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AAK-TYPE THEOREMS FOR HANKEL OPERATORS ON WEIGHTED SPACES

FREDRIK ANDERSSON, MARCUS CARLSSON, AND KARL-MIKAEL PERFEKT

Abstract. We consider weighted sequence spaces on $\mathbb{N}$ with increasing weights. Given a fixed integer k and a Hankel operator Γ on such a space, we show that the k:th singular vector generates an analytic function with precisely k zeroes in the unit disc, in analogy with the classical AAK-theory of Hardy spaces. We also provide information on the structure of the singular spectrum for Hankel operators, applicable for instance to operators on the Dirichlet and Bergman spaces. Finally, we show by example that the connection between the classical AAK-theorem and rational approximation fails for the Dirichlet space.

1. Introduction and Statement of Main Result

In the classical setting, a Hankel operator on a Hilbert space X is one which has the following matrix representation

$$\Gamma \sim \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots \\ \gamma_2 & \gamma_3 & \gamma_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \gamma_n \in \mathbb{C}, \tag{1.1}$$

with respect to some canonical basis. Note that this structure is perturbed by basis changes, and hence being Hankel is a property depending on the particular basis used. If $X = l^2(\mathbb{N})$, the basis is understood to be $\{e_j\}_{j=0}^\infty$ where $e_j(i) = (\delta_j(i))_{i=0}^\infty$ and δ_j denotes the Kronecker delta function. This definition can be recast in the setting of the standard Hardy space H^2 as follows; Let S denote the unilateral shift operator, i.e. $Sf(z) = zf(z)$, and let B denote the backward shift, $B = S^*$. An operator $\Gamma : H^2 \to H^2$ is then a Hankel operator if it satisfies

$$\Gamma S = B\Gamma. \tag{1.2}$$

This simply means that the matrix representation of Γ in the standard basis $(z^k)_{k=0}^\infty$ has the form (1.1).

We now introduce Hankel operators in a more abstract setting. Given a Hilbert space X we let $\mathcal{L}(X)$ denote the set of bounded operators on X. We follow a number of authors, see for example ([17], Vol 1, Part B, Sec 1.7), and generalize the definition of Hankel operators as follows.

Definition 1.1. Let X_1 and X_2 be Hilbert spaces and let $S \in \mathcal{L}(X_1)$ and $B \in \mathcal{L}(X_2)$ be given operators. A bounded operator $\Gamma : X_1 \to X_2$ will be called Hankel (with respect to S and B) if it satisfies (1.2).

In [21], Treil and Volberg consider the case when S is an expansive operator ($\|Sx\| \geq \|x\|$) and B a contractive operator ($\|Bx\| \leq \|x\|$). Our work will also be concerned with this setting, aiming to further develop the AAK-type theory of [21] for generalized Hankel operators. The next example

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gives concrete examples of Hankel operators on the type of spaces that we primarily will be concerned with.

Example 1.2. Let $w=(w_k)_{k=0}^\infty$ be a positive sequence that satisfies

$$\lim_{k\to\infty}\frac{w_{k+1}}{w_k}=1, \tag{1.3}$$

and define H^2_w as the completion of the holomorphic polynomials with respect to the norm

$$\|f\|^2_{H^2_w}=\sum_{k=0}^\infty |a_k|^2 w_k, \quad f(z)=\sum a_k z^k.$$

The condition (1.3) implies that H^2_w becomes a space of analytic functions on the unit disc $\mathbb{D}$. In particular, the Hardy space H^2 is obtained when $w=(1)_{k=0}^\infty$. Another example is the Dirichlet space, obtained when $w=(k+1)_{k=0}^\infty$. Given two sequences w and v, let S be the shift on H^2_w and B the backward shift on H^2_v,

$$Sf(z)=zf(z), \quad Bg(z)=\frac{g(z)-g(0)}{z}, \quad f\in H^2_w,\, g\in H^2_v. \tag{1.4}$$

Note that $\Gamma: H^2_w\to H^2_v$ is Hankel with respect to S and B if and only if its matrix representation has the form of (1.1) (with respect to $(z^k)_{k=0}^\infty$, considered as a spanning sequence in both H^2_w and H^2_v). Also note that S and B are expansive and contractive, respectively, if and only if w and v are increasing.

We now review the basics of AAK-theory. Let $\Gamma: X_1\to X_2$ be any bounded operator and recall that its singular values $\sigma_0,\sigma_1,\dots$ are defined by

$$\sigma_n=\inf\{\|\Gamma|_{\mathcal{M}}\| : \mathcal{M}\subset X_1 \text{ and } \mathsf{codim}\ \mathcal{M}=n\}=\inf_{K:X_1\to X_2}\{\|\Gamma-K\| : \mathsf{Rank}\ K=n\}, \tag{1.5}$$

where $\mathcal{M}\subset X_1$ means that $\mathcal{M}$ is a closed subspace of X_1 and $\Gamma|_{\mathcal{M}}$ denotes the restriction of Γ to $\mathcal{M}$. As usual, we denote by $\sigma_\infty=\lim_{n\to\infty}\sigma_n$ the essential norm of Γ. Whenever $\sigma_n>\sigma_\infty$, standard spectral theory (see e.g. [15]) implies that σ_n is an isolated point of the spectrum of $\sqrt{\Gamma^*\Gamma}: X_1\to X_1$, of finite multiplicity. The multiplicity of σ_n as an eigenvalue of $\sqrt{\Gamma^*\Gamma}$ is equal to the number of times it occurs in the sequence $(\sigma_n)_{n=0}^\infty$. The corresponding eigenvectors are called singular vectors. That is, a vector $u_n\in X_1$ is a σ_n-singular vector if $\|u_n\|=1$ and

$$\sigma_n^2 u_n=\Gamma^*\Gamma u_n.$$

Below we recall the celebrated result of Adamyan, Arov and Krein [1], which states that for a Hankel operator Γ on H^2, the best rank n approximation K of Γ is actually realized by a Hankel operator K. We call this result the AAK-theorem, although we note that preliminary versions of the result had also been obtained by D. N. Clark [9].

Theorem (AAK). *Let $\Gamma: H^2\to H^2$ be a bounded Hankel operator and let σ_n be its n:th singular value. Then there is a Hankel operator K of at most rank n such that*

$$\sigma_n=\|\Gamma-K\|.$$

The natural generalization of Theorem AAK to weighted spaces with increasing weights is in general false, which we show in Section 4. However, a key observation in AAK-theory, concerning the zeroes of the singular vectors, persists to the weighted setting, and this is the main result of the present paper. To explain our result, we begin by noting that the classical AAK-theorem is actually stronger than the statement above, in the sense that its proof provides a way of calculating the best rank-n Hankel approximation. This in turn is related to the curious fact that the n:th singular vector u_n has precisely n zeroes in the unit disc (see [5, 9]), assuming that $\sigma_{n+1}<\sigma_n<\sigma_{n-1}$. We now outline this in greater detail.

It is easy to see that a classical rank-1 Hankel operator has the form

$$\Gamma_{(z_0)} = \begin{pmatrix} 1 & z_0 & z_0^2 & \cdots \\ z_0 & z_0^2 & z_0^3 & \cdots \\ z_0^2 & z_0^3 & z_0^4 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad |z_0| < 1, \tag{1.6}$$

generated by the symbol $(1-z_0z)^{-1}$. That is, the entries of (1.6) are given by the Fourier coefficients (with positive index) of this function. In general, Kronecker's theorem states that any rank-n Hankel operator has a symbol of the form $r(z)$ where $zr(z)$ is a rational function of degree n with all poles lying in $\{z \in \mathbb{C} : |z| > 1\}$, (see e.g. [18]). In terms of applications, (see e.g. [4]), the power of the AAK-theorem comes from the fact that the location of these poles can be easily calculated using the singular vectors. For simplicity, let us assume that $(\sigma_n)_{n=0}^{\infty}$ is a strictly decreasing sequence (i.e. the σ_n's are distinct). Fix n and denote the corresponding singular vector by u_n. The proof of the AAK-theorem shows that u_n has precisely n roots $(z_j)_{j=1}^{n}$ in $\mathbb{D}$, counted with multiplicity, and that the poles of the rational symbol for the rank-n approximant K of the AAK-theorem are located at $(1/z_j)_{j=1}^{n}$, again counted with multiplicity. In particular, if u_n has distinct zeroes, then the best rank-n Hankel approximant of Γ is a linear combination of n matrices of the form (1.6) with z_0 replaced by z_j, $j = 1, \dots, n$. As mentioned above, we show in Section 4 that this type of result fails for the weighted spaces under consideration. However the main result of the present paper shows that the statement concerning the number of zeroes of u_n *does* extend to many weighted spaces.

For $f \in H^2$, the closed subspace generated by $\{S^m f : m \geq 0\}$ will be denoted by $[f]_S$, where S is the unilateral shift on H^2. Note that if $[f]_S$ has finite codimension n, then, by Beurling's theorem, f has precisely n zeroes in $\mathbb{D}$ (counted with multiplicity), and $[f]_S$ consists precisely of those functions that share the zeroes of f (to at least the same multiplicity as f). Using Beurling's and Nehari's theorem, a short argument shows that the AAK-theorem can be equivalently stated as follows.

Theorem (AAK*)**.** *Let $\Gamma : H^2 \to H^2$ be a Hankel operator and let σ_n be its n:th singular value. Then there is a singular vector u_n to σ_n such that* $\operatorname{codim}\, [u_n]_S \leq n$ *and* $\|\Gamma|_{[u_n]_S}\| = \sigma_n$.

We will now discuss S. Treil and A. Volberg's extension of the AAK*-theorem in [21]. We hence return to the general situation where X_1 and X_2 denote Hilbert spaces and $\Gamma : X_1 \to X_2$ a Hankel operator with respect to some operators $S \in \mathcal{L}(X_1)$ and $B \in \mathcal{L}(X_2)$. We give a slightly more specific statement of ([21], Theorem 3.2), which follows upon examination of its proof.

Theorem 1.3 (Treil, Volberg)**.** *Assume that S is expansive and that B is contractive and let $\Gamma : X_1 \to X_2$ be a Hankel operator. Let σ_n be a singular value of Γ. Then there exists an S-invariant subspace $\mathcal{M}$ with* $\operatorname{codim}\, \mathcal{M} \leq n$ *such that $\|\Gamma|_{\mathcal{M}}\| = \sigma_n$. If $\sigma_n < \sigma_{n-1}$ there always exists such a subspace with* $\operatorname{codim}\, \mathcal{M} = n$*.*

We remark that earlier extensions of AAK* and even AAK exist, see e.g. [11, 16]. However, these assume that S is isometric and that B is a compression of a unitary operator. Thus they typically apply to weighted Hardy spaces $H^2(\mu)$ where μ is a weight on the unit circle $\mathbb{T}$, but not to spaces of the form considered in Example 1.2.

Treil and Volberg's proof relies on a fixed point lemma of Ky Fan and does not give information concerning the singular vectors. In particular, it is not clear whether

$$\mathcal{M} = [u_n]_S, \tag{1.7}$$

or, which is weaker statement, whether $\mathcal{M}$ is determined by the zeroes of u_n. As before, $[u_n]_S$ is the closed linear span of $\{S^m u_n : m \geq 0\}$. Clearly (1.7) is not to be expected in the full generality of the above theorem. For instance, if X_1 is a vector valued Hardy space, e.g. $H^2 \oplus H^2$, and S is the shift operator (as defined in (1.4)), it is easy to see that $\operatorname{codim}\, [u]_S = \infty$ for all $u \in X_1$. On the other hand, for the concrete spaces H^2_w and H^2_v considered in Example 1.2, the question is very natural.

The expansivity and contractivity of S (the shift) and B (the backward shift), respectively, is in this case equivalent to w and v being increasing sequences. We will further impose that either S is strictly increasing ($\|Sx\| > \|x\|$, $x \neq 0$) or B is strictly decreasing ($\|Bx\| < \|x\|$, $x \neq 0$), meaning that either w or v should be strictly increasing.

In our main result the role of X_1 will be played by a general Hilbert space $\mathcal{H}$ of analytic functions on $\mathbb{D}$. That is, $\mathcal{H}$ should be continuously contained in $\mathrm{Hol}(\mathbb{D})$, the latter space equipped with the open-compact topology. The reproducing kernel of $\mathcal{H}$ at $\lambda \in \mathbb{D}$ will be denoted k^λ; we assume that k^λ does not vanish identically for any $\lambda \in \mathbb{D}$. From now on S will denote the the operator of multiplication by z, $Sf(z) = zf(z)$, $f \in \mathcal{H}$. We assume that $S : \mathcal{H} \to \mathcal{H}$ is bounded and that $\mathcal{H}$ has the division property. Namely, if $f \in \mathcal{H}$ and $f(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, then there exists $g \in \mathcal{H}$ such that $f = (S - \lambda)g$. It follows that $(S - \lambda)$ is a Fredholm operator with $\mathsf{ind}\,(S - \lambda) = -1$ for every $\lambda \in \mathbb{D}$. See [19] for a more thorough discussion.

Among these standard analytic reproducing kernel Hilbert spaces, our attention will be restricted to those that satisfy

$$\mathsf{cl}\,[(S - \lambda)\mathcal{H}] = \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{D}. \tag{1.8}$$

This condition is studied in [3]. We are content to point out that bounded point evaluations on $\mathbb{T} = \partial\mathbb{D}$ serve as the typical obstruction to the validity of (1.8). We also remark that in this setting, spaces of the form $[f]_S$ of finite codimension have the same characterization as in the H^2-case – they are completely determined by the zeroes of f in $\mathbb{D}$, see Proposition 2.1.

Definition 1.4. Let $\mathcal{H}$ be a Hilbert space of holomorphic functions in $\mathbb{D}$, on which S, multiplication by z, is a bounded operator. We say that $\mathcal{H}$ is of type (H) if it is continuously contained in $\mathrm{Hol}(\mathbb{D})$, zero-free (i.e. no reproducing kernel is identically zero), has the division property, and satisfies (1.8).

In Section 3, we give simple conditions for the spaces considered in Example 1.2 to be of type (H). Let us point out explicitly that the Dirichlet space is a space of type (H). We now state our main result.

Theorem 1.5. *Let $\mathcal{H}$ and X be Hilbert spaces, where $\mathcal{H}$ is of type (H). Suppose that the shift S is expansive and that $B : X \to X$ is a given contractive operator. Further assume that either S is strictly expansive, or B is strictly contractive.*

Let $\Gamma : \mathcal{H} \to X$ be a Hankel operator with respect to S and B, and let σ_n be a singular value such that $\sigma_n > \sigma_\infty$. Then σ_n has multiplicity 1. Moreover, if u_n is a corresponding singular vector, let $\lambda_j \in \mathbb{D}$ denote its zeroes in $\mathbb{D}$ with respective multiplicities $s_j \in \mathbb{N}$. Then $\sum_j s_j = n$ and if $\mathcal{M}$ is the codimension-n S-invariant subspace

$$\mathcal{M} = \{f \in \mathcal{H} : f \text{ has a zero at each } \lambda_j \text{ of multiplicity } \geq s_j\}$$

we have

$$\|\Gamma|_{\mathcal{M}}\| = \sigma_n.$$

Note that we trivially have $[u_n]_S \subset \mathcal{M}$. Whether equality holds is an open problem. However, even in concrete examples such as the Dirichlet space, the cyclic vectors in invariant subspaces are not completely understod, although partial results exist [12]. We also remark that the identity

$$\|\Gamma|_{[u_n]_S}\| = \sigma_n \tag{1.9}$$

was given a constructive proof in [7], relying on matrix inequalities.

Let us clarify the relationship between Theorem 1.3 and Theorem 1.5. If $\sigma_n > \sigma_\infty$, the latter theorem gives an explicit construction of a subspace $\mathcal{M}$ satisfying the conclusion of the former theorem. We do not know if $\mathcal{M}$ is always unique. In the case that $\sigma_n = \sigma_\infty = 0$, Γ is of finite rank and the sought subspace $\mathcal{M}$ is clearly given by the orthogonal complement of the first n singular vectors of Γ. In the remaining case $\sigma_n = \sigma_\infty > 0$, further information is given by the following theorem.

Theorem 1.6. *In the setting of Theorem 1.5, suppose that $\sigma_n = \sigma_\infty > 0$ for some $n \in \mathbb{N}$. Then the multiplicity of σ_n is 0 or 1. In the latter case, the conclusion of Theorem 1.5 still holds (with n the first integer such that $\sigma_n = \sigma_\infty$).*

Theorem 1.5 shows that unlike the classical AAK-theory of H^2, the only possible obstruction to having a strictly decreasing sequence of distinct singular numbers σ_n is that the sequence may eventually become stable at σ_∞, $\sigma_m = \sigma_\infty$ for all $m \geq n$, for some $n \in \mathbb{N}$. If it is not a strictly decreasing sequence and Γ is not of finite rank, σ_∞ can have multiplicity at most 1 as an eigenvalue of $\sqrt{\Gamma^*\Gamma}$, and is hence not an isolated point of the spectrum. In particular, if Γ is compact and not of finite rank, then $(\sigma_n)_{n=0}^{\infty}$ is a strictly decreasing sequence.

In Section 2 we give proofs of the above theorems. Section 3 is devoted to concrete examples and applications, and we develop Example 1.2 further. We also show that Theorem 1.5 is false if the conditions on S and B are not fulfilled, but that the statements concerning multiplicity of the singular vectors can be extended for example to Hankel operators on the Bergman space. Finally, in Section 4 we give remarks on rational approximation, to which the classical AAK-theory is strongly connected, as explained above. We conclude that the equivalent formulation of Theorem AAK in general fails in the weighted setting.

2. Proof of the Main Result

For Hilbert spaces of analytic functions of type (H), there is a natural characterization of the (closed) S-invariant subspaces with finite codimension. For an integer $s \geq 0$ and $\lambda \in \mathbb{D}$, let $k^{\lambda,s} \in \mathcal{H}$ be the function such that

$$f^{(s)}(\lambda) = \langle f, k^{\lambda,s} \rangle, \quad f \in \mathcal{H},$$

where $f^{(s)}$ denotes the s:th derivative of f.

Proposition 2.1 ([6]). *Let $m \in \mathbb{N}$ and let $\mathcal{M} \subset \mathcal{H}$ be a closed S-invariant subspace such that $\dim(\mathcal{H}/\mathcal{M}) = m$. Then there are a finite number of points $\lambda_j \in \mathbb{D}$ and integers $s_j \in \mathbb{N}$ such that $\sum_j s_j = m$ and*

$$\begin{aligned}\mathcal{M} &= \left(\cup_j \left\{k^{\lambda_j,t}\right\}_{t=0}^{s_j-1}\right)^{\perp} = \mathsf{Ran}\ \prod_j (S-\lambda_j)^{s_j} \\ &= \{f \in \mathcal{H} : f \text{ has a zero at each } \lambda_j \text{ of multiplicity } \geq s_j\}\end{aligned}$$

Conversely, any set of this form is a closed S-invariant subspace with codimension m.

We now give one proof that establishes both Theorem 1.5 and 1.6. Let $\mathcal{E}_\Gamma$ be the projection valued measure associated with $\sqrt{\Gamma^*\Gamma}$, as given by the spectral theorem (see e.g. [10]).

Proof. Consider a fixed n with $\sigma_n \neq 0$, and pick $u_k \in \mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_k\})$, $1 \leq k \leq p$, in such a way that $\{u_k\}_k$ is an orthonormal basis for the p-dimensional space $\mathsf{Ran}\ \mathcal{E}_\Gamma((\sigma_n, \infty))$. Let u_{p+1} be a unit vector in $\mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_n\})$ – if it does not exist there is nothing to prove. By Theorem 1.3, there exists an S-invariant subspace of codimension $p+1$ such that

$$\|\Gamma|_{\mathcal{M}}\| = \sigma_{p+1} = \sigma_n. \tag{2.1}$$

Since $\mathsf{Span}\ \{u_k\}_{k=0}^{p+1}$ is $(p+2)$-dimensional, it has a non-zero intersection with $\mathcal{M}$, so there are $c_0, \ldots, c_{p+1}$ such that

$$\sum_{k=0}^{p+1} c_k u_k \in \mathcal{M}.$$

By (2.1) we have

$$\sigma_{p+1}^2(\sum_{k=0}^{p+1}|c_k|^2) = \left\|\sigma_{p+1}\sum_{k=0}^{p+1}c_k u_k\right\|_{\mathcal{H}}^2 \geq \left\|\sqrt{\Gamma^*\Gamma}(\sum_{k=0}^{p+1}c_k u_k)\right\|_{\mathcal{H}}^2 = \left\|\sum_{k=0}^{p+1}\sigma_k c_k u_k\right\|_{\mathcal{H}}^2 = \sum_{k=0}^{p+1}\sigma_k^2|c_k|^2$$

which, since $\sigma_k > \sigma_{p+1}$ for $k < p+1$, is only possible if $c_k = 0$ for all $k < p+1$. We thus have $u_{p+1} \in \mathcal{M}$. Since u_{p+1} was an arbitrary unit vector in $\mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_n\})$, this gives

$$\mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_n\}) \subset \mathcal{M}. \tag{2.2}$$

Let $s_j \in \mathbb{N}$ and $\lambda_j \in \mathbb{D}$ characterize $\mathcal{M}$ as in Proposition 2.1. By (2.2) every $u \in \mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_n\})$ has a zero of multiplicity at least s_j at every point λ_j. Suppose there exists a non-zero $u \in \mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_n\})$ having a zero at some λ where either $\lambda \notin \{\lambda_j\}_j$ or where $\lambda = \lambda_{j_0}$ for some j_0 but the multiplicity of the zero λ is greater than s_{j_0}. We will show that this leads to a contradiction. Note that this also proves that σ_n has multiplicity 1, because if there would exist linearly independent $u, \tilde{u} \in \mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_n\})$, a linear combination $u + c\tilde{u} \in \mathsf{Ran}\ \mathcal{E}_\Gamma(\{\sigma_n\})$ would have $p+2$ zeros. Hence $(\sigma_m)_{m=0}^\infty$ is strictly decreasing, in particular forcing $p+1 = n$, until σ_∞ is reached by some finite m, if this happens.

Let $a \in \mathcal{H}$ be the element satisfying $u = (S-\lambda)a$. Note that $a \in \mathcal{M}$, by Proposition 2.1. Let also $b = \Gamma(a)/\sigma_{p+1}$. Combining $\|\Gamma|_\mathcal{M}\| = \sigma_{p+1}$ and the hypotheses of the theorem we have

$$\|Bb\|_X \leq \|b\|_X \leq \|a\|_\mathcal{H} \leq \|Sa\|_\mathcal{H}, \tag{2.3}$$

with one of the outer inequalities being strict. Moreover,

$$\langle (B-\lambda)b, b\rangle_X = \left\langle (B-\lambda)\frac{\Gamma(a)}{\sigma_{p+1}}, \frac{\Gamma(a)}{\sigma_{p+1}}\right\rangle_X = \left\langle \frac{\Gamma^*\Gamma((S-\lambda)a)}{\sigma_{p+1}^2}, a\right\rangle_\mathcal{H} = \langle u, a\rangle_\mathcal{H} = \langle (S-\lambda)a, a\rangle_\mathcal{H},$$

implying that

$$\mathsf{Re}\ \overline{\lambda}\langle Bb, b\rangle_X - |\lambda|^2\|b\|_X^2 = \mathsf{Re}\ \overline{\lambda}\langle Sa, a\rangle_\mathcal{H} - |\lambda|^2\|a\|_\mathcal{H}^2.$$

Combining this with (2.3), recalling that one of the inequalities was strict, we get

$$\begin{aligned}
\|u\|_\mathcal{H}^2 &= \|(S-\lambda)a\|_\mathcal{H}^2 = \|Sa\|_\mathcal{H}^2 - 2\mathsf{Re}\ \overline{\lambda}\langle Sa, a\rangle_\mathcal{H} + |\lambda|^2\|a\|_\mathcal{H}^2 \\
&= \|Sa\|_\mathcal{H}^2 - 2\mathsf{Re}\ \overline{\lambda}\langle Bb, b\rangle_X + 2|\lambda|^2\left(\|b\|_X^2 - \|a\|_\mathcal{H}^2\right) + |\lambda|^2\|a\|_\mathcal{H}^2 \\
&= \|Sa\|_\mathcal{H}^2 - \|Bb\|_X^2 + |\lambda|^2\left(\|b\|_X^2 - \|a\|_\mathcal{H}^2\right) + \|Bb\|_X^2 - 2\mathsf{Re}\ \overline{\lambda}\langle Bb, b\rangle_X + |\lambda|^2\|b\|_X^2 \\
&> \|Bb\|_X^2 - 2\mathsf{Re}\ \overline{\lambda}\langle Bb, b\rangle_X + |\lambda|^2\|b\|_X^2 = \|(B-\lambda)b\|_X^2 = \|\Gamma(u)/\sigma_{p+1}\|_X^2 = \|u\|_\mathcal{H}^2,
\end{aligned}$$

yielding a contradiction.

□

3. Examples and further results

In this section we revisit Example 1.2. We hence fix two positive increasing sequences w and v, one of which is strictly increasing. We assume additionally that the sequences satisfy

$$\lim_{k\to\infty}\frac{w_{k+1}}{w_k} = \lim_{k\to\infty}\frac{v_{k+1}}{v_k} = 1. \tag{3.1}$$

In order to be able to apply Theorem 1.5 we also impose that

$$\sum_{k=0}^\infty \frac{1}{w_k} = \infty,$$

which is easily checked to be precisely the description of those spaces H_w^2 such that $\lambda \to f(\lambda)$, $f \in H_w^2$, does not define a bounded point evaluation for $\lambda \in \mathbb{T}$.

Lemma 3.1. *Under the above assumptions H_w^2 is a Hilbert space of type (H).*

Proof. All the required properties are straightforward and standard to check. We give only the short argument that $\mathrm{cl}\,[(z-\lambda)H_w] = H_w$ for $|\lambda| = 1$. Suppose that $u = \sum_{k=0}^{\infty} u_k z^k \in H^2_w$ is orthogonal to $(z-\lambda)H_w$, and for each k, let h_k be the polynomial such that $z^k = \lambda^k + (z-\lambda)h_k$. Then

$$u_k w_k = \langle u, z^k \rangle_{H^2_w} = \langle u, \lambda^k \rangle_{H^2_w} = \bar{\lambda}^k u_0 w_0,$$

which implies that if $u \neq 0$, then $|u_k| \sim 1/w_k$ and hence

$$\sum_{k=0}^{\infty} \frac{1}{w_k} \sim \|u\|^2_{H^2_w} < \infty,$$

a contradiction. □

Letting S be the usual shift on H^2_w, $Sf(z) = zf(z)$, $f \in H^2_w$, and B the backward shift $Bg(z) = \frac{g(z)-g(0)}{z}$, $g \in H^2_v$, we see that Theorem 1.5 applies to any bounded Hankel operator $\Gamma : H^2_w \to H^2_v$ in this setting.

Note that our theorem a priori assumes that Γ is bounded. We refer to [21] for a description of the bounded Hankel operators $\Gamma : H^2_w \to H^2_v$ in the case that $(1/v_k)_{k=0}^{\infty}$ is generated by the moments of a positive measure. For boundedness conditions in the particular case of the Dirichlet space, see also [2].

Concrete examples of singular vectors are easily constructed using Hankel operators whose defining sequences $(\gamma_j)_{j=0}^{\infty}$ have finite support (in $\{0, \dots, N\}$, say). Then Γ is completely determined by the finite matrix

$$G = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_N \\ \gamma_1 & \gamma_2 & \iddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ \gamma_N & 0 & \cdots & 0 \end{pmatrix}$$

and Γ^* is represented by $I_v^{-1} G^* I_w$ where I_v is a diagonal matrix with the weights $(v_j)_{j=0}^{N}$ and G^* is the usual matrix adjoint of G. The singular vectors are thus eigenvectors of $I_v^{-1} G^* I_w G$. With these observations, singular vectors are easily computed using computer software. For example, the Hankel operator

$$G = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

acting on the Dirichlet space ($w = v = (j+1)_{j=0}^{\infty}$) has singular values (rounded) $22.72, 0.53, 0.08$ with corresponding singular vectors $\check{u}_0(z) = 0.97 + 0.25z + 0.08z^2$, $\check{u}_1(z) = -0.47 + 0.81z + 0.35z^2$ and $\check{u}_0(z) = 0.13 - 0.51z + 0.85z^2$. The zeroes in $\mathbb{D}$ are $\emptyset$, $\{0.48\}$ and $\{0.30 \pm 0.25i\}$, respectively, in accordance with Theorem 1.5.

A peculiar phenomenon which we have observed is that $I_w u_n$ also seems to generate polynomials with precisely n zeroes in $\mathbb{D}$. We have not been able to prove this, but note that its validity is related to the inequality (c.f. (1.9))

$$\|\Gamma|_{[u_n]_{I_w^{-1} S I_w}}\| \leq \sigma_n,$$

which also seems to be true according to our numerical tests.

Finally, some remarks on the case when the weights are not increasing. If we let G (as above) act on the Bergman space ($w = v = (\frac{1}{j+1})_{j=0}^{\infty}$), it is easily computed that all singular vectors generate 2 zeroes in $\mathbb{D}$ (although the singular values are distinct). In general, we have found no instances where one of the sequences w or v is not increasing, but where Theorem 1.5 seems to hold. In spite of this, the following corollary is easily obtained by a duality argument.

Corollary 3.2. *Let w and v be strictly decreasing weights that satisfy (1.3), and suppose that $\sum_{j=0}^{\infty} v_j = \infty$. Let Γ be a Hankel operator from $H^2_w \to H^2_v$ (with respect to S and B). If $\sigma_n > \sigma_\infty$, then σ_n has multiplicity 1. If $\sigma_n = \sigma_\infty$, then σ_n has multiplicity ≤ 1.*

Proof. With the unweighted pairing (also called the Cauchy pairing), it is easily seen that the dual of H^2_w is $H^2_{w^{-1}}$. Moreover, the dual operator of Γ with this pairing becomes a new Hankel operator $\Gamma^* : H^2_{v^{-1}} \to H^2_{w^{-1}}$ (with respect to the shift and backward shift). The conditions imposed on w and v, together with Lemma 3.1, show that Theorems 1.5 and 1.6 apply to Γ^*. The desired conclusion now follows from the elementary fact that Γ and Γ^* share the same singular values. □

In particular, the above corollary applies to the Bergman space [13].

4. Remarks on rational approximation

Given a function $\phi \in L^\infty(\mathbb{T})$, we let Γ_ϕ denote the Hankel operator on $l^2(\mathbb{N})$ whose defining sequence is given by the Fourier coefficients of positive index (i.e. $(\gamma_j)_{j=0}^\infty = (\widehat{\phi}_j)_{j=0}^\infty$ in (1.1)). ϕ will be called the symbol of Γ_ϕ. We denote by P the Riesz projection, the operator $P : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ such that $P(\phi)(z) = \sum_{j=0}^\infty \widehat{\phi}_j z^j$. Let R_n denote the set of rational functions $r = p/q$ where p and q are polynomials such that $\deg p < n$, $\deg q \leq n$ and q has no zeroes in $\overline{\mathbb{D}}$. One may think of R_n as the closure of functions of the form $\sum_{j=1}^n \frac{c_j}{1-\lambda_j z}$, where $\lambda_j \in \mathbb{D}$ and $c_j \in \mathbb{C}$.

Note that the matrices of the form (1.6) arise from symbols in R_1. Kronecker's theorem states that Γ_ϕ has rank n if and only if $P\phi \in R_n \setminus R_{n-1}$. Theorem AAK can thus be restated as

$$\inf_{r \in R_n} \|\Gamma_\phi - \Gamma_r\| = \sigma_n, \tag{4.1}$$

where σ_n is the n:th singular value of Γ_ϕ. Important for applications is that the minimizer $r_0 = p/q$ can be found explicitly, and the key observation behind this is that the poles of q are located at $\{\frac{1}{\lambda_j}\}_j$, where the λ_j's are as in Theorem 1.5. When this phenomenon holds also in the weighted setting, we refer to it as the "strong form" of the AAK-theorem.

In addition, by Nehari's theorem this can be reformulated as a result on best rational approximation with respect to a quotient norm in L^∞. More precisely, letting $(H^1)^\perp$ denote the subset of L^∞ with functions whose Fourier coefficients with index in $\mathbb{N}$ are zero, we have that

$$\inf_{r \in R_n} \|\phi - r\|_{L^\infty/(H^1)^\perp} = \sigma_n. \tag{4.2}$$

We refer to [8] for further details and applications to control theory.

It is known [2, 7, 17, 21] that no sharp version of Nehari's theorem exists in the weighted setting. Moreover, in [7] it is shown that the strong form of the AAK-theorem fails for Hankel operators between spaces H^2_w and H^2_v, as long as the weights are strictly increasing. Below we show that the weaker form, Theorem AAK, also fails – that is, even if we do not require the poles to be determined by the zeroes of the n:th singular vector – in the case of Hankel operators on the Dirichlet space.

Example 4.1. Let $H^2_w = H^2_v$ be the Dirichlet space D and consider the Hankel operator $\Gamma_z : D \to D$ with matrix representation (as in Section 3) given by

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then Γ_z^* is represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

so $\Gamma_z^*\Gamma_z$ is represented by $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and therefore $\sigma_0 = \sqrt{2}$, $u_0 = 1$, $\sigma_1 = 1/\sqrt{2}$, $u_1 = z/\sqrt{2}$. Thus if the AAK theorem were to hold in the Dirichlet space, we would have

$$\inf\left\{\left\|\Gamma_z - \Gamma_{\frac{c}{1-\lambda z}}\right\| : c \in \mathbb{C},\, \lambda \in \mathbb{D}\right\} = \sigma_1 = 1/\sqrt{2} < 1, \tag{4.3}$$

where the norm refers to the operator norm on D. We show below that

$$\left\|\Gamma_z - \Gamma_{\frac{c}{1-\lambda z}}\right\| \geq \sqrt{\frac{38}{27}} > 1, \tag{4.4}$$

in clear contrast with (4.3). To prove (4.4), note that

$$\left\|\left(\Gamma_z - \Gamma_{\frac{c}{1-\lambda z}}\right)u_0\right\|_D^2 / \|u_0\|_D^2 = \left(2 - 4\mathrm{Re}\,(c\lambda) + \frac{|c|^2}{(1-|\lambda|^2)^2}\right)/1,$$

where $c \in \mathbb{C}$ and $\lambda \in \mathbb{D}$. However, it is easy to see that the expression is minimal for real positive values of c and λ. Thus (4.4) follows if we show that

$$f(c,\lambda) = 2 - 4c\lambda + \frac{c^2}{(1-\lambda^2)^2}, \quad 0 \leq \lambda < 1,\ c \geq 0$$

is larger than $38/27$. Basic analysis yields that for fixed λ, the minimum in c is achieved at $c = 2\lambda(1-\lambda^2)^2$. Note that

$$f(2\lambda(1-\lambda^2)^2, \lambda) = 2 - 8\lambda^2(1-\lambda^2)^2 + \frac{4\lambda^2(1-\lambda^2)^4}{(1-\lambda^2)^2} = 2 - 4\lambda^2(1-\lambda^2)^2$$

Introducing the new variable $y = \lambda^2$ we see that $\inf_{c,\lambda} f = \inf_{0<y<1} g(y)$ where

$$g(y) = 2 - 4y(1-y)^2.$$

It is easy to deduce that g attains its minimum for $y = 1/3$, yielding

$$\inf_{0<y<1} g(y) = 2 - 4\frac{1}{3}(1-\frac{1}{3})^2 = \frac{38}{27},$$

as desired.

The above proof can obviously be extended to a greater range of weights than only those giving rise to the Dirichlet space.

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Centre for Mathematical Sciences, Lund University, Sweden

E-mail address: fa@maths.lth.se

Centre for Mathematical Sciences, Lund University, Sweden

E-mail address: marcus.carlsson@math.lu.se

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

E-mail address: karl-mikael.perfekt@math.ntnu.no