand, suppose that $K^m_{H'_i,v}$ is a hyperspecial subgroup of $H'_i(K_{i,v})$ for each place v of K_i lying above a prime p.

For each such subgroup, there exists a smooth group scheme $\mathbf{H}'_{i,\mathcal{O}_{K_{i,v}}}$ over $\mathcal{O}_{K_{i,v}}$ with generic fibre $H'_{i,K_{i,v}}$ such that

- $\mathbf{H}'_{i,k_{i,v}}$ is reductive,
- $\mathbf{H}'_{i,\mathcal{O}_{k_{i,v}}}(\mathcal{O}_{K_{i,v}})$ is equal to $K^m_{H'_i,v}$.

Let

$$\mathbf{H}_{i,\mathbb{Z}_p} := \prod_{v|p} \mathrm{Res}_{\mathcal{O}_{K_{i,v}}/\mathbb{Z}_p} \mathbf{H}'_{i,\mathcal{O}_{K_{i,v}}}.$$

Then, the generic fibre of $\mathbf{H}_{i,\mathbb{Z}_p}$ is $H_{i,\mathbb{Q}_p}$ and $\mathbf{H}_{i,\mathbb{Z}_p}(\mathbb{Z}_p) = K^m_{H_i,p}$. Furthermore, if p does not divide D_{K_i}, then

$$\mathbf{H}_{i,\mathbb{F}_p} = \prod_{v|p} \mathrm{Res}_{\mathcal{O}_{K_{i,v}}\otimes_{\mathbb{Z}_p}\mathbb{F}_p/\mathbb{F}_p} \mathbf{H}'_{i,\mathcal{O}_{K_{i,v}}\otimes_{\mathbb{Z}_p}\mathbb{F}_p} = \prod_{v|p} \mathrm{Res}_{k_{i,v}/\mathbb{F}_p} \mathbf{H}'_{i,k_{i,v}}$$

is a reductive group over $\mathbb{F}_p$. Therefore, the set $\Sigma(K^m_{H_i})$ is also contained in the union of the Σ_i and the set of primes p dividing D_{K_i}, from which we conclude there exists a uniform constant δ_3 such that

$$\mu_i(\Gamma^m_{H_i} \setminus H_i(\mathbb{R})) > c_6 \cdot \Pi(H_i, K^m_{H_i})^{\delta_3}.$$

However, the union of the $\Sigma(H_i)$ is equal to $\Sigma(H)$ and the union of the $\Sigma(K^m_{H_i})$ is equal to $\Sigma(K^m_{\widetilde{H}})$. □

We will require the following lemma in the proof of Lemma 6.6 and also to obtain suitable Hecke correspondences:

Lemma 6.5. *Let T be a maximal torus of $H_{\mathbb{Q}_p}$. There exists a basis of $X^*(T)$ such that the coordinates of the characters of T intervening in $\overline{\mathbb{Q}}_p^n$ are bounded in absolute value by a uniform constant.*

Proof. By the proof of [3, Proposition 2.1], since H is the generic Mumford–Tate group on X_H, there exists a dense set of special points X'_H in X_H such that, for $x \in X'_H$, the Mumford–Tate group $\mathrm{MT}(x)$ of x is a maximal torus in H. Choose an $x \in X'_H$ and let $M := \mathrm{MT}(x)$. Denote by L the splitting field of M and by R_L the torus $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{m,L}$.

The reciprocity morphism $r_x : R_L \to M$ corresponding to x (see [11, p. 104]) is surjective and induces an embedding

$$X^*(M) \hookrightarrow X^*(R_L).$$

Enumerate the elements $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$, thereby producing a basis $\mathcal{B} := \{b_\sigma\}$ of $X^*(R_L)$. By [23, Section 2], with respect to this basis, the characters of M intervening in $\overline{\mathbb{Q}}^n$ have coordinates bounded in absolute value by a uniform constant.

Since any two maximal tori of $H_{\overline{\mathbb{Q}}_p}$ are conjugate by an element of $H(\overline{\mathbb{Q}}_p)$, we may conjugate $r_{x,\overline{\mathbb{Q}}_p}$ by an element of $H(\overline{\mathbb{Q}}_p)$ to obtain a surjective morphism

$$r'_{x,\overline{\mathbb{Q}}_p} : R_{L,\overline{\mathbb{Q}}_p} \to T_{\overline{\mathbb{Q}}_p}.$$

Thus, we obtain an embedding

$$X^*(T_{\overline{\mathbb{Q}}_p}) \hookrightarrow X^*(R_{L,\overline{\mathbb{Q}}_p})$$

such that, with respect to the basis $\mathcal{B}$, the coordinates of the characters of T intervening in $\overline{\mathbb{Q}}_p^n$ are uniformly bounded in absolute value. Since our representation was faithful, these characters generate $X^*(T_{\overline{\mathbb{Q}}_p})$ and so there are only finitely many possibilities for this submodule of $X^*(R_{L,\overline{\mathbb{Q}}_p})$. For each such possibility, choose a basis for $X^*(T_{\overline{\mathbb{Q}}_p})$ and consider the maximum of the absolute values of coordinates of the characters intervening in $\overline{\mathbb{Q}}_p^n$ with respect to these bases. □

Lemma 6.6. *There exist uniform constants c_7 and c_8 such that, for any $p \notin \Sigma(\widetilde{H}, K^m_{\widetilde{H}})$ greater than c_7, such that $\widetilde{K}_{H,p} \subsetneq K^m_{\widetilde{H},p}$,*

$$[K^m_{\widetilde{H},p} : \widetilde{K}_{H,p}] > c_8 p.$$

Proof. We will imitate the proof of [19, Proposition 3.15]. Let $p \notin \Sigma(\widetilde{H}, K^m_{\widetilde{H}})$ be a prime. Since

$$K^m_{\widetilde{H},p} \subset \widetilde{H}(\mathbb{Q}_p)$$

is hyperspecial and $H_{\mathbb{Q}_p}$ is unramified, there exist smooth reductive group schemes $\widetilde{\mathbf{H}}$ and $\mathbf{H}$ over $\mathbb{Z}_p$, the generic fibres of which are $\widetilde{H}_{\mathbb{Q}_p}$ and $H_{\mathbb{Q}_p}$, such that

$$K^m_{\widetilde{H},p} = \widetilde{\mathbf{H}}(\mathbb{Z}_p).$$

By [20, Lemma 2.3.1], the central isogeny $\pi_{\mathbb{Q}_p}$ extends uniquely to a central isogeny

$$\pi_{\mathbb{Z}_p} : \widetilde{\mathbf{H}} \to \mathbf{H}$$

and we denote the kernel $\mathbf{Z}$.

The map

$$\widetilde{K}_{H,p} \setminus \widetilde{\mathbf{H}}(\mathbb{Z}_p) \to K_{H,p} \setminus \mathbf{H}(\mathbb{Z}_p)$$

is injective. However, recall from the proof of Lemma 6.3 that the cokernel is no larger than $H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p), \mathbf{Z}(\mathbb{Z}_p^{\mathrm{un}}))$. Furthermore, if F is the splitting field of Z and υ is a place of F lying above p, the kernel of the restriction map to $H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/F_\upsilon), \mathbf{Z}(\mathbb{Z}_p^{\mathrm{un}}))$ is uniformly bounded. However, as we have seen, $H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/F_v), \mathbf{Z}(\mathbb{Z}_p^{\mathrm{un}}))$ is itself uniformly bounded. Therefore, it suffices to show there exist uniform constants c_7 and c_8 such that

$$[\mathbf{H}(\mathbb{Z}_p) : K_{H,p}] > c_8 p$$

whenever $p > c_7$.

Let $\mathbf{T}$ be a maximal torus of $\mathbf{H}$. The group $(K_{H,p} \cap \mathbf{T}(\mathbb{Z}_p)) \setminus \mathbf{T}(\mathbb{Z}_p)$ is a subset of $K_{H,p} \setminus \mathbf{H}(\mathbb{Z}_p)$ and so the previous lower bound for the size of this group would suffice. Let T denote the generic fibre of $\mathbf{T}$ and note that, by [18, Section 3.8.2], the hyperspecial subgroup $\mathbf{T}(\mathbb{Z}_p)$ is the maximal compact subgroup of $T(\mathbb{Q}_p)$. Therefore, if $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, a condition satisfied for all primes p greater than a uniform constant, $T_{\mathbb{Z}_p}$ is only a torus if

$$K_{T,p} := \mathrm{GL}_n(\mathbb{Z}_p) \cap T(\mathbb{Q}_p) = K_{H,p} \cap T(\mathbb{Q}_p) = K_{H,p} \cap \mathbf{T}(\mathbb{Z}_p)$$

is equal to $\mathbf{T}(\mathbb{Z}_p)$.

We claim that it is possible to choose $\mathbf{T}$ such that $T_{\mathbb{Z}_p}$ is not a torus. In particular, since, by [7, Exposé XXII, Section 8], every semisimple element of $\mathbf{H}$ is contained in a maximal torus, we are claiming that $\mathbf{H}(\mathbb{Z}_p) \setminus K_{H,p}$ contains a semisimple element. To see this, note that, by [7, Exposé XXII, Corollaire 1.10], the functor of maximal tori of $\mathbf{H}$ is representable by $\mathbf{H}/\mathbf{N}$, where $\mathbf{N}$ is the normaliser of a maximal torus in $\mathbf{H}$. By [7, Exposé XXII, paragraph following Lemme 4.5], and by [7, Exposé XXI, Proposition 5.9], the universal maximal torus $\underline{\mathbf{T}}$ of $\mathbf{H}$ (see [7, Exposé XXII, Section 8]) has the same dimension as $\mathbf{H}$. However, the morphism $u : \underline{\mathbf{T}} \to \mathbf{H}$ is quasi-finite. Hence, by [7, Exposé XXII, Proposition 8.1], the semisimple elements of $\mathbf{H}$ constitute a constructible set of dimension $\dim \mathbf{H}$, which therefore contains a Zariski open set. On the other hand, $\mathbf{H}(\mathbb{Z}_p) \setminus K_{H,p}$ is open in $\mathbf{H}(\mathbb{Z}_p)$ for the p-adic topology and so the claim follows.

By [15, Section 3.3, p. 134], every maximal compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ is conjugate to $\mathrm{GL}_n(\mathbb{Z}_p)$ by an element of $\mathrm{GL}_n(\mathbb{Q}_p)$. Hence, there exists a $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ such that

$$\mathbf{T}(\mathbb{Z}_p) = g\mathrm{GL}_n(\mathbb{Z}_p)g^{-1} \cap T(\mathbb{Q}_p).$$

Let $T_0 := g^{-1}Tg$. Hence, $\mathrm{GL}_n(\mathbb{Z}_p) \cap T_0(\mathbb{Q}_p)$ is a maximal compact open subgroup $K^m_{T_0,p}$ of $T_0(\mathbb{Q}_p)$ and, since $K_{T,p} = \mathrm{GL}_n(\mathbb{Z}_p) \cap T(\mathbb{Q}_p)$, conjugation by g^{-1} establishes a bijection

$$K_{T,p} \setminus \mathbf{T}(\mathbb{Z}_p) \leftrightarrow (g^{-1}\mathrm{GL}_n(\mathbb{Z}_p)g \cap T_0(\mathbb{Q}_p)) \setminus K^m_{T_0,p}.$$

The latter index is the size of the orbit $K^m_{T_0,p} \cdot g^{-1}\mathbb{Z}_p^n$ in the space of lattices of $\mathbb{Q}_p^n$. Note that

$$K^m_{T_0,p} = T_{0,\mathbb{Z}_p}(\mathbb{Z}_p).$$

Since T splits over an unramified extension of $\mathbb{Q}_p$, so too does T_0 and so, by [18, Section 3.8.2], $K^m_{T_0,p}$ is a hyperspecial subgroup. Therefore, $T_{0,\mathbb{Z}_p}$ is a torus.

By [7, Exposé X, Lemme 4.1], there is a canonical isomorphism

$$X^*(T_{0,\overline{\mathbb{Q}}_p}) \cong X^*(T_{0,\overline{\mathbb{F}}_p})$$

identifying the characters intervening in $\overline{\mathbb{Q}}_p^n$ and $\overline{\mathbb{F}}_p^n$. Thus, with respect to the image of the basis obtained using Lemma 6.5, the coordinates of the characters of $T_{0,\mathbb{F}_p}$ intervening in $\overline{\mathbb{F}}_p^n$ are bounded in absolute value by a uniform constant.

Therefore, by [8, Lemma 4.4.1], for all subspaces W of $\overline{\mathbb{F}}_p^n$, the group of connected components of the stabiliser of W in $T_{0,\overline{\mathbb{F}}_p}$ is of order bounded by a uniform constant. Since $T_{\mathbb{Z}_p}$ is not a torus, $T_{0,\mathbb{Z}_p}$ does not fix the lattice $g^{-1}\mathbb{Z}_p^n$ in the sense of [8, Section 3.3]. Therefore, [8, Proposition 4.3.9] implies that there exists a uniform constant c_8 such that the size of the orbit $T_{0,\mathbb{Z}_p}(\mathbb{Z}_p)\cdot g^{-1}\mathbb{Z}_p^n$ is greater than $c_8 p$. □

Lemma 6.7. *There exists a uniform constant c_9 such that, if $p \notin \Sigma(\widetilde{H}, \widetilde{K}_H)$ is a prime greater than c_9, then $p \notin \Sigma(H, K_H)$.*

Proof. Since $H_{\mathbb{Q}_p}$ is unramified and $\widetilde{K}_{H,p}$ is hyperspecial, there exist smooth reductive group schemes $\widetilde{\mathbf{H}}$ and $\mathbf{H}$ over $\mathbb{Z}_p$, the generic fibres of which are $\widetilde{H}_{\mathbb{Q}_p}$ and $H_{\mathbb{Q}_p}$, such that

$$\widetilde{K}_{H,p} = \widetilde{\mathbf{H}}(\mathbb{Z}_p).$$

Therefore, $\mathbf{H}(\mathbb{Z}_p)$ is a hyperspecial subgroup of $H(\mathbb{Q}_p)$. By [20, Lemma 2.3.1], the central isogeny $\pi_{\mathbb{Q}_p}$ extends uniquely to a central isogeny $\pi_{\mathbb{Z}_p} : \widetilde{\mathbf{H}} \to \mathbf{H}$.

Let $K^m_{H,p}$ be a maximal compact open subgroup containing $K_{H,p}$. Therefore, $K^m_{H,p}$ contains the image of $\widetilde{K}_{H,p}$. Since, by [18, Section 3.8.2], $\widetilde{K}_{H,p}$ is maximal, [14, Proposition 3.3] implies that

$$K^m_{H,p} = \mathbf{H}(\mathbb{Z}_p).$$

Since $\widetilde{K}_{H,p} = \widetilde{\mathbf{H}}(\mathbb{Z}_p)$, the map

$$\widetilde{\mathbf{H}}(\mathbb{Z}_p) \to K_{H,p} \setminus \mathbf{H}(\mathbb{Z}_p)$$

is trivial and we have seen that the cokernel is uniformly bounded. On the other hand, the proof of Lemma 6.6 shows that, if p is greater than a uniform constant and $K_{H,p} \subsetneq \mathbf{H}(\mathbb{Z}_p)$, then $[\mathbf{H}(\mathbb{Z}_p) : K_{H,p}]$ is at least a uniform constant times p. □

Proof of Theorem 6.1. Recall that

$$\mu(\Gamma_H \setminus X_H^+) = \frac{[\mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}}) : \mathrm{ad} \circ \pi(\widetilde{\Gamma}_H)]}{[\mathrm{ad}(\Gamma_H) : \mathrm{ad} \circ \pi(\widetilde{\Gamma}_H)]} \cdot \mu(\mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}}) \setminus H^{\mathrm{ad}}(\mathbb{R})^+).$$

Therefore, by Lemmas 6.2 and 6.3, we have

$$\mu(\Gamma_H \setminus X_H^+) > c_3 c_4^{-1} C^{-|\Sigma(H,K_H)|} \cdot [K^m_{\widetilde{H}} : \widetilde{K}_H] \cdot \mu(\mathrm{ad} \circ \pi(\Gamma^m_{\widetilde{H}}) \setminus H^{\mathrm{ad}}(\mathbb{R})^+),$$

so, by Lemma 6.4,

$$\mu(\Gamma \setminus X_H^+) > c_3 c_4^{-1} c_5 C^{-|\Sigma(H,K_H)|} \cdot [K^m_{\widetilde{H}} : \widetilde{K}_H] \cdot \Pi(\widetilde{H}, K^m_{\widetilde{H}})^{\delta_1}.$$

Lemma 6.6 implies that there exist uniform constants c_{10} and δ_4 such that

$$\mu(\Gamma \setminus X_H^+) > c_{10} C^{-|\Sigma(H,K_H)|} \cdot \Pi(\widetilde{H}, \widetilde{K}_H)^{\delta_4}.$$

Therefore, the result follows from Lemma 6.7. □

Proof of Theorem 1.4. Follows from Theorem 5.1 and Theorem 6.1. □

7. Choosing a suitable Hecke correspondence

In this section, we prove an analogue of [9, Theorem 8.1], demonstrating the existence of suitable Hecke correspondences. Recall that, if (G, X) is a Shimura datum, G^{ad} decomposes into a product of simple factors, which we denote G_i. Thus, X_{ad} decomposes into a product of factors X_i and, if X^+ is a connected component of X, then it decomposes into a product of factors X_i^+. If K_{ad} is equal to a product of compact open subgroups $K_i \subset G_i(\mathbb{A}_f)$, then $S_{K_{\mathrm{ad}}}(G^{\mathrm{ad}}, X_{\mathrm{ad}})$ is equal to the product of the $S_{K_i}(G_i, X_i)$. If K is a compact open subgroup of $G(\mathbb{A}_f)$, equal to the product of compact open subgroups K_l of $G(\mathbb{Q}_l)$, we will use the notation K^p to denote the product $\prod_{l \neq p} K_l$.

Theorem 7.1. *Let (G', X') be a Shimura datum such that $G' = G'^{\mathrm{ad}}$, let K' be a neat compact open subgroup of $G'(\mathbb{A}_f)$, equal to a product of compact open subgroups K'_p of $G'(\mathbb{Q}_p)$, and fix a faithful representation*

$$\rho : G' \hookrightarrow \mathrm{GL}_n.$$

There exist positive integers k and f such that, if

- *V is a strongly special subvariety of $S_{K'}(G', X')$, defined by (H, X_H),*
- *$p \notin \Sigma(H, K_H)$ is a prime such that $K'_p = G'_{\mathbb{Z}_p}(\mathbb{Z}_p)$,*
- *(G, X) is a Shimura subdatum of (G', X') such that V is contained in $S_K(G, X)$, where $K := K' \cap G(\mathbb{A}_f)$,*

then there exist a compact open subgroup

$$I_p \subset K_p := K'_p \cap G(\mathbb{Q}_p)$$

and an element $\alpha \in G(\mathbb{Q}_p)$ such that

- *$[K_p : I_p] \leq p^f$,*
- *if $I \subset K$ is the compact open subgroup $K^p I_p \subset G(\mathbb{A}_f)$,*

$$\tau : \mathrm{Sh}_I(G, X) \to \mathrm{Sh}_K(G, X)$$

  is the natural morphism, and $\widetilde{V} \subset S_I(G, X)_{\mathbb{C}}$ is an irreducible component of $\tau^{-1}(V)$, then $\widetilde{V} \subset T_\alpha(\widetilde{V})$,
- *for every $k_1, k_2 \in I_p$, the image of $k_1 \alpha k_2$ generates an unbounded subgroup of $G_i(\mathbb{Q}_p)$ for each i,*
- *$[I_p : I_p \cap \alpha I_p \alpha^{-1}] < p^k$.*

In this section, we will use the term *uniform* to mean depending only on (G', X'), K' and ρ. Firstly, we will deal with the matter of including a strongly special subvariety in its image under a Hecke correspondence:

Lemma 7.2. *There exists a uniform integer A such that, for any $\alpha \in H(\mathbb{A}_f)$,*

$$V \subset T_{\alpha^A}(V).$$

Proof. By definition, the subvariety V is the image of $X_H^+ \times \{1\}$ in $\mathrm{Sh}_K(G,X)$. Thus, consider a point $\overline{(x,1)} \in V$ with $x \in X_H^+$. Let

$$\pi : \widetilde{H} \to H$$

be the simply connected covering, whose degree we denote d, and consider an $\alpha \in H(\mathbb{A}_f)$. Therefore, for any positive integer A divisible by d, there exists an element $\beta \in \widetilde{H}(\mathbb{A}_f)$ such that $\pi(\beta) = \alpha^A$. By strong approximation applied to $\widetilde{H}$, $\beta = qk$, where $q \in \widetilde{H}(\mathbb{Q})$ and $k \in \pi^{-1}(K)$. Note that, since π is proper, $\pi^{-1}(K)$ is a compact open subgroup of $\widetilde{H}(\mathbb{A}_f)$. Since $\widetilde{H}(\mathbb{R})$ is connected, $\pi(q) \in H(\mathbb{R})^+$ and $\pi(q) \cdot x \in X_H^+$.

Thus, consider the point

$$\overline{(\pi(q) \cdot x, \pi(\beta))} \in T_{\alpha^A}(V).$$

By the previous discussion, this is equal to $\overline{(x,1)}$. Since d is bounded by a uniform integer D, setting $A = D!$ finishes the proof. □

In order to find suitable Hecke correspondences, we will also need the following two results on maximal split tori:

Lemma 7.3. *Let $p \notin \Sigma(H, K_H)$ be a prime such that $K'_p = G'_{\mathbb{Z}_p}(\mathbb{Z}_p)$. Then there exists a maximal split torus $S \subset H_{\mathbb{Q}_p}$ such that $S_{\mathbb{Z}_p}$ is a torus.*

Proof. Since $p \notin \Sigma(H, K_H)$, there exists a smooth reductive group scheme $\mathbf{H}$ over $\mathbb{Z}_p$ such that $\mathbf{H}_{\mathbb{Q}_p} = H_{\mathbb{Q}_p}$ and $\mathbf{H}(\mathbb{Z}_p) = K_{H,p}$. Let $\mathbf{S}$ be a maximal split torus of $\mathbf{H}$ and let S denote its generic fibre. By [18, Section 3.8.1], $S_{\mathbb{Z}_p}$ is a torus if

$$S_{\mathbb{Z}_p}(\mathbb{Z}_p) := \mathrm{GL}_n(\mathbb{Z}_p) \cap S(\mathbb{Q}_p) = K_{H,p} \cap S(\mathbb{Q}_p)$$

is equal to $\mathbf{S}(\mathbb{Z}_p)$. However, since $K_{H,p} = \mathbf{H}(\mathbb{Z}_p)$, it follows that $S_{\mathbb{Z}_p}(\mathbb{Z}_p)$ contains $\mathbf{S}(\mathbb{Z}_p)$ and so, by [18, Section 3.8.2], they are equal. □

Lemma 7.4. *Assume $H_{\mathbb{Q}_p}$ is quasi-split and let $S \subset H_{\mathbb{Q}_p}$ be a maximal split torus. There exists a basis of $X^*(S)$ such that the coordinates of the characters of S that intervene in $\mathbb{Q}_p^n$ are uniformly bounded in absolute value.*

Proof. Let $T \subset H_{\mathbb{Q}_p}$ be the centraliser of S in $H_{\mathbb{Q}_p}$. Since $H_{\mathbb{Q}_p}$ is quasi-split, T is a maximal torus of $H_{\mathbb{Q}_p}$. By [22, Section 7.4], there exists an isogeny $T \to S \times A$, where A is the maximal anisotropic subtorus of T, and the degree d of this isogeny is bounded by $[L_T : \mathbb{Q}]^{\dim T}$, where L_T is the splitting field of T. Note that $\dim T$ is bounded by the absolute rank of G and that, by [6, Theorem 5.1], $[L_T : \mathbb{Q}]$ is bounded in terms of the dimension of T.

Consider the map of characters

$$\varphi : \chi \mapsto \chi_S + \chi_A : X^*(T) \to X^*(S) \oplus X^*(A)$$

induced by the inclusions $S \subset T$ and $A \subset T$. The characters of S intervening in $\mathbb{Q}_p^n$ are precisely the χ_S such that $\chi \in X^*(T)$ intervenes in $\overline{\mathbb{Q}}_p^n$.

Now consider the embedding

$$\phi : X^*(S) \oplus X^*(A) \hookrightarrow X^*(T)$$

induced by the above isogeny. By Lemma 6.5 there exists a basis $\{e_1, \dots, e_r\}$ of $X^*(T)$ such that the coordinates of the characters of T intervening in $\overline{\mathbb{Q}}_p^n$ are bounded in absolute value by a uniform constant B'. Given a character of T, its coordinates increase in absolute value by at most a factor of d under $\phi \circ \varphi$.

Thus, let $\{\chi_i\}$ be the characters of T intervening in $\overline{\mathbb{Q}}_p^n$ and let $\{\chi_{i,S} + \chi_{i,A}\}$ be their images in $X^*(S) \oplus X^*(A)$. Write the image of $\chi_{i,S} + \chi_{i,A}$ under ϕ as

$$\sum_{j=1}^{r} n_{i,j} e_j.$$

Hence, $|n_{i,j}| < B := dB'$ for all i and j and $n_{i,j} = n_{i,S,j} + n_{i,A,j}$, where

$$\sum_{j=1}^{r} n_{i,S,j} e_j \quad \text{and} \quad \sum_{j=1}^{r} n_{i,A,j} e_j$$

are the images of the $\chi_{i,S}$ and $\chi_{i,A}$ under ϕ, respectively. Therefore, either $|n_{i,S,j}| < B$ for all i and j, or there exist i and j such that $|n_{i,S,j}| \geq B$, in which case $n_{i,S,j}$ and $n_{i,A,j}$ are of opposite signs.

Assume the latter, letting χ_i denote the corresponding character and letting $n_{i,S,j}$ denote the coefficient with absolute value at least B. Since our representation of T was defined over $\mathbb{Q}_p$, for each $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\tau\chi_i$ also intervenes. Since S is split,

$$\tau\chi_{i,S} = \chi_{i,S} \quad \text{for every } \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p).$$

Therefore, the image of $\tau\chi_i$ in $X^*(S) \oplus X^*(A)$ varies over $\chi_{i,S} + \tau\chi_{i,A}$ and, since A is anisotropic,

$$\sum_{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \tau\chi_{i,A} = 0.$$

Thus, there exists a $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that the coefficient of e_j corresponding to the image of $\tau\chi_{i,A}$ under ϕ is of the opposite sign to $n_{i,A,j}$. But then this coefficient is of the same sign as $n_{i,S,j}$, which implies that the sum of these two coefficients has absolute value greater than or equal to B, which is a contradiction.

Therefore, with respect to the basis $\{e_1, \dots, e_r\}$ of $X^*(T)$, the coordinates of the characters of S intervening in $\mathbb{Q}_p^n$ are bounded in absolute value by B. Since our representation is faithful, these characters generate $X^*(S)$ and so, as a submodule of $X^*(T)$, there are only finitely many possibilities for $X^*(S)$. For each such possibility, choose a basis and consider the maximum of the absolute values of the coordinates of the characters intervening in $\mathbb{Q}_p^n$. □

Proof of Theorem 7.1. By Lemma 7.3, since $p \notin \Sigma(H, K_H)$, we can find a non-trivial, maximal, split torus $S \subset H_{\mathbb{Q}_p}$ such that $S_{\mathbb{Z}_p}$ is a torus. Furthermore, by Lemma 7.4, there is a basis of $X^*(S)$ such that the coordinates of the characters intervening in $\mathbb{Q}_p^n$ are uniformly bounded in absolute value. Let $\pi_i : G \to G_i$ denote the natural morphisms. By [13, SV3], it follows that $S_i := \pi_i(S)$ is a non-trivial, split torus.

As $K'_p = G'_{\mathbb{Z}_p}(\mathbb{Z}_p)$, the compact open subgroup $K_p := K'_p \cap G(\mathbb{Q}_p)$ of $G(\mathbb{Q}_p)$ is equal to $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ and

$$[K_p : K_p \cap \alpha K_p \alpha^{-1}] = [K_p : K_p \cap \alpha G_{\mathbb{Z}_p}(\mathbb{Z}_p)\alpha^{-1}] \quad \text{for any } \alpha \in S(\mathbb{Q}_p).$$

By [8, Lemma 7.4.3], for $q_i = \pi_{i|S}$ and $e = A$ (the positive integer given by Lemma 7.2), there exist a uniform constant k' and an element $\alpha \in S(\mathbb{Q}_p)$ such that no $\pi_i(\alpha)$ lies in a compact subgroup of $S_i(\mathbb{Q}_p)$ and

$$[K_p : K_p \cap \alpha^A G_{\mathbb{Z}_p}(\mathbb{Z}_p)\alpha^{-A}] < p^{k'}.$$

Next we define I_p following [9, Section 8.3.2]. Since S is a split torus and $S_{\mathbb{Z}_p}$ is a torus, it follows that $G_{\mathbb{Z}_p}(\mathbb{Z}_p) = K_p$ is in good position with respect to S (using the terminology of [9, Section 4.1.6]).

Let f be the constant, defined in [9, Lemma 8.1.6 (b)], for the group G'. We claim that there exists an Iwahori subgroup I_p^1 of $G(\mathbb{Q}_p)$ such that

$$[K_p : K_p \cap I_p^1] < p^f.$$

To see this we let K_p^1 be any maximal compact subgroup of $G(\mathbb{Q}_p)$ containing K_p. Since K_p is in good position with respect to S, so too is K_p^1. Thus, by [9, Lemma 8.1.6 (b) (i)], there exists an Iwahori subgroup $I_p^1 \subset K_p^1$ in good position with respect to S satisfying $[K_p^1 : I_p^1] < p^f$. Thus,

$$[K_p : K_p \cap I_p^1] < p^f.$$

Let S' be a maximal split torus of $G_{\mathbb{Q}_p}$ containing S such that I_p^1 is in good position with respect to S'. Let M be the centraliser of S' in $G_{\mathbb{Q}_p}$. Let $\mathcal{B}$ be the (extended) Bruhat–Tits building of G and $\mathcal{A} \subset \mathcal{B}$ the apartment of $\mathcal{B}$ associated to S'.

The group $M(\mathbb{Q}_p)$ acts on $\mathcal{A}$ as follows: we denote by

$$\mathrm{ord}_M : M(\mathbb{Q}_p) \to X_*(M)_{\mathbb{Q}_p}$$

the homomorphism characterised by

$$\langle \mathrm{ord}_M(m), \chi \rangle = \mathrm{ord}_p(\chi(m)) \quad \text{for all } \chi \in X^*(M)_{\mathbb{Q}_p},$$

where ord_p is the normalised additive valuation on $\mathbb{Q}_p^*$ and $X_*(M)_{\mathbb{Q}_p}$ (resp. $X^*(M)_{\mathbb{Q}_p}$) denotes the group of cocharacters (resp. characters) of M defined over $\mathbb{Q}_p$. Let $\Lambda \subset X_*(M)_{\mathbb{Q}_p}$ be the free $\mathbb{Z}$-module $\mathrm{ord}_M(M(\mathbb{Q}_p))$. Then $M(\mathbb{Q}_p)$ acts on $\mathcal{A}$ via Λ-translations.

Let K_p^m be a special compact subgroup containing I_p^1 and let $x \in \mathcal{A}$ be the unique special vertex fixed by K_p^m. Recall the element $\alpha \in S(\mathbb{Q}_p)$ chosen above. The vector $\mathrm{ord}_M(\alpha) \in \Lambda$ is non-trivial. Let C be the chamber of $\mathcal{A}$ fixed pointwise by I_l^1 (it contains x in its closure). Consider the chamber $C' = C + \mathrm{ord}_M(\alpha)$. Let $\mathcal{C} \subset \mathcal{A}$ be the unique Weyl chamber with apex x containing C'. Finally, let I_p^2 be the Iwahori subgroup of $G(\mathbb{Q}_p)$ fixing the unique chamber of $\mathcal{C}$ containing x in its closure.

Define I_p as the intersection $K_p \cap I_p^1 \cap I_p^2$. Since I_p^2 stabilises a chamber in $\mathcal{A}$ it is also in good position with respect to S' and, therefore, S. Thus, I_p is in good position with respect to S. It follows from [9, Lemma 8.1.6 (b) (ii)] that

$$\begin{aligned} [K_p : I_p] &= [K_p : K_p \cap I_p^1 \cap I_p^2] \\ &\leq [K_p^1 : I_p^1 \cap I_p^2] \\ &< p^f, \end{aligned}$$

which is the first condition of Theorem 7.1. By Lemma 7.2, we have $\widetilde{V} \subset T_{\alpha^A}(\widetilde{V})$, which is the second condition of Theorem 7.1.

Let $S_i' := \pi_i(S')$ and denote by $M_i := \pi_i(M)$ its centraliser. Let C_i be the unique chamber of the Bruhat–Tits building $\mathcal{B}_i$ of $G_{i,\mathbb{Q}_p}$ fixed by the Iwahori subgroup $\pi_i(I_p^2)$ and let x_i be the vertex in the closure of C_i fixed by $\pi_i(K_p^m)$. Finally, let $\mathcal{C}_i$ be the unique Weyl chamber of the apartment $\mathcal{A}_i$ corresponding to S_i' with apex x_i and containing C_i.

For M_i we have a homomorphism

$$\mathrm{ord}_{M_i} : M_i(\mathbb{Q}_p) \to X_*(M_i)_{\mathbb{Q}_p},$$

defined analogously to ord_M. We denote the image $\mathrm{ord}_{M_i}(M_i(\mathbb{Q}_p)$ by Λ_i. Thus, $M_i(\mathbb{Q}_p)$ acts on $\mathcal{A}_i$ by Λ_i-translations. We denote by $\Lambda_i^+ \subset \Lambda_i$ the positive cone stabilising $\mathcal{C}_i$. By virtue of our choice of I_p^2, since $\pi_i(\alpha)$ does not lie in a compact subgroup of $S_i(\mathbb{Q}_p)$, it follows that $\mathrm{ord}_{M_i}(\pi(\alpha))$ lies in $\Lambda_i^+ \setminus \{0\}$. Hence, $\mathrm{ord}_{M_i}(\pi_i(\alpha^A))$ must also belong to $\Lambda_i^+ \setminus \{0\}$. Thus, by [9, Proposition 8.1.4], for any $k_1, k_2 \in I_p^2$ (in particular for any $k_1, k_2 \in I_p$), $\pi_i(k_1\alpha^A k_2)$ generates an unbounded subgroup of $G_i(\mathbb{Q}_p)$. This is the third condition of Theorem 7.1.

Finally, from the previous discussion we have

$$\begin{aligned}
[I_p : I_p \cap \alpha^A I_p \alpha^{-A}] &= [I_p : I_p \cap \alpha^A K_p \alpha^{-A}] \cdot [I_p \cap \alpha^A K_p \alpha^{-A} : I_p \cap \alpha^A I_p \alpha^{-A}] \\
&\leq [K_p : K_p \cap \alpha^A K_p \alpha^{-A}] \cdot [K_p : I_p] \\
&\leq [K_p : K_p \cap \alpha^A G_{\mathbb{Z}_p}(\mathbb{Z}_p) \alpha^{-A}] \cdot [K_p : I_p] \\
&\leq p^{k'+f} := p^k.
\end{aligned}$$

This is the fourth condition of Theorem 7.1. □

8. The geometric criterion

In this section, we explain the procedure via which we replace strongly special subvarieties with higher dimensional, strongly special subvarieties given the existence of suitable Hecke correspondences:

Theorem 8.1. *Let (G, X) be a Shimura datum and let $K \subset G(\mathbb{A}_f)$ be a neat compact open subgroup, the product of compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. Let X^+ be a connected component of X and let V be a special subvariety of $S_K(G, X)$. Suppose that V is properly contained in a Hodge generic, irreducible subvariety Z of $S_K(G, X)$ and assume that there exist a prime p and an $\alpha \in G(\mathbb{Q}_p)$ such that*

- *$Z \subset T_\alpha(Z)$,*
- *for every $k_1, k_2 \in K_p$, the element $k_1\alpha k_2$ generates an unbounded subgroup of $G_i(\mathbb{Q}_p)$ for each i.*

Then Z contains a special subvariety V' containing V properly. Moreover, if V is strongly special, then V' is strongly special.

This theorem is very similar to [9, Theorem 7.2.1] and the proof here is nearly a carbon copy of the proof found in there. Our situation is slightly simplified by the fact that Z is geometrically irreducible. Ensuring that V' properly contains V is where we require the stronger condition on α.

Lemma 8.2. *If the conclusion of Theorem* 8.1 *holds for all Shimura data* (G, X) *with G semisimple of adjoint type, then it holds for all Shimura data.*

Proof. Consider the situation in Theorem 8.1. We have a finite morphism of Shimura varieties

$$f : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K_{\mathrm{ad}}}(G^{\mathrm{ad}}, X_{\mathrm{ad}}).$$

Let Z_{ad} be the image of Z under this morphism. Similarly, let V_{ad} be the image of V. Thus, V_{ad} is a special subvariety of $S_{K_{\mathrm{ad}}}(G^{\mathrm{ad}}, X_{\mathrm{ad}})$.

Let α_{ad} denote the image of α in $G^{\mathrm{ad}}(\mathbb{Q}_p)$. The inclusion $Z \subset T_\alpha(Z)$ implies that

$$Z_{\mathrm{ad}} \subset T_{\alpha_{\mathrm{ad}}}(Z_{\mathrm{ad}}).$$

As K_{ad} is a product of compact open subgroups $K_{\mathrm{ad},p} \subset G^{\mathrm{ad}}(\mathbb{Q}_p)$, the second condition of Theorem 8.1 implies the analogous condition for α_{ad} and $K_{\mathrm{ad},p}$.

As irreducible components of the preimage of a special subvariety by a finite morphism of Shimura varieties are special, it is enough to show that Z_{ad} contains a special subvariety V'_{ad} containing V_{ad} properly. □

Thus, in this section, we henceforth assume that G is semisimple of adjoint type. We fix a $\mathbb{Z}$-structure on G by choosing a finitely generated free $\mathbb{Z}$-module W, choosing a faithful representation

$$\xi : G \hookrightarrow \mathrm{GL}(W_{\mathbb{Q}})$$

and taking the Zariski closure of G in $\mathrm{GL}(W)$. We may choose ξ in such a way that K is contained in $\mathrm{GL}(W_{\hat{\mathbb{Z}}})$. This canonically induces a $\mathbb{Z}$-variation of Hodge structures $\mathcal{F}$ on $\mathrm{Sh}_K(G, X)$ and, in particular, on $S_K(G, X)$ (see [8, Section 3.2]).

Let z be a Hodge generic point of the smooth locus Z^{sm} of Z. Let $\pi_1(Z^{\mathrm{sm}}, z)$ be the topological fundamental group of Z^{sm} at the point z. We choose a point $x \in X$ lying above z. This choice canonically identifies the fibre at z of the locally constant sheaf underlying $\mathcal{F}$ with the $\mathbb{Z}$-module W. The action of $\pi_1(Z^{\mathrm{sm}}, z)$ on this fibre is described by the monodromy representation

$$\rho : \pi_1(Z^{\mathrm{sm}}, z) \to \pi_1(S_K(G, X), z) = G(\mathbb{Q})_+ \cap K \hookrightarrow \mathrm{GL}(W).$$

Since Z is Hodge generic in $\mathrm{Sh}_K(G, X)$, the Mumford–Tate group of $\mathcal{F}_{|Z^{\mathrm{sm}}}$ at z is G. Thus, by [12, Section 1.4], given that the group G is adjoint, the group $\rho(\pi_1(Z^{\mathrm{sm}}, z))$ is Zariski dense in G. Having fixed a prime p (as in Theorem 8.1), [9, Proposition 4.2.1] implies that the p-adic closure of $\rho(\pi_1(Z^{\mathrm{sm}}, z))$ in $G(\mathbb{Z}_p)$ is a compact open subgroup $K'_p \subset K_p$.

We have a Galois, pro-étale cover

$$\pi_{K_p} : \mathrm{Sh}_{K^p}(G, X) \to \mathrm{Sh}_K(G, X),$$

with group K_p, as defined in [9, Section 4.1.3]. Let $\widetilde{Z}$ be an irreducible component of the preimage of Z in $\mathrm{Sh}_{K^p}(G, X)$ and let $\widetilde{V}$ be an irreducible component of the preimage of V in $\widetilde{Z}$. By [9, Lemma 7.2.3], we have

Lemma 8.3. *The variety* $\widetilde{Z}$ *is stabilised by the group* K'_p *and the set of irreducible components of* $\pi^{-1}_{K_p}(Z)$ *is naturally identified with the finite set* K_p/K'_p.

The inclusion $Z \subset T_\alpha(Z)$ implies that $\widetilde{Z}$ is an irreducible component of $\pi_{K_p}^{-1}(T_\alpha(Z))$. However, these components are of the form $\widetilde{Z} \cdot k_1\alpha k_2$ for $k_1, k_2 \in K_p$. Therefore, there exist $k_1, k_2 \in K_p$ such that $\widetilde{Z} = \widetilde{Z} \cdot k_1\alpha k_2$.

Corollary 8.4. *Let U_p be the group generated by K'_p and $k_1\alpha k_2$. The variety $\widetilde{Z}$ is stabilised by the group U_p.*

We now conclude the proof of Theorem 8.1. Again, let $\pi_i : G \to G_i$ denote the natural morphisms. By the condition placed on α, the group $\pi_i(U_p)$ is unbounded in $G_i(\mathbb{Q}_p)$ for all i. Let $G_{1,\mathbb{Q}_p} = \prod_i H_i$ be the decomposition of $G_{1,\mathbb{Q}_p}$ into simple factors. Up to renumbering, we can assume that the projection of U_p to $H_1(\mathbb{Q}_p)$ is unbounded in $H_1(\mathbb{Q}_p)$. Let

$$\tau : \widetilde{H}_1 \to H_1$$

be the universal cover of H_1. We have ([9, Lemma 7.2.6]):

Lemma 8.5. *The group $U_p \cap H_1(\mathbb{Q}_p)$ contains the group $\tau(\widetilde{H}_1(\mathbb{Q}_p))$ with finite index.*

Let $K_{p,1}$ be the compact open subgroup $\pi_1(K_p)$ of $G_{1,\mathbb{Q}_p}$ and let $K_{p,>1}$ be the projection of K to

$$G_{>1,\mathbb{Q}_p} := \prod_{i>1} G_{i,\mathbb{Q}_p}.$$

As U_p is an open subgroup of $G(\mathbb{Q}_p)$, it contains a compact open subgroup of $G_{1,\mathbb{Q}_p}$ and, in particular, a compact open subgroup $U_{p,1}$ of $K_{p,1} \cap \prod_{i>1} H_i(\mathbb{Q}_p)$. Similarly, U_p contains a compact open subgroup $U_{p,>1}$ of $K_{p,>1}$. By the previous lemma, U_p contains the unbounded subgroup $\tau(\widetilde{H}_1(\mathbb{Q}_p)) \cdot U_{p,1} \cdot U_{p,>1}$. We make the definition ([9, Definition 7.2.7]):

Definition 8.6. We replace U_p by its subgroup $\tau(\widetilde{H}_1(\mathbb{Q}_p)) \cdot U_{p,1} \cdot U_{p,>1}$. We denote by V' the Zariski closure of $\pi_{K_p}(\widetilde{V} \cdot U_p)$.

Since $\widetilde{Z}$ is stabilised by the group U_p, the variety V' is a subvariety of Z. Therefore, let $K_i := \pi_i(K)$ and let $\mathcal{K}$ be the neat compact open subgroup $\prod_i K_i$. We have the natural finite morphism

$$f : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{\mathcal{K}}(G, X)$$

of Shimura varieties and we let $\mathcal{V}' := f(V')$ and $\mathcal{V} := f(V)$. The proof of [9, Lemma 7.2.8] demonstrates that

$$\mathcal{V}' = S_{K_1}(G_1, X_1) \times \mathcal{V}'_{>1},$$

where $\mathcal{V}'_{>1}$ is the special subvariety of $\prod_{i>1} S_{K_i}(G_i, X_i)$ given by the projection of $\mathcal{V}'$. Hence, $\mathcal{V}'$ is a strongly special subvariety of $S_{\mathcal{K}}(G, X)$ and, therefore, since f is a finite morphism of Shimura varieties, V' is a strongly special subvariety of $S_K(G, X)$. Furthermore, after possibly renumbering the G_i (which we are free to do due to the condition placed on α), we may assume that $\mathcal{V}'$ properly contains $\mathcal{V}$. Therefore, V is properly contained in V', which concludes the proof of Theorem 8.1.

9. Proof of Theorem 1.3

In this section we will prove Theorem 1.3. In fact, we will prove the following, equivalent statement:

Theorem 9.1. *Let S be a Shimura variety and let Σ be a set of strongly special subvarieties contained in S. Let Z be an irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ in S. Then Z is a strongly special subvariety of S.*

Lemma 9.2. *Theorem* 9.1 *is equivalent to Theorem* 1.3.

Proof. Consider the situation described in Theorem 9.1. If we assume that Theorem 1.3 holds, then there exists a finite set $\{V_1, \dots, V_k\}$ of strongly special subvarieties contained in Z such that, for every $V \in \Sigma$, V is contained in one of the V_i. Therefore, $\boldsymbol{\Sigma}$ is contained in the union of the V_i, which is itself contained in Z. Since Z is an irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$, it must be equal to one of the V_i, proving Theorem 9.1.

Now consider the situation described in Theorem 1.3 and consider the set Σ of all strongly special subvarieties of S contained in Z. If we assume that Theorem 9.1 holds, the Zariski closure of $\boldsymbol{\Sigma}$ is a union of finitely many strongly special subvarieties $V_1, \dots, V_k$. Thus, any strongly special subvariety contained in Z is contained in one of the V_i, proving Theorem 1.3. □

Note that, in order to prove Theorem 9.1, we may assume that the elements of Σ are of equal dimension. We first prove Theorem 9.3, following the proof of [9, Theorem 9.2.1]:

Theorem 9.3. *Let (G', X') be a Shimura datum such that $G' = G'^{\mathrm{ad}}$ and fix a faithful representation*

$$\rho : G' \hookrightarrow \mathrm{GL}_n.$$

Let K' be a neat compact open subgroup of $G'(\mathbb{A}_f)$, equal to a product of compact open subgroups $K'_p \subset G'(\mathbb{Q}_p)$, such that $K' \subset \mathrm{GL}_n(\hat{\mathbb{Z}})$. Let k and f be the positive integers given by Theorem 7.1.

Let Σ be a set of strongly special subvarieties contained in $S_{K'}(G', X')$. Assume that the elements of Σ are of equal dimension d and that the Zariski closure Z of $\boldsymbol{\Sigma}$ is irreducible. For each $V \in \Sigma$, let (H_V, X_V) be the Shimura subdatum defining V and put $\Pi_V := \Pi(H_V, K_H)$.

Let (G, X) be a Shimura subdatum of (G', X') such that Z is contained and Hodge generic in $S_K(G, X)$, where $K := K' \cap G(\mathbb{A}_f)$. Let $r := \dim Z - d > 0$ and make one *of the following assumptions:*

- *The Π_V are bounded as V ranges through Σ.*
- *For each $V \in \Sigma$, there exists a prime p not dividing Π_V such that $K'_p = G'_{\mathbb{Z}_p}(\mathbb{Z}_p)$ and*

$$p^{(k+2f)\cdot 2^r} \cdot (\deg_{\mathcal{L}_K} Z)^{2^r} < c \cdot \Pi_V^{\delta}.$$

Then, for each $V \in \Sigma$, Z contains a strongly special subvariety of $S_{K'}(G', X')$ containing V properly.

In this section, we will use the term *uniform* to mean depending only on (G', X'), K' and ρ.

Proof. Firstly, we consider the case that, as V ranges through Σ, Π_V is bounded. That is to say, the primes dividing any given Π_V belong to a fixed, finite set, whose product we denote Π.

By Theorem 7.1, for any prime p not dividing Π such that $K'_p = G'_{\mathbb{Z}_p}(\mathbb{Z}_p)$, there exists a compact open subgroup

$$I_p \subset K_p := K'_p \cap G(\mathbb{Q}_p) = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$$

and an element $\alpha \in G(\mathbb{Q}_p)$ satisfying the four requirements of Theorem 7.1, for each $V \in \Sigma$. However, in this case we will choose these objects slightly more precisely: recall that, by Lemma 7.3, for each $V \in \Sigma$, there exists a non-trivial, maximal, split torus $S_V \subset H_{V,\mathbb{Q}_p}$ such that $S_{V,\mathbb{Z}_p}$ is a torus. Since S_V is split, it is conjugate via an element of $\mathrm{GL}_n(\mathbb{Q}_p)$ to a subtorus of the diagonal matrices. By Lemma 7.4, after possibly replacing $\boldsymbol{\Sigma}$ by a Zariski dense subset, we may assume that this torus is fixed, i.e. that the S_V are all conjugate by elements of $\mathrm{GL}_n(\mathbb{Q}_p)$ to a fixed torus $S := S_{V_0}$ for some $V_0 \in \Sigma$. Let $I_p \subset K_p$ and $\alpha \in G(\mathbb{Q}_p)$ be the objects given by Theorem 7.1 applied to V_0.

Now consider another $V \in \Sigma$ and let $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ be such that $gS_Vg^{-1} = S$. Since $S_{V,\mathbb{Z}_p}$ is a torus, S stabilises the lattice $g\mathbb{Z}_p^n$. Therefore, by [8, Lemma 3.3.1], since $S_{\mathbb{Z}_p}$ is a torus, there exists an element $c \in Z_G(S)(\mathbb{Q}_p)$ such that $g\mathbb{Z}_p^n = c\mathbb{Z}_p^n$, where $Z_G(S)$ is the centraliser of S in G. Therefore, there exists an element $k \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $g = ck$ and so the S_V are all conjugate by elements of $\mathrm{GL}_n(\mathbb{Z}_p)$. If we further assume that p is a prime such that $G_{\mathbb{F}_p}$ is smooth, the final paragraph of the proof of [8, Proposition 7.3.1] explains that, again, after possibly replacing $\boldsymbol{\Sigma}$ by a Zariski dense subset, we may assume that the S_V are all conjugate by elements of K_p and, therefore, by elements of I_p.

Therefore, for each $V \in \Sigma$, we let $g_V \in I_p$ be such that $S_V = g_V S g_V^{-1}$. It follows that I_p and $\alpha_V := g_V \alpha g_V^{-1}$ satisfy the requirements of Theorem 7.1 applied to V. Furthermore, if we let $I \subset K$ be the compact open subgroup $K^p I_p \subset G(\mathbb{A}_f)$, then the Hecke correspondences T_{α_V} on $\mathrm{Sh}_I(G,X)$ all coincide with T_α.

Let

$$\tau : \mathrm{Sh}_I(G,X) \to \mathrm{Sh}_K(G,X)$$

be the induced morphism of Shimura varieties and let $\widetilde{Z}$ be an irreducible component of the preimage $\tau^{-1}(Z)$. For each $V \in \Sigma$, let $\widetilde{V} \subset S_I(G,X)$ be an irreducible component of the preimage $\tau^{-1}(V)$ contained in $\widetilde{Z}$. Each $\widetilde{V}$ is a strongly special subvariety of $S_I(G,X)$ defined by the Shimura subdatum (H_V, X_V). Denote the set of the $\widetilde{V}$ by $\widetilde{\Sigma}$. By the second requirement of Theorem 7.1, we have $\widetilde{V} \subset T_\alpha(\widetilde{V})$ for every $\widetilde{V} \in \widetilde{\Sigma}$. Hence, $\widetilde{\boldsymbol{\Sigma}}$ is contained in $\widetilde{Z} \cap T_\alpha(\widetilde{Z})$ and, therefore, $\widetilde{Z} \subset T_\alpha(\widetilde{Z})$.

As α satisfies the third requirement of Theorem 7.1, we can apply Theorem 8.1 to this α and conclude that, for each $\widetilde{V} \in \widetilde{\Sigma}$, there exists a special subvariety $\widetilde{V}' \subset \widetilde{Z}$ containing $\widetilde{V}$ properly whose image in $\mathrm{Sh}_{K'}(G',X')$ is strongly special. As τ preserves the property of being special, exhibiting a special subvariety $V' \subset Z$ containing V properly is equivalent to exhibiting a special subvariety $\widetilde{V}' \subset \widetilde{Z}$ containing $\widetilde{V}$ properly.

Thus, we consider the case that Π_V is unbounded as V ranges through Σ. Hence, we may assume that Π_V is larger than any uniform constant. We proceed by induction on r. Consider first the case $r = 1$ and let $V \in \Sigma$.

By the second assumption of Theorem 9.3, there exists a compact open subgroup $I_p \subset K_p$ and an element $\alpha \in G(\mathbb{Q}_p)$ satisfying the four requirements of Theorem 7.1 applied to V.

Let $I \subset K$ be the compact open subgroup $K^p I_p \subset G(\mathbb{A}_f)$ and let

$$\tau : \mathrm{Sh}_I(G, X) \to \mathrm{Sh}_K(G, X)$$

be the induced morphism of Shimura varieties. It follows from the first requirement of Theorem 7.1 that the degree of τ is bounded above by p^f.

Let $\widetilde{V} \subset S_I(G, X)$ be an irreducible component of the preimage $\tau^{-1}(V)$. It is a strongly special subvariety of $S_I(G, X)$ defined by the Shimura subdatum (H_V, X_V) of (G, X). By the projection formula (see [9, Proposition 5.3.2 (1)]) and Theorem 1.4,

$$\deg_{\mathcal{L}_I} \widetilde{V} \geq \deg_{\mathcal{L}_K} V > c \cdot \Pi_V^{\delta}.$$

Let $\widetilde{Z}$ be an irreducible component of the preimage $\tau^{-1}(Z)$ containing $\widetilde{V}$. Thus, $\widetilde{Z}$ is Hodge generic in $\mathrm{Sh}_I(G, X)$ and

$$\deg_{\mathcal{L}_I} \widetilde{Z} \leq p^f \cdot d_Z.$$

As τ preserves the property of being special, exhibiting a special subvariety $V' \subset Z$ containing V properly is equivalent to exhibiting a special subvariety $\widetilde{V}' \subset \widetilde{Z}$ containing $\widetilde{V}$ properly.

By the second requirement of Theorem 7.1, we have $\widetilde{V} \subset T_\alpha(\widetilde{V})$. Hence, $\widetilde{V} \subset \widetilde{Z} \cap T_\alpha(\widetilde{Z})$. Given their dimensions, if $\widetilde{Z}$ and $T_\alpha(\widetilde{Z})$ intersect properly, then $\widetilde{V}$ is an irreducible component of the intersection. Thus,

$$\begin{aligned} c \cdot \Pi_V^{\delta} < \deg_{\mathcal{L}_I} \widetilde{V} &\leq \deg_{\mathcal{L}_I}(\widetilde{Z} \cap T_\alpha(\widetilde{Z})) \\ &\leq (\deg_{\mathcal{L}_I} \widetilde{Z})^2 \cdot [I_p : I_p \cap \alpha I_p \alpha^{-1}] \\ &< p^{k+2f} \cdot d_Z^2, \end{aligned}$$

contradicting the second assumption of the theorem. Therefore, the intersection cannot be proper. Thus, $\widetilde{Z} \subset T_\alpha(\widetilde{Z})$ and, since α satisfies the second condition of Theorem 8.1, there exists a special subvariety $\widetilde{V}' \subset \widetilde{Z}$ containing $\widetilde{V}$ properly whose image in $\mathrm{Sh}_{K'}(G', X')$ is strongly special.

Therefore, we consider the case $r > 1$. Suppose that the conclusion of Theorem 9.3 holds for all subvarieties V and Z of $\mathrm{Sh}_K(G, X)$ as in the statement of Theorem 9.3 such that $0 < \dim Z - d < r$ and consider the case that $\dim Z = d + r$. We have $\widetilde{V}$, $\widetilde{Z}$, a compact open subgroup $I \subset K$ and an $\alpha \in G(\mathbb{Q}_p)$, constructed as in the case $r = 1$, where

$$\deg_{\mathcal{L}_I} \widetilde{V} > c \cdot \Pi_V^{\delta} \quad \text{and} \quad \deg_{\mathcal{L}_I} \widetilde{Z} \leq p^f \cdot d_Z.$$

Suppose that $\widetilde{Z} \subset T_\alpha(\widetilde{Z})$. In this case we can apply Theorem 8.1 to deduce that there exists a special subvariety $\widetilde{V}' \subset \widetilde{Z}$ containing $\widetilde{V}$ properly whose image in $\mathrm{Sh}_{K'}(G', X')$ is strongly special.

Therefore, suppose that the intersection $\widetilde{Z} \cap T_\alpha(\widetilde{Z})$ is proper. By the second requirement of Theorem 7.1, $\widetilde{V} \subset \widetilde{Z} \cap T_\alpha(\widetilde{Z})$. Choose an irreducible component $\widetilde{Y} \subset S_I(G, X)$ of $\widetilde{Z} \cap T_\alpha(\widetilde{Z})$ containing $\widetilde{V}$ and denote its image in $\mathrm{Sh}_K(G, X)$ by Y. Thus, Y is irreducible and satisfies $r_Y := \dim Y - d < r$. To show that $r_Y > 0$ it suffices to check that $\widetilde{V}$ is not a component of $\widetilde{Z} \cap T_\alpha(\widetilde{Z})$. However, if this were true, we would have

$$c \cdot \Pi_V^{\delta} < p^{k+2f} \cdot d_Z^2,$$

as in the case $r = 1$, contradicting the second assumption of Theorem 9.3.

Let (P, X_P) be a Shimura datum of (G, X), defining the smallest special subvariety of $S_I(G, X)$ containing $\widetilde{Y}$. Let $X_P^+ \subset X^+$ be the corresponding connected component of X_P. Define $K_P := K \cap P(\mathbb{A}_F)$ and $I_P := I \cap P(\mathbb{A}_f)$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{I_P}(P, X_P) & \xrightarrow{q} & \mathrm{Sh}_I(G, X) \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \mathrm{Sh}_{K_P}(P, X_P) & \xrightarrow{q} & \mathrm{Sh}_K(G, X). \end{array}$$

Let $\widetilde{V}_P$ be an irreducible component of $q^{-1}(\widetilde{V})$ contained in $S_{I_P}(P, X_P)$; let $V_P := \tau(\widetilde{V}_P)$.

Let $\widetilde{Y}_P \subset S_{I_P}(P, X_P)$ be an irreducible component of $q^{-1}(\widetilde{Y})$ containing $\widetilde{V}_P$. In particular, $\widetilde{Y}_P$ is a Hodge generic subvariety of $S_{I_P}(P, X_P)$. Define $Y_P := \tau(\widetilde{Y}_P)$, a Hodge generic subvariety of $S_{K_P}(P, X_P)$.

We have

$$\deg_{\mathfrak{L}_{K_P}} Y_P \leq \deg_{\mathfrak{L}_{I_P}} \widetilde{Y}_P \leq \deg_{\mathfrak{L}_I} \widetilde{Y} \leq \deg_{\mathfrak{L}_I}(\widetilde{Z} \cap T_\alpha(\widetilde{Z})) < p^{k+2f} \cdot d_Z^2,$$

where the first inequality comes from the projection formula, the second comes from [9, Proposition 5.3.10], the third is due to the fact that $\widetilde{Y}$ is an irreducible component of $\widetilde{Z} \cap T_\alpha(\widetilde{Z})$, and the last inequality was demonstrated previously.

Lemma 9.4. *The data P, X_P, X_P^+, K_P, V_P and Y_P satisfy the conditions of Theorem* 9.3 *(in place of G, X, X^+, K, V and Z, respectively).*

Proof. Firstly, note that the image of V_P in $\mathrm{Sh}_{K'}(G', X')$ is strongly special since it is still defined by the Shimura datum (H_V, X_V). Let

$$r_P := \dim Y_P - \dim V_P.$$

Thus, $r_P = r_Y > 0$. We must verify that P, X_P, X_P^+, K_P, V_P and Y_P satisfy the second condition of Theorem 9.3 for the same prime p.

From the above inequalities we have

$$p^{(k+2f)\cdot 2^{r_P}} \cdot (\deg_{\mathfrak{L}_{K_P}} Y_P)^{2^{r_P}} \leq p^{(k+2f)\cdot 2^{r_P+1}} \cdot d_Z^{2^{r_P+1}}$$

and, as $r_P + 1 \leq r$, we deduce from the second assumption of Theorem 9.3 that

$$p^{(k+2f)\cdot 2^{r_P}} \cdot (\deg_{\mathfrak{L}_{K_P}} Y_P)^{2^{r_P}} < c \cdot \Pi_V^{\delta}. \qquad \square$$

As $r_P < r$, by the induction hypothesis, we can apply Theorem 9.3 to P, X_P, X_P^+, K_P, V_P and Y_P. Thus Y_P contains a special subvariety V_P', which contains V_P properly and whose image in $\mathrm{Sh}_{K'}(G', X')$ is strongly special. This implies that Z contains a special subvariety V', which contains V properly and whose image in $\mathrm{Sh}_{K'}(G', X')$ is strongly special. $\square$

Therefore, in order to prove Theorem 9.1, it suffices to prove the following lemma:

Lemma 9.5. *Let $V \in \Sigma$. There exists a uniform constant c_{11} such that, if $\Pi_V > c_{11}$, then there exists a prime p not dividing Π_V such that $K_p' = G_{\mathbb{Z}_p}'(\mathbb{Z}_p)$ and*

$$p^{(k+2f)\cdot 2^r} \cdot (\deg_{\mathfrak{L}_K} Z)^{2^r} < c \cdot \Pi_V^{\delta}.$$

Proof. By a theorem of Chebyshev, there exist absolute positive constants c_{12} and c_{13} such that the number of primes $\pi(x)$ less than a given real number $x \geq 2$ is bounded below by $c_{12}\frac{x}{\log x}$ and above by $c_{13}\frac{x}{\log x}$. Therefore, for any fixed $\gamma > \epsilon > 0$,

$$\pi(\Pi_V^\gamma) \gg \frac{\Pi_V^\gamma}{\log \Pi_V^\gamma} \gg \Pi_V^{\gamma-\epsilon}.$$

If we denote by $\omega(\Pi_V)$ the number of primes dividing Π_V, we have the trivial estimate

$$\omega(\Pi_V) \leq \frac{\log \Pi_V}{\log 2} \ll \Pi_V^\epsilon.$$

Note that $K'_p = G'_{\mathbb{Z}_p}(\mathbb{Z}_p)$ holds for all primes p greater than a uniform constant. Therefore, if we set

$$\gamma = \frac{\delta}{(k+2f)2^r} - \epsilon > 2\epsilon > 0,$$

provided Π_V is larger than a uniform constant, we can find a prime p satisfying the requirements of the lemma. □

10. The André–Oort conjecture

We will prove the following theorem, which appears as [9, Theorem 1.2.2]. The difference between our proof and the one appearing there is that ours does not depend on any results from ergodic theory.

Theorem 10.1. *Let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Let Σ be a set of special subvarieties in $\mathrm{Sh}_K(G, X)$ and let Z be an irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ in $\mathrm{Sh}_K(G, X)$. We make* one *of the following assumptions:*

- *Assume the generalised Riemann hypothesis for CM fields.*
- *Assume that there exists a faithful representation $G \hookrightarrow \mathrm{GL}_n$ such that, with respect to this representation, the generic Mumford–Tate groups MT_V of the $V \in \Sigma$ lie in one $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class.*

Then Z is a special subvariety of $\mathrm{Sh}_K(G, X)$.

Proof. Fix a connected component X^+ of X. We may assume that Z lies in the connected component $S_K(G, X)$. Now, [9, Theorem 2.5.3] produces a dichotomy: either the subvarieties V have Galois orbits whose degrees are bounded from below by an invariant unbounded as we range through Σ or there exists a finite set $\{T_1, \dots, T_r\}$ of subtori of G, anisotropic over $\mathbb{R}$, such that each $V \in \Sigma$ is T_i-special for some $i \in \{1, \dots, r\}$ (see [19, Definition 3.1 and Definition 3.2] for the definition of T-special).

If the former occurs then [9, Theorem 3.2.1] implies Theorem 10.1. Otherwise, we may assume that every $V \in \Sigma$ is T-special for some fixed subtorus T of G such that $T_{\mathbb{R}}$ is anisotropic. Thus, by [19, Lemma 3.3 and Lemma 3.5], there exist $q \in G(\mathbb{Q})$, $\theta \in G(\mathbb{A}_f)$ and, for each $V \in \Sigma$, a qTq^{-1}-Shimura subdatum (H_V, X_V) of (G, X), where H_V is the generic Mumford–Tate group of X_V, such that V is the image of $X_V^+ \times \{\theta\}$ in $S_K(G,X)$ (see [19, Definition 3.1] for the definition of a T-Shimura subdatum). Hence, after replacing Z by an irreducible component of its image under a suitable Hecke correspondence, we may assume that each V is a standard T-special subvariety of $S_K(G,X)$, associated to a T-Shimura subdatum (H_V, X_V), with $H = \mathrm{MT}(X_V)$ (see [19, Definition 3.2] for the definition of a standard T-special subvariety).

Thus, by [19, Lemma 3.6 and Lemma 3.7], for every $V \in \Sigma$, (H_V, X_V) is a Shimura subdatum of a fixed T-Shimura subdatum (L, X_L). Therefore, we may assume that $\mathbf{\Sigma}$ is contained in $S_{L(\mathbb{A}_f)\cap K}(L, X_L)$. Let $(L^{\mathrm{ad}}, X_{L,\mathrm{ad}})$ be the adjoint Shimura datum and let K_L be a compact open subgroup of $L^{\mathrm{ad}}(\mathbb{A}_f)$ containing the image of $L(\mathbb{A}_f)\cap K$. Thus, we have an induced morphism of Shimura varieties

$$f : \mathrm{Sh}_{L(\mathbb{A}_f)\cap K}(L, X_L) \to \mathrm{Sh}_{K_L}(L^{\mathrm{ad}}, X_{L,\mathrm{ad}}).$$

Let V^{ad} be the image of V under f. Since T is the connected centre of H_V and T is contained in the centre of L, V^{ad} is defined by a Shimura subdatum (H'_V, X'_V) of $(L^{\mathrm{ad}}, X_{L^{\mathrm{ad}}})$ such that H'_V is semisimple. Since, by [8, Proposition 2.2], Z is special if and only if its image under f is special, we have reduced Theorem 10.1 to Theorem 1.3. □

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