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(joint work with Jinbo Ren)

Using o-minimality, the Pila-Zannier strategy combines results from arithmetic and functional transcendence to prove finiteness theorems in arithmetic geometry. It originated in the paper [6] of Pila and Zannier in which the authors gave a new proof of the Manin-Mumford conjecture for abelian varieties.

The objective of [2] was to extend the Pila-Zannier strategy to the Zilber-Pink conjecture for (pure) Shimura varieties. The method generalises that of Habegger and Pila who in [3] obtained results for abelian varieties and products of modular curves.

Let S be a mixed Shimura variety. For any (irreducible) subvariety W of S, there exists a smallest special subvariety $\langle W\rangle$ of S containing W. In [7], Pink defined the defect of W to be

$$\delta(W) := \dim\langle W\rangle - \dim W.$$

In their article [3], Habegger and Pila made the following definition.

Definition 1. *Fix a subvariety V of S. A subvariety W of V is optimal in V if for any subvariety Y of V strictly containing W we have*

$$\delta(Y) > \delta(W).$$

This clearly generalises the notion of a maximal special subvariety. The André-Oort conjecture predicts that any subvariety of S contains only finitely many maximal special subvarieties. Hence, the following formulation of the Zilber-Pink conjecture is a natural generalisation of that statement.

Conjecture 1 (Zilber-Pink)**.** *Let V be a subvariety of S. Then V contains only finitely many subvarieties that are optimal in V.*

This formulation is equivalent to Zilber's conjecture regarding atypical intersections and it implies the conjecture of Pink in which V is intersected with special subvarieties of codimension exceeding the dimension of V.

1. Shimura varieties

By a Shimura variety, we refer to a variety of the form $\Gamma\backslash X$, where X is a hermitian symmetric domain and Γ is a congruence subgroup. For us, however, it is more useful to view X as the $G(\mathbb{R})^+$ conjugacy class of a morphism

$$\mathbb{C}^\times \to G(\mathbb{R})$$

of real Lie groups, where G is an algebraic group over $\mathbb{Q}$.

Special subvarieties arise as follows. Let x be any point on X and let M be the smallest subvariety of G defined over $\mathbb{Q}$ with the property that $x(\mathbb{C}^\times)$ is contained in $M(\mathbb{R})$. The $M(\mathbb{R})^+$ conjugacy class X_M of x is a hermitian symmetric subdomain of X and its image under

$$\pi : X \to \Gamma\backslash X$$

is an algebraic subvariety of $\Gamma\backslash X$. We refer to such a subvariety as a special subvariety of $\Gamma\backslash X$ and we refer to X_M as a pre-special subvariety of X.

Since $x(\mathbb{C}^\times)$ is contained in $M(\mathbb{R})$ and

$$M(\mathbb{R})^+ \to M^{\mathrm{ad}}(\mathbb{R})^+$$

is surjective, we have that X_M is equal to the $M^{\mathrm{ad}}(\mathbb{R})^+$ conjugacy class of x. In particular, for any direct product decomposition $M_1 \times M_2$ of M^{ad} defined over $\mathbb{Q}$, we have

$$X_M = X_1 \times X_2$$

and, for any point $x_1 \in X_1$, the image of $\{x_1\} \times X_2$ in $\Gamma\backslash X$ is again an algebraic subvariety of $\Gamma\backslash X$. We refer to such a subvariety as a weakly special subvariety of $\Gamma\backslash X$. In particular, any special subvariety of $\Gamma\backslash X$ is weakly special. We refer to $\{x_1\} \times X_2$ as a pre-weakly special subvariety of X.

2. The strategy

As in all instances of the Pila-Zannier strategy, the method can be broken into two parts. Both parts rely on o-minimality, though only the latter relies on the Pila-Wilkie counting theorems. The first part is geometric in nature and relies on a result from functional transcendence. We prove the following theorem that was obtained for products of modular curves by Habegger and Pila.

Theorem 1 (cf. [3], Proposition 6.6)**.** *Let V be a subvariety of S. There exists a finite set Σ of pre-special subvarieties of X such that if W is a subvariety of V that is optimal in V then W is an irreducible component of*

$$V \cap \pi(\{x_1\} \times X_2),$$

for some $x_1 \in X_1$, where $X_1 \times X_2 \in \Sigma$.

It is then straightforward to show that Conjecture 1 follows from...

Conjecture 2. *Let V be a subvariety of S. Then V contains only finitely many points that are optimal in V.*

At this point we enter the second phase of the strategy. Using the uniform (in families) version of Pila-Wilkie, we are able to show that Conjecture 2 holds under certain arithmetic hypotheses. The first of which is the so-called large Galois orbits conjecture (LGO), which says that the Galois orbit of an optimal point P should grow at least as quickly as a uniform positive power of the complexity of $\langle P\rangle$. Habegger and Pila obtained this conjecture in [4] for certain curves in products of modular curves but it is otherwise completely open.

The remaining hypotheses are concerned with the parametrisation of special subvarieties and optimal points. To use the Pila-Wilkie theorem, we need to control the heights of certain elements as well as the degrees of certain fields of definition. The height of a pre-special point in a fundamental set was bounded by Orr and the author in [1] but two hypotheses remain outstanding, though we are able to verify them both in a product of modular curves and hence give a new proof of Conjecture 1 under the LGO in that case.

3. Proof of Theorem 1

Recall that X can be realised as a bounded symmetric domain in $\mathbb{C}^N$ for some $N \in \mathbb{N}$. We define a subvariety of X to be any irreducible analytic component of the intersction of X with a subvariety of $\mathbb{C}^N$.

Fix a subvariety V of S. We say that a subset A of $\pi^{-1}(V)$ is an intersection component if it is an irreducible analytic component of the intersection of $\pi^{-1}(V)$ with a subvariety of X. If A is an intersection component, we let $\langle A\rangle_{\mathrm{Zar}}$ denote the smallest subvariety of X containing A i.e. the Zariski closure of A. We say that A is Zariski optimal if for any intersection component B strictly containing A, we have

$$\delta_{\mathrm{Zar}}(B) > \delta_{\mathrm{Zar}}(A).$$

The following conjecture is a problem in functional transcendence. Pila and Tsimerman gave a proof in [5] for the case when S is a product of modular curves. A proof of the full conjecture has recently been anounced by Mok, Pila, and Tsimerman.

Conjecture 3 (weak hyperbolic Ax-Schanuel)**.** *Let A be a Zariski optimal intersection component. Then $\langle A\rangle_{\mathrm{Zar}}$ is pre-weakly special.*

Using Conjecture 3, we can show that if a subvariety W of V is optimal in V then any irreducible analytic component of $\pi^{-1}(W)$ is a Zariski optimal intersection component. In particular, such a component is an irreducible component of the intersection of its Zariski closure with $\pi^{-1}(V)$. By Conjecture 3, the Zariski closure is pre-weakly special and, by virtue of the fact that the restriction of π to a fundamental set is definable in $\mathbb{R}_{\mathrm{an,exp}}$, we can choose Σ is bijection with a definable set. However, since the set of pre-special subvarieties is countablea and $\mathbb{R}_{\mathrm{an,exp}}$ is o-minimal, Σ must therefore be finite.

References

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