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1 **A multiscale asymptotic theory of extratropical wave, mean-flow interaction**

2 Lina Boljka\* and Theodore G. Shepherd

3 *Department of Meteorology, University of Reading, Reading, United Kingdom*

4 \**Corresponding author address:* Lina Boljka, Department of Meteorology, University of Reading,

5 P.O. Box 243, Reading, RG6 6BB, United Kingdom.

6 E-mail: l.boljka@pgr.reading.ac.uk

## ABSTRACT

7 Multiscale asymptotic methods are used to derive wave-activity equations  
8 for planetary and synoptic scale eddies and their interactions with a zonal  
9 mean flow. The eddies are assumed to be of small amplitude, and the  
10 synoptic-scale zonal and meridional length scales are taken to be equal. Under  
11 these assumptions, the zonal-mean and planetary-scale dynamics are plane-  
12 tary geostrophic (i.e. dominated by vortex stretching), and the interaction be-  
13 tween planetary and synoptic scale eddies occurs only through the zonal mean  
14 flow or through diabatic processes. Planetary scale heat fluxes are shown to  
15 enter the angular momentum budget through meridional mass redistribution.  
16 After averaging over synoptic length and time scales, momentum fluxes dis-  
17 appear from the synoptic-scale wave-activity equation whilst synoptic-scale  
18 heat fluxes disappear from the baroclinicity equation, leaving planetary-scale  
19 heat fluxes as the only adiabatic term coupling the baroclinic and barotropic  
20 components of the zonal mean flow. In the special case of weak planetary  
21 waves, the decoupling between the baroclinic and barotropic parts of the flow  
22 is complete with momentum fluxes driving the barotropic zonal mean flow,  
23 heat fluxes driving the wave activity, and diabatic processes driving baroclin-  
24 icity. These results help explain the apparent decoupling between the baro-  
25 clinic and barotropic components of flow variability recently identified in ob-  
26 servations, and may provide a means of better understanding the link between  
27 thermodynamic and dynamic aspects of climate variability and change.

## 28 **1. Introduction**

29 The interaction between jet variability and eddies is a long-studied topic, but the interaction  
30 is not yet understood well enough to identify causal mechanisms for variability or sources of  
31 systematic errors in models. There are well-developed theoretical frameworks for the zonally  
32 homogeneous case (e.g. annular-mode variability), however zonally asymmetric analyses includ-  
33 ing planetary scale interactions are more complicated and only partial theories for this case exist  
34 (Hoskins et al. 1983; Plumb 1985, 1986). Yet longitudinal variations and synoptic-planetary scale  
35 interactions are important for the location and strength of the storm tracks and blocking episodes  
36 (Hoskins et al. 1983; Luo 2005; Simpson et al. 2014). These phenomena strongly affect the re-  
37 gional climate and its climate change. As the dynamical aspects of climate are not yet well under-  
38 stood, there is low confidence in circulation patterns simulated by global and regional models and  
39 their response to climate change (Shepherd 2014).

40 An important aspect of wave-mean flow interaction concerns barotropic and baroclinic processes  
41 and their links through eddy momentum and heat fluxes. It has recently been shown from obser-  
42 vations for the Southern and Northern Annular Modes in Thompson and Woodworth (2014) and  
43 Thompson and Li (2015) that the zonal mean flow is affected only by momentum fluxes and not  
44 by heat fluxes, while the opposite is true for a so-called baroclinic annular mode (BAM) that is  
45 based on eddy kinetic energy (EKE). This decoupling goes against the usual Transformed Eulerian  
46 Mean (TEM) perspective, first introduced by Andrews and McIntyre (1976), within which both  
47 heat and momentum fluxes affect the zonal mean flow tendency through the Eliassen-Palm (EP)  
48 flux divergence. The decoupling was further investigated in Thompson and Barnes (2014), who  
49 found an oscillating relationship between EKE and heat flux with time periods of 20-30 days. A

50 similar relationship was found between wave activity and heat flux in Wang and Nakamura (2015,  
51 2016).

52 To derive a theoretical framework for understanding planetary-synoptic scale interactions and  
53 the apparent decoupling of the baroclinic and barotropic parts of the flow, we use multi-scale  
54 asymptotic methods as introduced in Dolaptchiev and Klein (2009, 2013) (hereafter DK09 and  
55 DK13, respectively). This approach is taken as such methods provide a self-consistent (albeit ide-  
56 alised) framework for studying interactions between processes on different length and time scales,  
57 starting from a minimal set of assumptions. While the derived theory using these methods may  
58 not be quantitatively accurate for the atmosphere, it can still provide qualitative value, especially  
59 when trying to determine the causal relationships that are so elusive in standard budget calcula-  
60 tions. This is analogous to the use of the quasi-geostrophic approximation, which provides a clear  
61 qualitative picture of the large scale flow and both planetary and synoptic scale eddies, however for  
62 accurate representation of the flow (e.g. in weather prediction), the primitive equations are used.  
63 Therefore, the aim of this work is to find a theoretical framework by which to better understand  
64 the emergent properties of observations and model behavior, rather than developing a predictive  
65 theory.

66 DK13 used a separation of length scales in the meridional and zonal directions, with an isotropic  
67 scaling for the synoptic scales, as well as a temporal scale separation between the synoptic and  
68 planetary waves. Isotropic scaling for the synoptic scales is standard in quasi-geostrophic (QG)  
69 theory (Pedlosky 1987), and a meridional scale separation has been argued to be a useful and  
70 physically realizable idealization of baroclinic instability (Haidvogel and Held 1980). These as-  
71 sumptions allowed DK13 to study planetary and synoptic scale interactions. However, they did not  
72 derive a wave activity equation or develop explicit equations for the interaction with a zonal mean  
73 flow. These aspects are the focus of this paper. For simplicity, we derive the asymptotic equations

74 for the case of small-amplitude eddies evolving in the presence of a zonal mean flow, which is an  
75 important special case of the DK13 framework. As well as giving a theoretical description for the  
76 interaction of a zonal mean flow with planetary and synoptic scale waves, this setting also allows  
77 a study of the link between baroclinic and barotropic processes, and the relative importance of  
78 planetary and synoptic scale waves for these processes.

79 The outline of the paper is as follows. Section 2 gives the equations and assumptions used to  
80 derive the potential vorticity (section 3), wave activity and mean flow equations (section 4), and  
81 the angular momentum budget for the zonal mean flow (section 5). The momentum, continuity,  
82 thermodynamic and vorticity equations at different asymptotic orders, which are needed for the  
83 derivations, are given in Appendix A. Further details on the derivations of the mean flow and  
84 angular momentum equations, and the non-acceleration theorem, are given in Appendices B, C  
85 and D. The zonally homogeneous case with weak planetary scale waves is discussed in section 6,  
86 and conclusions are given in section 7.

## 87 **2. The multiscale asymptotic model**

### 88 *a. Nondimensional compressible flow equations*

89 The asymptotic system of equations is derived starting from the nondimensionalised compress-  
90 ible flow equations in spherical coordinates with a small parameter  $\varepsilon^1$  (DK09). To obtain the  
91 nondimensional equations the DK09 and DK13 scaling parameters<sup>2</sup> are used, based on the as-  
92 sumption that the waves are not propagating faster than the speed of sound. In this process,

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<sup>1</sup> $\varepsilon$  is defined as  $(a^*\Omega^2g^{-1})^{1/3}$  (global atmospheric aspect ratio), where  $\Omega$  is Earth's rotation rate,  $a^*$  is Earth's radius and  $g$  the Earth's gravitational acceleration.  $\varepsilon$  is a constant within the range 1/8 to 1/6.

<sup>2</sup>Pressure  $p_{ref} = 10^5$  Pa, air density  $\rho_{ref} = 1.25$  kg m<sup>-3</sup>, characteristic flow velocity  $u_{ref} = 10$  m s<sup>-1</sup>, scale height  $h_{sc} = p_{ref}/g\rho_{ref} \approx 10$  km, gravitational acceleration  $g \approx 10$  m s<sup>-2</sup>, and time scale  $t_{ref} = h_{sc}/u_{ref} \approx 20$  min.

93 the following nondimensional numbers appear (DK09): Rossby<sup>3</sup> ( $Ro_{QG} = u_{ref}/2\Omega L_{QG}$  with  
 94  $L_{QG} = \varepsilon^{-2}h_{sc}$ ), Mach ( $M = u_{ref}/\sqrt{p_{ref}/\rho_{ref}}$ ), Froude ( $Fr = u_{ref}/\sqrt{gh_{sc}}$ ) and the ratio of density  
 95 and potential temperature scale heights  $\sqrt{h_{sc}/H_\theta}$ . These are related to the small parameter  $\varepsilon$  ac-  
 96 cording to  $\sqrt{M} \approx \sqrt{Fr} \approx Ro_{QG} \approx \sqrt{h_{sc}/H_\theta} \approx \varepsilon$  (DK09). This procedure yields the system (the  
 97 full derivation is given in DK09):

$$\frac{Du}{Dt} - \varepsilon^3 \left( \frac{uv \tan \phi}{R} - \frac{uw}{R} \right) + \varepsilon(w \cos \phi - v \sin \phi) = -\frac{\varepsilon^{-1}}{R \rho \cos \phi} \frac{\partial p}{\partial \lambda} + S_u \quad (1a)$$

$$\frac{Dv}{Dt} + \varepsilon^3 \left( \frac{u^2 \tan \phi}{R} + \frac{vw}{R} \right) + \varepsilon u \sin \phi = -\frac{\varepsilon^{-1}}{R \rho} \frac{\partial p}{\partial \phi} + S_v \quad (1b)$$

$$\frac{Dw}{Dt} - \varepsilon^3 \left( \frac{u^2}{R} + \frac{v^2}{R} \right) - \varepsilon u \cos \phi = -\frac{\varepsilon^{-4}}{\rho} \frac{\partial p}{\partial z} - \varepsilon^{-4} + S_w \quad (1c)$$

$$\frac{D\theta}{Dt} = S_\theta \quad (1d)$$

$$\frac{D\rho}{Dt} + \frac{\varepsilon^3 \rho}{R \cos \phi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial(v \cos \phi)}{\partial \phi} \right) + \rho \frac{\partial w}{\partial z} + \frac{\varepsilon^3 2w\rho}{R} = 0 \quad (1e)$$

$$\rho \theta = p^{1/\gamma} \quad (1f)$$

98 where  $S$  denotes source-sink terms ( $S_{u,v,w}$  are the frictional terms, while  $S_\theta$  represents diabatic  
 99 effects),  $\sin \phi = f$  is the nondimensional Coriolis parameter,  $p$  is nondimensional pressure,  $\theta$   
 100 is nondimensional potential temperature,  $\rho$  is nondimensional density,  $(u, v, w)$  represent the  
 101 nondimensional 3-D velocity field,  $R = \varepsilon^3 r$ ,  $r = \varepsilon^{-3}a + z$  where  $z$  is altitude from the ground,  
 102  $a = a^* \varepsilon^3 / h_{sc}$  is nondimensional Earth's radius,  $\phi$  is latitude,  $\lambda$  is longitude,  $t$  is time, all param-  
 103 eters are nondimensional, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\varepsilon^3 u}{R \cos \phi} \frac{\partial}{\partial \lambda} + \frac{\varepsilon^3 v}{R} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial z}. \quad (2)$$

104 Note that the shallow-atmosphere limit  $R \rightarrow a$  is used here unless otherwise stated (this approxi-  
 105 mation is used as it holds well to leading order). Expanding  $R$ , the material derivative (2) involves

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<sup>3</sup>Note that the Rossby number ( $Ro$ ) used in DK09 and DK13 is  $\varepsilon^{-2}Ro_{QG}$  as they used the vertical instead of the horizontal length scale to define

it.

106 horizontal advection terms  $-a^{-1}\varepsilon^6_z(u\{a\cos\phi_p\}^{-1}\partial/\partial\lambda + va^{-1}\partial/\partial\phi)$  that become relevant at  
107  $5^{th}$  and higher orders.

108 *b. Assumptions for multiscale asymptotic methods*

109 In order to derive the multiscale asymptotic version of the equations, some assumptions must be  
110 made. In particular, we assume small-amplitude eddies in the presence of a zonal mean flow. This  
111 approximation is made in order to gain qualitative insight into the behavior of the system, and to  
112 allow connection with previous theories of wave, mean-flow interaction. This can be considered a  
113 special case of DK13, with the eddies (but not the zonal mean flow) scaled down by one order of  $\varepsilon$ .  
114 The assumptions for the scale separation between the synoptic, planetary and mean flow in time,  
115 height, latitude and longitude are given in Table 1 (following DK13), where the subscripts  $m$ ,  $p$   
116 and  $s$  represent mean, planetary and synoptic scales, respectively. Note that  $\phi_s \gg \phi_p$  (similarly for  
117 other coordinates) since the same meridional distance is a much larger number when measured on  
118 synoptic scales compared to planetary or zonal mean scales. Here  $\lambda_m$  is not considered as the zonal  
119 mean flow is uniform in longitude,  $\lambda_p$  and  $\phi_p$  represent variations of the flow on planetary scales  
120 (those of order  $a^*$ ),  $\lambda_s$  and  $\phi_s$  represent variations on synoptic scales (of order 1000 km), and the  
121 time scales are well separated between the mean flow, planetary and synoptic scale eddies, where  
122  $t_s$  is of order a day,  $t_p$  is of order a week and  $t_m$  is a seasonal timescale. The time scales emerge  
123 naturally from the equations;  $t_m$  is  $\varepsilon^2$  slower than  $t_p$  because the eddy fluxes driving the zonal mean  
124 flow changes are quadratic in eddy amplitude. (In the finite-amplitude theory of DK13, there is  
125 no distinction between the two timescales.) As this is the small-amplitude limit of the system,  
126 we expect that in practice the zonal mean flow time scale would be shorter. Note that from the  
127 above assumptions we see that there is a separation of scales in the meridional direction, which  
128 has implications for the final results (see further discussion in sections 3, 4 and 6).



129 Using these scales, we can write asymptotic series for all variables; an example for potential  
 130 temperature (which provides stratification) is (following DK09, DK13):

$$\theta(\lambda, \phi, z, t) = 1 + \varepsilon^2 \theta^{(2)}(\phi_p, t_m, z) + \varepsilon^3 \theta^{(3)}(\mathbf{X}_p, z) + \varepsilon^4 \theta^{(4)}(\mathbf{X}_p, \mathbf{X}_s, z) + \dots \quad (3)$$

131 where the number in parentheses in superscript represents the order of the variable,  $\mathbf{X}_p =$   
 132  $(\lambda_p, \phi_p, t_p)$  and  $\mathbf{X}_s = (\lambda_s, \phi_s, t_s)$ . Here the first order term has been omitted as  $h_{sc}/H_\theta \propto \Delta\theta/\theta_0 \approx$   
 133  $\varepsilon^2$ ; to make this  $\mathcal{O}(\varepsilon)$  would lead to stronger wind variations (of order  $70 \text{ m s}^{-1}$ ) (DK09), which  
 134 would require a different treatment. Note that here the leading order variation in potential temper-  
 135 ature  $\theta^{(2)}$  depends on  $\phi_p$  and  $z$ , not only on  $z$  as is the case for the static stability parameter in QG  
 136 theory.

137 In order to have a well defined asymptotic expansion (3) the sublinear growth condition (DK13)  
 138 is required. This means that variables at any order grow slower than linearly in any of the synoptic  
 139 coordinates, which effectively means that any averaging over the synoptic scales ( $\mathbf{X}_s$ ) sets the  
 140 derivatives over synoptic scales to zero (for more details see DK13).

141 The full set of equations at different asymptotic orders using the assumptions from this section  
 142 is given in Appendix A. This includes the momentum, thermodynamic and continuity equations,  
 143 thermal wind, hydrostatic balance and the vorticity equation. These equations are used in the  
 144 following sections to derive potential vorticity, wave activity and mean flow equations.

### 145 3. Potential vorticity equation

146 To derive the potential vorticity (PV) equation, a vorticity equation has to be derived first. To  
 147 do so (see Appendix A for the full derivation), take  $\nabla_s \times \mathcal{O}(\varepsilon^3)$  momentum equation (A6) and use  
 148 the  $\mathcal{O}(\varepsilon^4)$  continuity equation (A15), which yields

$$\frac{\partial}{\partial t_s} \zeta^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_s \zeta^{(1)} - \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \rho^{(0)} w^{(4)} \right) + \beta v^{(1)} = S_\zeta \quad (4)$$

149 where  $\nabla_s = ((a \cos \phi_p)^{-1} \partial / \partial \lambda_s, a^{-1} \partial / \partial \phi_s)$ ,  $\mathbf{u}^{(0)} = u^{(0)} \mathbf{e}_\lambda$  is horizontal velocity of the mean flow,  
 150  $\beta = a^{-1} \partial f / \partial \phi_p$ ,  $\zeta^{(1)} = \zeta^{(1)} \mathbf{e}_r = \nabla_s \times \mathbf{u}^{(1)}$  is relative vorticity,  $\mathbf{u}^{(1)} = (u^{(1)}, v^{(1)})$  is horizontal  
 151 velocity at first order,  $S_\zeta = \mathbf{e}_r \cdot \nabla_s \times \mathbf{S}_u^{(3)}$ , and  $w^{(4)}$  is known from the  $\mathcal{O}(\varepsilon^6)$  thermodynamic  
 152 equation (A11)

$$w^{(4)} = -\frac{1}{\partial \theta^{(2)} / \partial z} \left[ \frac{\partial \theta^{(3)}}{\partial t_p} + \frac{\partial \theta^{(4)}}{\partial t_s} + \mathbf{u}^{(0)} \cdot \nabla_p \theta^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_s \theta^{(4)} + \mathbf{u}^{(1)} \cdot \nabla_p \theta^{(2)} - S_\theta^{(6)} \right] \quad (5)$$

153 where  $\nabla_p = ((a \cos \phi_p)^{-1} \partial / \partial \lambda_p, a^{-1} \partial / \partial \phi_p)$ . Substituting (5) into (4) gives

$$\begin{aligned} \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \frac{\rho^{(0)}}{\partial \theta^{(2)} / \partial z} \left[ \frac{\partial \theta^{(3)}}{\partial t_p} + \frac{\partial \theta^{(4)}}{\partial t_s} + \mathbf{u}^{(0)} \cdot \nabla_p \theta^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_s \theta^{(4)} + \mathbf{u}^{(1)} \cdot \nabla_p \theta^{(2)} - S_\theta^{(6)} \right] \right) \\ + \frac{\partial}{\partial t_s} \zeta^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_s \zeta^{(1)} + \beta v^{(1)} = S_\zeta. \quad (6) \end{aligned}$$

154 The first term in brackets on the left-hand-side of (6) can be simplified. Firstly notice that  $\rho^{(0)}$ ,  
 155  $\theta^{(2)}$  and  $f$  do not depend on  $t_s$ , thus  $\partial / \partial t_s$  can be brought outside the brackets. The other terms in  
 156 the first term can be simplified using thermal wind balance (A9a, A9b). This leads to cancellation  
 157 of terms with  $\partial u^{(0)} / \partial z$ ,  $\partial \mathbf{u}_s^{(1)} / \partial z$ , or  $\partial \mathbf{u}_p^{(1)} / \partial z$  (with  $\mathbf{u}_p^{(1)}$  and  $\mathbf{u}_s^{(1)}$  as the horizontal velocities for  
 158 planetary and synoptic scales, respectively), which means that velocities can be taken out of the  
 159  $\partial / \partial z$  derivative. This yields the potential vorticity equation

$$\left( \frac{\partial}{\partial t_s} + u_m^{(0)} \frac{1}{a \cos \phi_p} \frac{\partial}{\partial \lambda_s} \right) q_s^{(4)} + \left( \frac{\partial}{\partial t_p} + u_m^{(0)} \frac{1}{a \cos \phi_p} \frac{\partial}{\partial \lambda_p} \right) q_p^{(3)} + (v_s^{(1)} + v_p^{(1)}) \hat{\beta} = S^{PV} \quad (7)$$

160 where

$$q_s^{(4)}(\mathbf{X}_p, \mathbf{X}_s, z) = \frac{1}{f} \nabla_s^2 \pi^{(4)} + \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \frac{\rho^{(0)} \theta^{(4)}}{\partial \theta^{(2)} / \partial z} \right), \quad (8a)$$

$$q_p^{(3)}(\mathbf{X}_p, z) = \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \frac{\rho^{(0)} \theta^{(3)}}{\partial \theta^{(2)} / \partial z} \right), \quad (8b)$$

$$\hat{\beta}(\phi_p, t_m, z) = \beta + \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \frac{\frac{\partial}{\partial \phi_p} (\rho^{(0)} \theta^{(2)})}{\partial \theta^{(2)} / \partial z} \right), \quad (8c)$$

$$S_p^{PV} = \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \frac{\rho^{(0)} \overline{S_\theta^{(6)}}^{x_s, t_s, y_s}}{\partial \theta^{(2)} / \partial z} \right), \quad (8d)$$

$$S_s^{PV} = \mathbf{e}_r \cdot \nabla_s \times \mathbf{S}_u^{(3)} + \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \frac{\rho^{(0)} \left( S_\theta^{(6)} - \overline{S_\theta^{(6)}}^{x_s, t_s, y_s} \right)}{\partial \theta^{(2)} / \partial z} \right), \quad (8e)$$

161  $S^{PV} = S_s^{PV} + S_p^{PV}$ ,  $u_m^{(0)} = u^{(0)}$  is the zonal velocity of the zonal mean flow, here  $\theta^{(3)}$  and  $\theta^{(4)}$   
 162 correspond to planetary and synoptic scale potential temperature, respectively,  $\theta^{(2)}$  is the leading  
 163 order potential temperature of the mean flow,  $\pi^{(i)} = p^{(i)} / \rho^{(0)}$ ,  $\theta^{(i=2,3,4)} = \partial \pi^{(i=2,3,4)} / \partial z$ ,  $q_p^{(3)}$  is  
 164 planetary scale PV,  $q_s^{(4)}$  is synoptic scale PV,  $\hat{\beta}$  is the effective background PV gradient,  $\zeta^{(1)} =$   
 165  $f^{-1} \nabla_s^2 \pi^{(4)}$  is relative vorticity on the synoptic scale, and  $S^{PV}$ ,  $S_s^{PV}$  and  $S_p^{PV}$  represent the source-  
 166 sink terms for the full PV, synoptic scale PV and planetary scale PV, respectively. A similar  
 167 equation to (7) can be obtained by linearising (A5) in DK13, though without the planetary scale  
 168 PV as it is then absorbed in the background PV gradient as the zonal mean flow. Similarly, (9)  
 169 below can be linked to (44) in DK13.

170 Equation (7) can then be split into planetary and synoptic PV equations, by averaging over  
 171 synoptic scales: only the planetary scale terms remain, and the residual represents the synoptic  
 172 scale equation (DK13). This yields

$$\left( \frac{\partial}{\partial t_s} + u_m^{(0)} \frac{1}{a \cos \phi_p} \frac{\partial}{\partial \lambda_s} \right) q_s^{(4)} + v_s^{(1)} \hat{\beta} = S_s^{PV} \quad (9)$$

173 for synoptic scales, and

$$\left( \frac{\partial}{\partial t_p} + u_m^{(0)} \frac{1}{a \cos \phi_p} \frac{\partial}{\partial \lambda_p} \right) q_p^{(3)} + v_p^{(1)} \hat{\beta} = S_p^{PV} \quad (10)$$

174 for planetary scales. The synoptic scale PV equation (9) closely resembles the QG PV equation,  
 175 with the main differences arising in the background PV gradient.

176 The background PV gradient  $\hat{\beta}$  resembles the background PV gradient used in Charney's baro-  
 177 clinic instability model (e.g. Hoskins and James 2014). However, in Charney's model (and also  
 178 in the QG model) there is no dependence of the static stability  $N^2$  (linked to background potential  
 179 temperature) on latitude ( $\phi_p$ ), as there is here since  $\theta^{(2)} = \theta^{(2)}(\phi_p, t_m, z)$ . The QG background  
 180 PV gradient, on the other hand, includes the mean flow relative vorticity gradient ( $-\partial^2 u_m^{(0)} / \partial \phi_p^2$ ),  
 181 which is not present here due to the planetary scaling. This means that  $\hat{\beta}$  represents planetary  
 182 geostrophy (e.g. Phillips 1963, DK09), but it is more realistic than in QG due to the dependence  
 183 of background PV gradient on latitude.

184 The planetary scale PV equation (10) also resembles the QG PV equation, however the planetary  
 185 scale PV (8b) only includes the stretching term (again due to the planetary scaling we chose).  
 186 Note that the planetary and synoptic scale PV equations are independent of each other in this  
 187 small amplitude limit, which implies no direct interaction between planetary and synoptic scales  
 188 — their interaction only occurs via source-sink terms, the mean flow, or at higher order. This  
 189 independence is not present in DK13's finite amplitude theory where the synoptic and planetary  
 190 scale waves interact at leading order.

191 This analysis suggests that the QG approximation can be used locally for both planetary and  
 192 synoptic scale PV. Note, however, that this is only true in this small amplitude case (in the finite  
 193 amplitude theory of DK13 this approach is not applicable for the planetary scales).

194 The potential vorticity equation can be written in a different form (the one used in DK13 for the  
 195 planetary scale), with a vertical advection term in the PV equation, starting from (6). Following  
 196 the derivations in DK09 and DK13, we get

$$\frac{\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \left[ \left( \mathbf{u}^{(1)} \cdot \nabla_m + w^{(4)} \frac{\partial}{\partial z} \right) q_m^{(2)} + \left( \frac{\partial}{\partial t_s} + \mathbf{u}_m^{(0)} \cdot \nabla_s \right) q_{s,2}^{(4)} + \left( \frac{\partial}{\partial t_p} + \mathbf{u}_m^{(0)} \cdot \nabla_p \right) q_{p,2}^{(3)} \right] = S^{PV2} \quad (11)$$

197 where

$$q_{s,2}^{(4)} = \frac{\zeta^{(1)}}{\rho^{(0)}} \frac{\partial\theta^{(2)}}{\partial z} + \frac{f}{\rho^{(0)}} \frac{\partial\theta^{(4)}}{\partial z},$$

$$q_{p,2}^{(3)} = \frac{f}{\rho^{(0)}} \frac{\partial\theta^{(3)}}{\partial z},$$

$$q_m^{(2)} = \frac{f}{\rho^{(0)}} \frac{\partial\theta^{(2)}}{\partial z}, \text{ and}$$

$$S^{PV2} = S_\zeta + \frac{f}{\partial\theta^{(2)}/\partial z} \frac{\partial S_\theta^{(6)}}{\partial z}.$$

201 Here  $q_{s,2}^{(4)}$ ,  $q_{p,2}^{(3)}$ ,  $q_m^{(2)}$ , and  $S^{PV2}$  are the DK synoptic, planetary and mean flow PVs, and the corre-  
 202 sponding PV source term, respectively.

203 The PV equation (11) is closely related to the Ertel PV equation. However, it includes vertical  
 204 advection, which is problematic with respect to obtaining a QG wave activity equation. As shown  
 205 in (7) we can eliminate the vertical advection term by including it in the stretching term of the  
 206 synoptic or planetary scale PV. This is similar to the classical QG approximation of Charney and  
 207 Stern (1962), in which they point out that the QG PV equation is the QG approximation to the  
 208 PV equation, however the QG PV is not the QG approximation to the Ertel PV (because the QG  
 209 PV equation only includes horizontal advection). Notice that in (11) there is also the mean flow  
 210 PV, whereas equation (7) only has the background PV gradient that came from this mean flow  
 211 PV (but not via the direct meridional derivative of  $q_m^{(2)}$ , i.e.  $\hat{\beta} \neq \partial q_m^{(2)}/\partial y_p$ ). This means that the

212 QG approximation of PV would not work for the zonal mean flow, which is consistent with the  
 213 arguments above on the relation between the QG PV and the Ertel PV.

#### 214 4. Wave activity equation and the equations for the mean flow

##### 215 a. Wave activity equation

216 Wave activity is a quantity that is quadratic in amplitude and is conserved in the absence of  
 217 forcing and dissipation (e.g. Vallis 2006). To derive an equation for wave activity, known as the  
 218 Eliassen-Palm (EP) relation, we multiply the PV equations (9) and (10) by  $q_s^{(4)}$  and  $q_p^{(3)}$ , respec-  
 219 tively, and divide them by  $\hat{\beta}$  (as done in e.g. Plumb 1985). This yields

$$\frac{\partial \mathcal{A}_s}{\partial t_s} + \nabla_s^{3D} \cdot \mathbf{F}_s = S_s^{wa} \quad (12)$$

220

$$\frac{\partial \mathcal{A}_p}{\partial t_p} + \nabla_p^{3D} \cdot \mathbf{F}_p = S_p^{wa} \quad (13)$$

221 where

$$\mathcal{A}_s = \frac{\rho^{(0)} q_s^{(4)^2}}{2\hat{\beta}},$$

222

$$\mathcal{A}_p = \frac{\rho^{(0)} q_p^{(3)^2}}{2\hat{\beta}}$$

223 are synoptic and planetary scale wave activities, respectively,  $S_s^{wa} = S_s^{PV} \rho^{(0)} q_s^{(4)} / \hat{\beta}$  and  $S_p^{wa} =$   
 224  $S_p^{PV} \rho^{(0)} q_p^{(3)} / \hat{\beta}$  are wave activity source-sink terms,

$$\mathbf{F}_s = \left( u_m^{(0)} \mathcal{A}_s + \frac{\rho^{(0)}}{2} \left( v_s^{(1)^2} - u_s^{(1)^2} - \frac{\theta^{(4)^2}}{\partial \theta^{(2)} / \partial z} \right), -\rho^{(0)} v_s^{(1)} u_s^{(1)}, \rho^{(0)} f \frac{v_s^{(1)} \theta^{(4)}}{\partial \theta^{(2)} / \partial z} \right),$$

225

$$\mathbf{F}_p = \left( u_m^{(0)} \mathcal{A}_p - \frac{\rho^{(0)}}{2} \frac{\theta^{(3)^2}}{\partial \theta^{(2)} / \partial z}, 0, \rho^{(0)} f \frac{v_p^{(1)} \theta^{(3)}}{\partial \theta^{(2)} / \partial z} \right)$$

226 are synoptic and planetary Eliassen-Palm (EP) fluxes, respectively, and  $\nabla^{3D}$  means that the gradi-  
 227 ent includes the vertical derivative.

228 Note how the planetary scale EP flux does not have a meridional component (no momentum  
 229 flux), and that the synoptic scale EP flux closely resembles Plumb (1985)'s total flux  $\mathbf{B}^{(T)}$ , with  
 230 the main difference, again, arising in  $\hat{\beta}$ . Also,  $u_s^{(1)}$  is actually composed of  $u_s^{(1)} = [u]_s^{(1)} + u_s^{*(1)}$   
 231 (with  $[.]$  as zonal mean and  $*$  as perturbation from zonal mean), which is another difference to  
 232 Plumb's  $\mathbf{B}^{(T)}$  flux.

233 We can also relate these expressions to Hoskins et al. (1983)'s E-vector, where the difference  
 234 is in the zonal component of the E-vector, which lacks the wave activity advection ( $[u]\mathcal{A}$ ) and  
 235 potential temperature ( $\propto -\theta^{*2}$ ) terms.

236 Nonetheless, the synoptic scale EP flux is similar to the QG form of EP flux (e.g. Edmon  
 237 et al. 1980), especially if zonally averaged. The planetary scale wave activity implies that the  
 238 momentum fluxes and hence barotropic processes at those scales are less important than heat  
 239 fluxes and baroclinic processes. Also, this emphasises the fact that planetary and synoptic scales  
 240 do not interact directly, but rather through other processes (source-sink terms or the mean flow)  
 241 as the two wave activity equations are at different orders and have no ‘‘cross’’ terms. The wave  
 242 activity equations are at different orders as the planetary (10) and synoptic (9) PV equations are  
 243 multiplied by  $q_p^{(3)}$  and  $q_s^{(4)}$ , respectively, which are of different orders. This is because they have  
 244 different horizontal derivatives associated with them ( $q_s$  has synoptic and  $q_p$  has planetary).

245 Averaging over synoptic scales ( $\lambda_s, \phi_s, t_s$ ; denoted by overline and  $s$ ) in (12) and over planetary  
 246 scales ( $\lambda_p, t_p$ ; denoted by overline and  $p$ ) in (13) gives

$$\frac{\partial}{\partial z} \left( \rho^{(0)} f \frac{\overline{v_s^{(1)} \theta^{(4)}}^s}{\partial \theta^{(2)} / \partial z} \right) = \overline{S_s^{wa}}^s \approx -r_s \overline{\mathcal{A}_s}^s \quad (14)$$

247

$$\frac{\partial}{\partial z} \left( \rho^{(0)} f \frac{\overline{v_p^{(1)} \theta^{(3)}}^p}{\partial \theta^{(2)} / \partial z} \right) = \overline{S_p^{wa}}^p \approx -r_p \overline{\mathcal{A}_p}^p \quad (15)$$

248 where  $r_{s,p}$  are effective damping coefficients. Note that the approximation  $\overline{S_{s,p}^{wa,s,p}} \approx -r_{s,p} \overline{\mathcal{A}_{s,p}^{s,p}}$   
 249 does not follow from the equations themselves, but is a heuristic relation used as a device to help  
 250 us better understand the physical interpretation of the equations. These equations imply that under  
 251 these averages both synoptic and planetary scale wave activities change via heat flux terms on  
 252 timescales longer than  $t_s$  or  $t_p$  (as we averaged over those) - e.g. timescale  $\varepsilon^4 t$  (between  $t_p$  and  $t_m$ )  
 253 or  $t_m$ . Averaging only over the zonal and time dimensions, the synoptic scale wave activity would  
 254 still be influenced by the synoptic scale momentum fluxes.

### 255 *b. Barotropic equation*

256 As the wave activity equation represents the equation for the eddies, we need additional equa-  
 257 tions for the mean flow to get the influence from the eddies on the mean flow. The barotropic  
 258 pressure equation is derived (following DK13) from the  $\mathcal{O}(\varepsilon^5)$  momentum equation (A8) using  
 259 the relevant thermodynamic, hydrostatic, thermal wind, momentum and continuity equations av-  
 260 eraged not only over  $t_s$ ,  $\lambda_s$ ,  $\phi_s$  and  $t_p$ ,  $\lambda_p$ , but also over  $z$  (denoted by overline and  $z$ ). This yields  
 261 momentum equation (B6) (see Appendix B for more details), which can be used to derive the  
 262 barotropic pressure equation, taking  $\partial/\partial\tilde{y}_p$  of (B6), eliminating the term  $\partial\left(\overline{v^{(4)}\rho^{(0)}^{s,p,z}}\right)/\partial\tilde{y}_p$   
 263 via (B5), multiplying it by  $f$  and recalling (A4):

$$\frac{\partial}{\partial t_m} \left( \frac{\partial}{\partial \tilde{y}_p} \frac{1}{f} \frac{\partial}{\partial y_p} \overline{p^{(2)}^{s,p,z}} - \frac{\beta}{f^2} \frac{\partial}{\partial y_p} \overline{p^{(2)}^{s,p,z}} - f \overline{p^{(2)}^{s,p,z}} \right) - \frac{\partial}{\partial \tilde{y}_p} N_1 + \frac{\beta}{f} N_1 - f N_2 = -S_{barotropic} \quad (16)$$

264 with

$$N_1 = \frac{\partial}{\partial \tilde{y}_p} \left( \overline{\rho^{(0)} v_p^{(1)} u_p^{(1)} + \rho^{(0)} v_s^{(1)} u_s^{(1)}} \right)^{s,p,z} - \frac{\tan \phi_p}{a} \left( \overline{\rho^{(0)} v_p^{(1)} u_p^{(1)} + \rho^{(0)} v_s^{(1)} u_s^{(1)}} \right)^{s,p,z},$$

265

$$N_2 = \frac{\partial}{\partial \tilde{y}_p} \left( \overline{\rho^{(0)} v_p^{(1)} \theta^{(3)}} \right)^{s,p,z},$$



$$\begin{aligned}
S_{barotropic} &= \overline{f\rho^{(0)}S_{\theta}^{(7)}}^{s,p,z} + f \frac{\partial}{\partial \tilde{y}_p} \left( \overline{(\rho^{(2)} + \rho^{(0)}\theta^{(2)}) \frac{S_u^{(3)}}{f}}^{s,p,z} \right) \\
&+ \left( \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f} \right) \left[ \overline{\rho^{(0)}S_u^{(5)}}^{s,p,z} + \left\{ \frac{\partial}{\partial \tilde{y}_p} - \frac{\tan \phi_p}{a} \right\} \left( \overline{\frac{S_u^{(3)}}{f} u^{(0)} \rho^{(0)}}^{s,p,z} \right) - \frac{\rho^{(0)} \overline{S_{\theta}^{(6)}}^{s,p,z} \cos \phi_p}{f \partial \theta^{(2)} / \partial z} \right]
\end{aligned}$$

267 where the underlined terms represent eddy forcing of the mean flow,  $\partial/\partial \tilde{y}_p \equiv$   
268  $(a \cos \phi_p)^{-1} \partial \cos \phi_p / \partial \phi_p$ , and  $\partial/\partial y_p \equiv a^{-1} \partial/\partial \phi_p$ . This evolution equation (16) for  $p^{(2)}$   
269 on the  $t_m$  scale is similar to DK13's  $p^{(2)}$  evolution on the  $t_p$  scale when no source terms are  
270 considered. Using geostrophic balance for  $u^{(0)}$ , (16) can be rewritten as

$$\left( \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f} \right) \frac{\partial \overline{\rho^{(0)} u^{(0)}}^{s,p,z}}{\partial t_m} + f \frac{\partial \overline{p^{(2)}}^{s,p,z}}{\partial t_m} + \left( \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f} \right) N_1 + f N_2 = S_{barotropic}. \quad (17)$$

271 This equation implies that although both the synoptic and planetary scale momentum fluxes  
272 affect the barotropic part of the mean flow, only the planetary scale heat fluxes  $N_2$  are relevant.

273 The zonal mean flow equations at different orders can be further written in TEM form (Andrews  
274 and McIntyre 1976; Edmon et al. 1980), from which a non-acceleration theorem can be derived  
275 using the wave activity equations. This is addressed in Appendix D. Note that an evolution equa-  
276 tion for  $p^{(3)}$  can also be derived, however under the  $\lambda_p, \lambda_s, t_s, \phi_s, z$  average it only evolves through  
277 diabatic and frictional processes (D9).

### 278 *c. Baroclinic equation*

279 The barotropic equation (17) shows how barotropic processes affect the zonal mean flow, how-  
280 ever we are also interested in the baroclinic processes. Therefore, a baroclinic equation for the  
281 zonal mean flow (i.e. equation for baroclinicity  $\propto \partial u^{(0)} / \partial z$ ) is derived from the  $\mathcal{O}(\epsilon^7)$  thermo-  
282 dynamic equation (A12), using the relevant continuity and momentum equations averaged over  $t_s$ ,  
283  $\lambda_s, t_p, \lambda_p$  (denoted with overline), and taking  $\partial/\partial y_p$  of the resulting equation (B7b). The relevant

284 equations (and their derivations) are given in Appendix B, hence using (B10-B14) yields:

$$\begin{aligned}
& -\frac{\partial}{\partial t_m} \left( \overline{f \rho^{(0)} \frac{\partial u^{(0)}}{\partial z}}^{\lambda_s, t_s, P} \right) + \frac{\partial}{\partial y_p} \left[ \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v_p^{(1)} \rho^{(0)} \theta^{(3)}}^{\lambda_s, t_s, P} \right) + \frac{\partial}{\partial \tilde{y}_s} \left( \overline{v_s^{(1)} \rho^{(0)} \theta^{(4)}}^{\lambda_s, t_s, P} \right) \right] \\
& - \frac{\partial}{\partial y_p} \left[ \frac{\partial}{\partial z} \left( \overline{v_p^{(1)} \rho^{(0)} \theta^{(3)}}^{\lambda_s, t_s, P} \frac{\partial \theta^{(2)} / \partial y_p}{\partial \theta^{(2)} / \partial z} \right) - \overline{\rho^{(0)} u_s^{(1)} \frac{\partial \theta^{(3)}}{\partial x_p}}^{\lambda_s, t_s, P} \right] \\
& - \frac{\partial}{\partial y_p} \left[ \frac{\partial \theta^{(2)}}{\partial z} \int_0^{z_{max}} \rho^{(0)} \frac{\partial}{\partial \tilde{y}_s} \left( \frac{\partial}{\partial \tilde{y}_s} \left( \overline{v_s^{(1)} u_s^{(1)}}^{\lambda_s, t_s, P} \right) \right) dz \right] = S_{baroclinic} \quad (18)
\end{aligned}$$

285 with

$$\begin{aligned}
S_{baroclinic} = & \frac{\partial}{\partial y_p} \left[ \frac{\partial}{\partial \tilde{y}_s} \left( \overline{\frac{S_u^{(3)}}{f} \rho^{(0)} \theta^{(3)}}^{\lambda_s, t_s, P} \right) - \frac{\rho^{(0)} \overline{\theta^{(3)} S_\theta^{(6)}}^{\lambda_s, t_s, P}}{\partial \theta^{(2)} / \partial z} + \rho^{(0)} \frac{\overline{S_u^{(3)}}^{\lambda_s, t_s, P}}{f} \frac{\partial \theta^{(2)}}{\partial y_p} \right] \\
& + \frac{\partial}{\partial y_p} \left[ \overline{S_\theta^{(7)} \rho^{(0)}}^{\lambda_s, t_s, P} - S_{w5} \frac{\partial \theta^{(2)}}{\partial z} - \frac{\partial}{\partial \tilde{y}_s} \left( \frac{z \overline{S_u^{(3)}}^{\lambda_s, t_s, P}}{a f} \right) + \frac{\partial}{\partial z} \left( \frac{z \overline{S_\theta^{(6)}}^{\lambda_s, t_s, P}}{a \partial \theta^{(2)} / \partial z} \right) \right],
\end{aligned}$$

$$\begin{aligned}
S_{w5} = & - \int_0^{z_{max}} \left[ \frac{\partial}{\partial \tilde{y}_s} \left( \rho^{(0)} \left\{ \frac{\overline{S_\theta^{(6)}}^{\lambda_s, t_s, P}}{f} \frac{\partial u^{(0)} / \partial z}{\partial \theta^{(2)} / \partial z} - \frac{\overline{S_u^{(4)}}^{\lambda_s, t_s, P}}{f} \right\} \right) - \frac{\partial}{\partial \tilde{y}_p} \left( \rho^{(0)} \frac{\overline{S_u^{(3)}}^{\lambda_s, t_s, P}}{f} \right) \right] dz,
\end{aligned}$$

287 where the terms with  $z/a$  come from corrections to the shallow-atmosphere approximation of the  
288 thermodynamic and continuity equations. Averaging (18) over the synoptic meridional scale ( $\phi_s$ )  
289 gives

$$\begin{aligned}
& -\frac{\partial}{\partial t_m} \left( \overline{f \rho^{(0)} \frac{\partial u^{(0)}}{\partial z}}^{s, P} \right) + \frac{\partial}{\partial y_p} \left[ \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v_p^{(1)} \rho^{(0)} \theta^{(3)}}^{s, P} \right) - \frac{\partial}{\partial z} \left( \overline{v_p^{(1)} \rho^{(0)} \theta^{(3)}}^{s, P} \frac{\partial \theta^{(2)} / \partial y_p}{\partial \theta^{(2)} / \partial z} \right) \right] \\
& = \overline{S_{baroclinic}}^{\phi_s} \quad (19)
\end{aligned}$$

290 which implies that baroclinicity is not affected by the synoptic scale heat fluxes ( $\rho^{(0)} v_s^{(1)} \theta^{(4)}$ ), only  
291 by baroclinic source terms ( $S_{baroclinic}$ ) and planetary scale heat fluxes ( $\rho^{(0)} v_p^{(1)} \theta^{(3)}$ ). The absence  
292 of a synoptic scale heat flux contribution to the baroclinicity tendency is discussed in section 6.

293 **5. Angular momentum conservation**

294 Apart from the mean flow equations (baroclinic and barotropic) and the eddy equations (wave  
 295 activity), angular momentum conservation provides additional information about the transfer of  
 296 angular momentum between the earth and the atmosphere, which has implications for the surface  
 297 easterlies in the tropics and westerlies in the midlatitudes (e.g. Holton 2004). Hence, it is important  
 298 to show that such a budget can be found also in the asymptotic model.

299 Generally, the angular momentum for the hydrostatic primitive equations takes the form (e.g.  
 300 Holton 2004)

$$M = au \cos \phi + a^2 \Omega \cos^2 \phi \quad (20)$$

301 where  $a$  is the radius of the Earth,  $\Omega$  is the Earth's rotation rate,  $\phi$  is meridional coordinate,  $u$  is  
 302 zonal velocity, and  $M$  is angular momentum per unit mass.

303 In the asymptotic regime, a nondimensional version of angular momentum must be used. To  
 304 derive the nondimensional version of (20), define nondimensional terms (similarly as in section  
 305 2):  $u = u^* u_{ref}$ ,  $a = a^* \varepsilon^{-3} h_{sc}$ ,  $\Omega = \frac{1}{2} \Omega^* (2\Omega_{ref})$  and  $M = M^* u_{ref} h_{sc} \varepsilon^{-3}$ , where  $u_{ref}$  and  $h_{sc}$  were  
 306 defined in section 2,  $\Omega_{ref}$  is the Earth's rotation rate (previously denoted  $\Omega$ ),  $M \propto \varepsilon^{-3}$  as it needs  
 307 to be of the same order as other terms, and the asterisk (\*) denotes nondimensional parameters.  
 308 Now divide (20) by  $u_{ref} h_{sc}$  to get nondimensional angular momentum

$$\varepsilon^{-3} M^* = a^* \varepsilon^{-3} u^* \frac{u_{ref} h_{sc}}{u_{ref} h_{sc}} \cos \phi + (\varepsilon^{-3})^2 (a^*)^2 \frac{1}{2} \Omega^* \frac{h_{sc}}{h_{sc}} \frac{h_{sc} 2\Omega_{ref}}{u_{ref}} \cos^2 \phi. \quad (21)$$

309 Cancelling out a few terms, setting  $\Omega^*$  to unity, recognising that<sup>4</sup>  $h_{sc} 2\Omega_{ref} / u_{ref} = Ro^{-1} \approx \varepsilon$ , and  
 310 omitting asterisks for simplicity, yields the nondimensional angular momentum

$$\varepsilon^{-3} M = \varepsilon^{-3} au \cos \phi + \varepsilon^{-3} \varepsilon^{-2} \frac{1}{2} a^2 \cos^2 \phi. \quad (22)$$

---

<sup>4</sup>Here the Rossby number used is the same as the one defined in DK09, DK13:  $Ro^{-1} \approx Ro_{QG} \approx \varepsilon$ .

311 Taking the total derivative (2) of  $M$  in (22) gives the nondimensional angular momentum equation

$$\varepsilon^{-3} \frac{DM}{Dt} = \varepsilon^{-3} a \cos \phi \frac{Du}{Dt} - uv \sin \phi - \varepsilon^{-2} a f v \cos \phi \quad (23)$$

312 using  $\partial/\partial t = \varepsilon^5 \partial/\partial t_m$ ,  $w^{(0)} = w^{(1)} = w^{(2)} = w^{(3)} = 0$  (as derived in Appendix A), and all param-  
313 eters are nondimensional. Notice that

$$\frac{\partial \cos^2 \phi}{\partial \phi} = -2 \cos \phi \sin \phi,$$

314 which means that the factor 2 from this equation cancels out the factor 1/2 in  $M$  (22). Here

$$v = \varepsilon^{-3} a \frac{D\phi}{Dt} = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \dots,$$

315

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots$$

316 The angular momentum equation and its conservation for the zonal mean flow ( $u^{(0)}$ ) are derived  
317 in Appendix C. The second order angular momentum equation is

$$\begin{aligned} \rho \frac{DM}{Dt_m} &= a \cos \phi_p \rho^{(0)} \frac{Du^{(0)}}{Dt_m} - (\rho^{(0)} u^{(1)} v^{(1)} + \rho^{(0)} u^{(0)} v^{(2)}) \sin \phi_p \\ &\quad - f (\rho^{(0)} v^{(4)} + \rho^{(2)} v^{(2)} + \rho^{(3)} v^{(1)}) a \cos \phi_p, \end{aligned} \quad (24)$$

318 from which it is shown (Appendix C) that  $M$  is conserved (using the 5<sup>th</sup> order momentum equation  
319 A8) in the absence of source-sink terms and orography, yielding

$$\iiint_{V_p} \frac{\partial \overline{(\rho M)^{(2)}^{s,t,p}}}{\partial t_m} dV_p = 0 \quad (25)$$

320 where  $V_p$  is volume on planetary scales ( $\lambda_p, \phi_p, z$ ).

321 The barotropic pressure equation (17) can now be rewritten using the angular momentum equa-  
322 tion (Appendix C) as

$$\left( \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f} \right) \left\{ \frac{\rho}{a \cos \phi_p} \frac{DM^{s,p,z}}{Dt_m} \right\} - f \frac{\partial \overline{\rho^{(2)}^{s,p,z}}}{\partial t_m} = -f \frac{\partial \overline{p^{(2)}^{s,p,z}}}{\partial t_m} - f \frac{\partial}{\partial \tilde{y}_p} \left( \overline{\rho^{(0)} v_p^{(1)} \theta^{(3)}^{s,p,z}} \right) \quad (26)$$

323 where the overbar denotes average over  $t_s, t_p, \lambda_s, \lambda_p, \phi_s, z$ . This shows that the two quantities are  
 324 directly linked.

325 Note that the surface pressure tendency  $\overline{\partial p^{(2)}{}^{s,p,z}}/\partial t_m$  in (17) and (26) reflects the response of  
 326 planetary angular momentum to an imposed torque, via mass redistribution, and is an essential  
 327 component of the angular momentum equation at planetary scales (Haynes and Shepherd 1989).  
 328 The present analysis has shown further that the planetary-scale meridional heat flux contributes to  
 329 this meridional mass redistribution. That the synoptic scale heat flux does not so contribute can be  
 330 anticipated from the scaling arguments of Haynes and Shepherd (1989).

## 331 6. The zonally homogeneous case

332 If there are no forced planetary scale waves in the system, then there is no justification for  
 333 separate  $\lambda_p$  and  $t_p$  scales. If the zonal and synoptic scale (including  $\phi_s$ ) average is taken in such a  
 334 case, then the wave activity, barotropic and baroclinic equations become:

$$\frac{\partial}{\partial z} \left( \rho^{(0)} f \frac{\overline{v_s^{(1)} \theta^{(4)}}^s}{\partial \theta^{(2)}/\partial z} \right) \approx -r_s \overline{\mathcal{A}_s^s}, \quad (27a)$$

$$\left( \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f} \right) \frac{\overline{\partial \rho^{(0)} u^{(0)}{}^{s,p,z}}}{\partial t_m} - f \frac{\overline{\partial p^{(2)}{}^{s,p,z}}}{\partial t_m} + \left( \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f} \right) \overline{N_1}{}^{s,p,z} = \overline{S_{barotropic}}{}^{s,p,z}, \quad (27b)$$

$$-\frac{\partial}{\partial t_m} \left( \overline{f \rho^{(0)} \frac{\partial u^{(0)}{}^{s,p}}{\partial z}} \right) = \overline{S_{baroclinic}}{}^{s,p}. \quad (27c)$$

337 These equations imply that under synoptic scale averaging, and to leading order, the wave activity  
 338 is only affected by the heat fluxes through a quasi-steady balance, the barotropic part of the zonal  
 339 mean flow tendency is only affected by the momentum fluxes (in  $N_1$ ), and the baroclinicity ten-  
 340 dency is only affected by source-sink terms. The latter can, however, be related to the source-sink  
 341 terms in the wave activity and barotropic pressure equations. The most surprising of these rela-  
 342 tions are (27a) and (27c), which depend crucially on the averaging over  $\phi_s$ . When the equations

343 are not averaged over  $\phi_s$ , then momentum fluxes appear in the wave activity equation and heat  
 344 fluxes appear in the baroclinicity tendency equation.

345 These findings may help explain the empirical results of Thompson and Woodworth (2014), who  
 346 found that the barotropic and baroclinic parts of the Southern Hemisphere (SH) flow variability  
 347 were decoupled, with the barotropic part of the flow (characterised by the Southern Annular Mode  
 348 (SAM), based on zonal mean zonal wind) being only affected by the momentum fluxes, and the  
 349 baroclinic part of the flow (characterised by the baroclinic annular mode (BAM), based on EKE)  
 350 being only affected by the heat fluxes. We assume here that the wave activity is closely linked to  
 351 EKE. Indeed, Wang and Nakamura (2015, 2016) found that wave activity during the SH summer  
 352 is only affected by the heat fluxes under an average over a few latitudinal bands (approximately  
 353  $10^\circ$ ), giving an equation similar to (27a). Here we put this view into a self-consistent mathematical  
 354 perspective.

355 In a separate study, Thompson and Barnes (2014) found an oscillating relationship between the  
 356 EKE and the heat fluxes with a timescale of 20-30 days. In their model, baroclinicity is affected  
 357 by synoptic scale heat fluxes, through the assumption that

$$\frac{\partial^2 [v^* T^*]}{\partial y^2} = -l^2 [v^* T^*],$$

358 where  $l$  is meridional wave number,  $T$  is temperature,  $[\cdot]$  represents zonal mean and asterisk ( $*$ )  
 359 represents perturbations therefrom. This relation is not present here due to the chosen scaling  
 360 and the averaging over synoptic scales. Equation (18) does in fact have the heat fluxes, acting  
 361 on synoptic scales, which due to the sublinear growth condition (DK13) disappear in (27c), as  
 362 mentioned above.

363 Pfeffer (1987, 1992) argued that heat fluxes (vertical EP fluxes) grow in the part of the domain  
 364 with low stratification parameter  $S$ . Pfeffer's  $S$  can be related to  $\varepsilon$  as  $S = (L_R/a^*)^2 \approx \varepsilon^2$ , where

365  $L_R \approx \varepsilon a^*$  is Rossby deformation radius (a typical synoptic scale) and  $a^*$  is Earth's radius (a typical  
366 planetary scale). Since here we consider the case with  $\varepsilon \ll 1$ , we are then in a regime where  $S \ll 1$   
367 and hence the heat fluxes act to drive the residual meridional circulation rather than the zonal mean  
368 flow, and the vertical derivative of the zonal mean flow (i.e. baroclinicity) is not related to EP flux  
369 divergence to leading order (see equations (6)-(9) in Pfeffer 1992). This suggests a barotropic  
370 response of the zonal mean flow to eddy fluxes after averaging over synoptic scales, which is  
371 consistent with (27b) and (27c).

372 Zurita-Gotor (2017) showed further that there is a low frequency suppression of heat fluxes (at  
373 periods longer than 20-30 days) and concluded that at longer timescales (considered here) the  
374 meridional circulation and diabatic processes are more important for the baroclinicity than the  
375 synoptic scale heat fluxes (consistent with (27c)).

## 376 7. Conclusions

377 In this paper we have provided a theoretical framework for planetary-synoptic-zonal mean flow  
378 interactions in the small amplitude limit with a scale separation in the meridional direction, as well  
379 as in the zonal direction, between planetary and synoptic scales. Thus the synoptic scale eddies  
380 are assumed to be isotropic (which is the case also in QG theory). These assumptions allow us to  
381 derive strong results, e.g. a lack of direct interaction between the planetary and synoptic waves,  
382 and a lack of a direct link between the baroclinic and barotropic components of the flow when only  
383 synoptic scale fluxes are considered.

384 We derived planetary and synoptic scale PV equations (9, 10), and equations for the eddies  
385 (wave activity equations (14-15)), the barotropic part of the zonal mean flow (17) and the baro-  
386 clinic part of the zonal mean flow (19). A crucial step in deriving these equations was finding a  
387 form of the PV equation that eliminated the effect of vertical advection. The synoptic scale PV

388 then resembled QG PV and the planetary PV resembled that of planetary geostrophy, i.e. with only  
389 stretching vorticity representing PV on planetary scales (e.g. Phillips 1963). These equations pro-  
390 vide an alternative view to the conventional Transformed Eulerian Mean (TEM) framework (first  
391 introduced in Andrews and McIntyre 1976), which combines all components into two equations  
392 that are linked through the Eliassen-Palm flux.

393 The background PV gradient (8c) that emerged from the equations lacks the relative vorticity  
394 term as in planetary geostrophy (Phillips 1963), implying the dominance of baroclinic processes  
395 for eddy generation. Thus this PV gradient resembles that of Charney's baroclinic instability  
396 model (e.g. Hoskins and James 2014), but is more general as it includes variations in static stability  
397 in both the vertical and meridional directions. The latter should be stressed as this is the main  
398 difference to QG dynamics in this model.

399 In terms of the baroclinic life cycle (Simmons and Hoskins 1978), the barotropic pressure equa-  
400 tion (17) would be relevant in the breaking region of the storm track and the baroclinic equation  
401 (19) would be more relevant in the source region. We also showed that only the planetary scale  
402 heat fluxes affect the baroclinicity (19), that both planetary and synoptic scale momentum fluxes,  
403 as well as planetary scale heat fluxes, affect the barotropic zonal mean flow (17), and that the  
404 planetary waves and synoptic scale eddies only interact via the zonal mean flow, the source-sink  
405 terms or at higher order approximations. Since both the barotropic (17) and baroclinic (19) parts  
406 of the zonal mean flow are affected by the planetary scale heat fluxes, the latter could provide  
407 a link between upstream and downstream development of storm tracks. The barotropic equation  
408 (17) was also directly linked to the angular momentum equation (26), which has not been noted in  
409 previous work. This linkage revealed the importance of planetary scale heat fluxes (via meridional  
410 mass transport) for the angular momentum budget (Haynes and Shepherd 1989).



411 The importance of planetary scale waves was also noted in Kaspi and Schneider (2011, 2013),  
412 who found that the termination of storm tracks downstream is related to stationary waves and the  
413 baroclinicity associated with them. Stationary waves are especially important locally in contribut-  
414 ing to heat fluxes, which enhance temperature gradients upstream, and reduce them downstream.

415 When considering only the synoptic scale eddies (when planetary scale eddies are weak, as  
416 e.g. in aquaplanet simulations or in the Southern Hemisphere), we find that under synoptic scale  
417 averaging the barotropic zonal mean flow (27b) is only affected by the momentum fluxes, the  
418 baroclinicity (27c) is only affected by the source-sink terms, and wave activity (27a) is only related  
419 to heat fluxes (as in Thompson and Woodworth 2014). This suggests that the baroclinicity is  
420 primarily diabatically driven. Understanding the decoupling of the baroclinic and barotropic parts  
421 of the flow (in the case of weak planetary scale waves) is addressed in a companion study (Boljka  
422 et al. 2018), where it is shown that at timescales longer than synoptic the EKE is only affected by  
423 the heat fluxes and not momentum fluxes, confirming relation (27a).

424 As well as helping to understand a variety of previous results in the literature, one potential  
425 use of the theory presented here could be to help understand the barotropic response to climate  
426 change, which is fundamentally thermally driven. In general, we need a better understanding of  
427 the interaction between the baroclinic and barotropic parts of the flow, where planetary scale heat  
428 fluxes and diabatic processes may play an important role.

429 This theoretical framework could be extended by allowing finite amplitude eddies (as in DK13)  
430 and by relaxing the assumption of a separation of scales in latitude (e.g. Dolaptchiev 2008).

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## 436 APPENDIX A

### 437 **The Multiscale Asymptotic Version of the Primitive Equations**

438 Using the assumptions from section 2b the momentum, thermodynamic, continuity, hydrostatic  
 439 and thermal wind balance equations at different orders ( $\mathcal{O}(i)$ ) can be derived following DK09,  
 440 DK13.

#### 441 *Hydrostatic balance*

442 Up to 4th order:

$$443 \quad \rho^{(i)} = -\frac{\partial p^{(i)}}{\partial z} \quad ; \quad i = 0, \dots, 4. \quad (\text{A1})$$

443 There is also a relationship between  $p$  and  $\theta$  as defined in (47) in (DK09):

$$444 \quad \frac{\partial \pi^{(i)}}{\partial z} = \theta^{(i)} \quad ; \quad i = 2, 3, 4 \quad (\text{A2})$$

444 where  $\pi^{(i)} = p^{(i)}/\rho^{(0)}$ . This identity at the fourth order only holds if  $\frac{\partial}{\partial \phi_s}$  of  $\theta$  is taken (and this  
 445 relationship will only be used in this case).

446 Using (A2) and (A1) one gets a relationship between  $\rho$ ,  $p$  and  $\theta$ :

$$447 \quad \rho^{(i)} = p^{(i)} - \rho^{(0)}\theta^{(i)} \quad ; \quad i = 2, 3 \quad (\text{A3})$$

447 where an assumption is made that  $\rho^{(0)} = \exp(-z)$ .

448 *Momentum equations*

449 Below is the list of all momentum equations up to 5th order. Note that we derive the PV and wave  
 450 activity equations from the 3rd order momentum equation, and we obtain a barotropic equation  
 451 for the mean flow from the 5th order momentum equation.

452  $\mathcal{O}(\varepsilon^1)$  - geostrophic balance for zonal mean wind:

$$f\mathbf{e}_r \times \mathbf{u}^{(0)} = f\mathbf{e}_r \times \mathbf{u}_m^{(0)} = -\nabla_p \pi^{(2)} = -\frac{\partial}{\partial y_p} \pi^{(2)} \mathbf{e}_\phi \quad (\text{A4})$$

453 where subscript  $m$  refers to the mean flow -  $\mathbf{u}^{(0)}$  is related to the zonal mean zonal velocity. Note  
 454 that  $v^{(0)} = 0$ .

455  $\mathcal{O}(\varepsilon^2)$  - geostrophic balance for 1st order wind (planetary and synoptic scale perturbations to  
 456 zonal mean):

$$f\mathbf{e}_r \times \mathbf{u}^{(1)} = -\left(\nabla_p \pi^{(3)} + \nabla_s \pi^{(4)}\right) \quad (\text{A5})$$

457 where  $\mathbf{u}^{(1)} = \mathbf{u}_p^{(1)} + \mathbf{u}_s^{(1)}$  (with subscripts  $p$  and  $s$  referring to planetary and synoptic waves, re-  
 458 spectively), such that  $f\mathbf{e}_r \times \mathbf{u}_p^{(1)} = -\nabla_p \pi^{(3)}$  and  $f\mathbf{e}_r \times \mathbf{u}_s^{(1)} = -\nabla_s \pi^{(4)}$ .

459  $\mathcal{O}(\varepsilon^3)$  - first nontrivial order, used to derive PV equations:

$$\begin{aligned} \frac{\partial \mathbf{u}^{(1)}}{\partial t_s} + \mathbf{u}^{(0)} \cdot \nabla_s \mathbf{u}^{(1)} + f\mathbf{e}_r \times \mathbf{u}^{(2)} + \mathbf{e}_\phi \frac{u^{(0)} u^{(0)} \tan \phi_p}{a} = \\ -\nabla_p \pi^{(4)} + \frac{\rho^{(2)}}{\rho^{(0)}} \nabla_p \pi^{(2)} - \nabla_s \pi^{(5)} + \mathbf{S}_u^{(3)} \end{aligned} \quad (\text{A6})$$

460  $\mathcal{O}(\varepsilon^4)$  - we require only the  $u$ -momentum equation:

$$\begin{aligned} \frac{\partial u^{(2)}}{\partial t_s} + \frac{\partial u^{(1)}}{\partial t_p} + \mathbf{u}^{(1)} \cdot \nabla_s u^{(1)} + \frac{\partial}{\partial \tilde{x}_s} \left( u^{(0)} u^{(2)} \right) + \frac{\partial}{\partial \tilde{x}_p} \left( u^{(0)} u^{(1)} \right) + \\ v^{(1)} \frac{\partial}{\partial y_p} u^{(0)} + w^{(4)} \frac{\partial}{\partial z} u^{(0)} - f v^{(3)} - \frac{u^{(0)} v^{(1)} \tan \phi_p}{a} = \\ -\frac{\partial}{\partial x_p} \pi^{(5)} + \frac{\partial}{\partial x_p} \left( \frac{\rho^{(2)}}{\rho^{(0)}} \pi^{(3)} \right) - \frac{\partial}{\partial x_s} \pi^{(6)} + \frac{\partial}{\partial x_s} \left( \frac{\rho^{(2)}}{\rho^{(0)}} \pi^{(4)} \right) + S_u^{(4)} \end{aligned} \quad (\text{A7})$$

461  $\mathcal{O}(\varepsilon^5)$  - again we require only the  $u$ -momentum equation, used to derive the barotropic pressure

462 equation (equation for the zonal mean zonal flow):

$$\begin{aligned}
& \frac{\partial u^{(0)}}{\partial t_m} + \frac{\partial u^{(3)}}{\partial t_s} + \frac{\partial u^{(2)}}{\partial t_p} + \mathbf{u}^{(1)} \cdot \nabla_s u^{(2)} + \mathbf{u}^{(2)} \cdot \nabla_s u^{(1)} + \frac{\partial}{\partial \tilde{x}_s} \left( u^{(0)} u^{(3)} \right) + \frac{\partial}{\partial \tilde{x}_p} \left( u^{(0)} u^{(2)} \right) + \\
& \quad \mathbf{u}^{(1)} \cdot \nabla_p u^{(1)} + v^{(2)} \frac{\partial}{\partial y_p} u^{(0)} + w^{(4)} \frac{\partial}{\partial z} u^{(1)} + w^{(5)} \frac{\partial}{\partial z} u^{(0)} - f v^{(4)} \\
& - \frac{u^{(0)} v^{(2)} \tan \phi_p}{a} - \frac{u^{(1)} v^{(1)} \tan \phi_p}{a} + w^{(4)} \cos \phi_p = - \frac{\partial}{\partial x_p} \pi^{(6)} + \frac{\partial}{\partial x_p} \left( \frac{\rho^{(2)}}{\rho^{(0)}} \pi^{(4)} \right) + \frac{\rho^{(3)}}{\rho^{(0)}} \frac{\partial}{\partial x_p} \pi^{(3)} - \\
& \quad \frac{\partial}{\partial x_s} \pi^{(7)} + \frac{\partial}{\partial x_s} \left( \frac{\rho^{(2)}}{\rho^{(0)}} \pi^{(5)} \right) + \frac{\partial}{\partial x_s} \left( \frac{\rho^{(3)}}{\rho^{(0)}} \pi^{(4)} \right) + S_u^{(5)}
\end{aligned} \tag{A8}$$

463 where in all equations  $\frac{\partial}{\partial y_{p,s}} = \frac{1}{a} \frac{\partial}{\partial \phi_{p,s}}$ ,  $\frac{\partial}{\partial \tilde{y}_{p,s}} = \frac{1}{a \cos \phi_p} \frac{\partial \cos \phi_p}{\partial \phi_{p,s}}$ ,  $\frac{\partial}{\partial \tilde{x}_{p,s}} = \frac{\partial}{\partial x_{p,s}} = \frac{1}{a \cos \phi_p} \frac{\partial}{\partial \lambda_{p,s}}$ ,  $\nabla_p$  and  $\nabla_s$

464 are the horizontal gradients in a spherical coordinate system (with the above  $x$  and  $y$  coordinates,

465 tilde is used when  $\nabla$  is used as curl or divergence), and  $\mathbf{e}_\phi$  and  $\mathbf{e}_r$  are the unit vectors in the

466 latitudinal and vertical directions respectively.

467 *Thermal wind balance*

468 Using (A5) and (A2):

$$\frac{\partial}{\partial z} u^{(0)} = - \frac{1}{f} \frac{\partial \theta^{(2)}}{\partial y_p}, \tag{A9a}$$

$$\frac{\partial}{\partial z} \mathbf{u}^{(1)} = \frac{1}{f} \mathbf{e}_r \times \left( \nabla_p \theta^{(3)} + \nabla_s \theta^{(4)} \right). \tag{A9b}$$

469 *Thermodynamic ( $\theta$ ) equations*

470 Below is the list of all needed thermodynamic equations. Note that all orders below  $\mathcal{O}(\varepsilon^5)$  give

471 nothing, thus the first order that appears below is  $\mathcal{O}(\varepsilon^5)$ .

472  $\mathcal{O}(\varepsilon^5)$ :

$$w^{(3)} = \frac{S_\theta^{(5)}}{\partial \theta^{(2)} / \partial z} = 0 \tag{A10}$$

473  $\mathcal{O}(\varepsilon^6)$ :

$$\frac{\partial \theta^{(3)}}{\partial t_p} + \frac{\partial \theta^{(4)}}{\partial t_s} + \frac{\partial}{\partial \tilde{x}_p} \left( u^{(0)} \theta^{(3)} \right) + \frac{\partial}{\partial \tilde{x}_s} \left( u^{(0)} \theta^{(4)} \right) + v^{(1)} \frac{\partial \theta^{(2)}}{\partial y_p} + w^{(4)} \frac{\partial \theta^{(2)}}{\partial z} = S_\theta^{(6)} \quad (\text{A11})$$

474  $\mathcal{O}(\varepsilon^7)$ :

$$\begin{aligned} \frac{\partial \theta^{(4)}}{\partial t_p} + \frac{\partial \theta^{(5)}}{\partial t_s} + \frac{\partial \theta^{(2)}}{\partial t_m} + \frac{\partial}{\partial \tilde{x}_p} \left( u^{(0)} \theta^{(4)} \right) + \mathbf{u}^{(1)} \cdot \nabla_p \theta^{(3)} + \mathbf{u}^{(1)} \cdot \nabla_s \theta^{(4)} \\ + \frac{\partial}{\partial \tilde{x}_s} \left( u^{(0)} \theta^{(5)} \right) + v^{(2)} \frac{\partial \theta^{(2)}}{\partial y_p} + w^{(4)} \frac{\partial \theta^{(3)}}{\partial z} + w^{(5)} \frac{\partial \theta^{(2)}}{\partial z} = S_\theta^{(7)} \end{aligned} \quad (\text{A12})$$

475 *Continuity equations*

476 This is the set of all continuity equations (also the trivial ones as they give us information about  
477 vertical velocities).

478  $\mathcal{O}(\varepsilon^0)$ ,  $\mathcal{O}(\varepsilon^1)$  &  $\mathcal{O}(\varepsilon^2)$ :

$$\frac{\partial w^{(i)}}{\partial z} = 0 \quad ; \quad i = 0, 1, 2 \quad (\text{A13})$$

479  $\mathcal{O}(\varepsilon^3)$  (here note that  $w^{(3)} = 0$  from the thermodynamic equation (A10) and that  $\nabla_s \cdot \mathbf{u}^{(1)} = 0$  by  
480 definition):

$$\nabla_p \cdot \mathbf{u}^{(0)} = 0 \quad (\text{A14})$$

481  $\mathcal{O}(\varepsilon^4)$ :

$$\nabla_p \cdot \left( \mathbf{u}^{(1)} \rho^{(0)} \right) + \nabla_s \cdot \left( \mathbf{u}^{(2)} \rho^{(0)} \right) + \frac{\partial}{\partial z} \left( w^{(4)} \rho^{(0)} \right) = 0 \quad (\text{A15})$$

482  $\mathcal{O}(\varepsilon^5)$ :

$$\nabla_p \cdot \left( \mathbf{u}^{(2)} \rho^{(0)} \right) + \nabla_s \cdot \left( \mathbf{u}^{(3)} \rho^{(0)} \right) + \frac{\partial}{\partial z} \left( w^{(5)} \rho^{(0)} \right) = 0 \quad (\text{A16})$$

483  $\mathcal{O}(\varepsilon^6)$ :

$$\begin{aligned} \frac{\partial \rho^{(3)}}{\partial t_p} + \frac{\partial \rho^{(4)}}{\partial t_s} + \nabla_p \cdot \left( \mathbf{u}^{(3)} \rho^{(0)} + \mathbf{u}^{(1)} \rho^{(2)} + \mathbf{u}^{(0)} \rho^{(3)} \right) + \\ \nabla_s \cdot \left( \mathbf{u}^{(4)} \rho^{(0)} + \mathbf{u}^{(2)} \rho^{(2)} + \mathbf{u}^{(0)} \rho^{(4)} - \mathbf{u}^{(1)} \rho^{(0)} \frac{z}{a} \right) + \frac{\partial}{\partial z} \left( w^{(4)} \rho^{(2)} + w^{(6)} \rho^{(0)} \right) = 0 \end{aligned} \quad (\text{A17})$$

484  $\mathcal{O}(\varepsilon^7)$ :

$$\begin{aligned} & \frac{\partial \rho^{(2)}}{\partial t_m} + \frac{\partial \rho^{(4)}}{\partial t_p} + \frac{\partial \rho^{(5)}}{\partial t_s} + \nabla_p \cdot \left( \mathbf{u}^{(4)} \rho^{(0)} + \mathbf{u}^{(2)} \rho^{(2)} + \mathbf{u}^{(1)} \rho^{(3)} + \mathbf{u}^{(0)} \rho^{(4)} - \mathbf{u}^{(1)} \rho^{(0)} \frac{z}{a} \right) \\ & + \nabla_s \cdot \left( \mathbf{u}^{(5)} \rho^{(0)} + \mathbf{u}^{(3)} \rho^{(2)} + \mathbf{u}^{(2)} \rho^{(3)} + \mathbf{u}^{(1)} \rho^{(4)} + \mathbf{u}^{(0)} \rho^{(5)} - \mathbf{u}^{(2)} \rho^{(0)} \frac{z}{a} \right) \\ & + \frac{\partial}{\partial z} \left( w^{(4)} \rho^{(3)} + w^{(5)} \rho^{(2)} + w^{(7)} \rho^{(0)} \right) = 0 \quad (\text{A18}) \end{aligned}$$

485 where terms with  $z/a$  come from corrections to the shallow-atmosphere approximation at higher  
486 orders. Note that these terms vanish in the zonal mean and/or synoptic scale average.

### 487 *Vorticity Equation*

488 To derive the vorticity equation, take  $\nabla_s \times \mathcal{O}(\varepsilon^3)$  momentum equation (A6), and note that terms  
489 with  $\nabla_s \times \nabla_s$  and synoptic scale derivatives of terms  $(\pi, \rho, \theta)$  that do not depend on synoptic  
490 scales (up to 3<sup>rd</sup> order) are zero. This yields (following DK13):

$$\frac{\partial}{\partial t_s} \zeta^{(1)} + \nabla_s \times \left( \mathbf{u}^{(0)} \cdot \nabla_s \mathbf{u}^{(1)} \right) + \nabla_s \times \left( f \mathbf{e}_r \times \mathbf{u}^{(2)} \right) = -\nabla_s \times \nabla_p \pi^{(4)} + \nabla_s \times \mathbf{S}_u^{(3)} \quad (\text{A19})$$

491 where  $\nabla_s = ((a \cos \phi_p)^{-1} \partial / \partial \lambda_s, a^{-1} \partial / \partial \phi_s)$ ,  $\nabla_p = ((a \cos \phi_p)^{-1} \partial / \partial \lambda_p, a^{-1} \partial / \partial \phi_p)$ , the num-  
492 bers in superscripts denote orders of variables,  $\mathbf{u} = (u, v)$  is horizontal velocity,  $\pi = p/\rho$ ,  $\zeta^{(1)} =$   
493  $\nabla_s \times \mathbf{u}^{(1)}$  is relative vorticity, and as  $\nabla_s$  and  $\mathbf{u}^{(1)}$  have only horizontal components  $\zeta^{(1)} = \zeta^{(1)} \mathbf{e}_r$ .  
494 The source term  $\mathbf{S}_u^{(3)}$  represents frictional processes. Note that  $\nabla_s \times \nabla_p \pi^{(4)} = (0, 0, \nabla_p \cdot (f \mathbf{u}_s^{(1)}))$ .

495 Taking  $\mathbf{e}_r \cdot$  of (A19) and applying the vector identities as in DK09 and DK13, we get:

$$\frac{\partial}{\partial t_s} \zeta^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_s \zeta^{(1)} + f \nabla_s \cdot \mathbf{u}^{(2)} = -\nabla_p \cdot (f \mathbf{u}_s^{(1)}) + \mathbf{e}_r \cdot \nabla_s \times \mathbf{S}_u^{(3)} \quad (\text{A20})$$

496 where  $S_\zeta = \mathbf{e}_r \cdot \nabla_s \times \mathbf{S}_u^{(3)}$  and  $\nabla_p \cdot (f \mathbf{u}^{(1)}) = f \nabla_p \cdot \mathbf{u}^{(1)} + v^{(1)} \cos \phi_p / a$  with  $a^{-1} \cos \phi_p =$   
497  $a^{-1} \partial f / \partial \phi_p = \beta$ . Since  $\mathbf{u}^{(2)}$  is not known, we use the  $\mathcal{O}(\varepsilon^4)$  continuity equation (A15) to ob-  
498 tain the vorticity equation:

$$\frac{\partial}{\partial t_s} \zeta^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_s \zeta^{(1)} - \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left( \rho^{(0)} w^{(4)} \right) + \beta v^{(1)} = S_\zeta \quad (\text{A21})$$

499 where  $w^{(4)}$  is known from the  $\mathcal{O}(\varepsilon^6)$  thermodynamic equation (A11), which can be used to derive  
 500 the potential vorticity equation. This vorticity equation resembles the QG vorticity equation (e.g.  
 501 Holton 2004), but now there are different scales represented in the equation.

## 502 APPENDIX B

### 503 Derivation of the Mean Flow Equations

#### 504 a. Barotropic equation

505 This section shows the steps in deriving the barotropic pressure equation - combining the correct  
 506 thermodynamic, hydrostatic, thermal wind, momentum and continuity equations (see Appendix A)  
 507 with the  $\mathcal{O}(\varepsilon^5)$  momentum equation (A8) averaged over  $t_s, \lambda_s, \phi_s, t_p, \lambda_p, z$  (denoted with overline).  
 508 Note that the vertical mean assumes  $w = 0$  at the top and bottom boundaries. This section modifies  
 509 the momentum (A8) and thermodynamic (A12) equations, which can then be used to derive the  
 510 barotropic equations in section 4b (following DK13).

511 First average the flux forms of all equations mentioned:

512 Momentum Equations at  $\mathcal{O}(\varepsilon^3), \mathcal{O}(\varepsilon^4), \mathcal{O}(\varepsilon^5)$ :

$$513 \quad \overline{v^{(2)}} = -\frac{\overline{S_u^{(3)}}^{s,p,z}}{f}, \quad (\text{B1a})$$

$$514 \quad \overline{v^{(3)}} = -\frac{\overline{S_u^{(4)}}^{s,p,z}}{f}, \quad (\text{B1b})$$

$$515 \quad \begin{aligned} & \frac{\partial \overline{u^{(0)} \rho^{(0)}}^{s,p,z}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v^{(1)} u^{(1)} \rho^{(0)}}^{s,p,z} + \overline{v^{(2)} u^{(0)} \rho^{(0)}}^{s,p,z} \right) \\ & - \frac{\tan \phi_p}{a} \left( \overline{v^{(1)} u^{(1)} \rho^{(0)}}^{s,p,z} + \overline{v^{(2)} u^{(0)} \rho^{(0)}}^{s,p,z} \right) - \overline{\rho^{(0)} v^{(4)} f}^{s,p,z} \\ & + \overline{\rho^{(0)} w^{(4)}}^{s,p,z} \cos \phi_p = \overline{\rho^{(3)} \frac{\partial \pi^{(3)}}{\partial x_p}}^{s,p,z} + \overline{\rho^{(0)} S_u^{(5)}}^{s,p,z}. \end{aligned} \quad (\text{B1c})$$

515 Continuity equations at  $\mathcal{O}(\varepsilon^4)$ ,  $\mathcal{O}(\varepsilon^5)$ ,  $\mathcal{O}(\varepsilon^6)$ ,  $\mathcal{O}(\varepsilon^7)$ :

$$\frac{\partial}{\partial \tilde{y}_p} \left( \overline{v^{(1)} \rho^{(0)}{}^{s,p,z}} \right) = 0, \quad (\text{B2a})$$

516

$$\frac{\partial}{\partial \tilde{y}_p} \left( \overline{v^{(2)} \rho^{(0)}{}^{s,p,z}} \right) = 0, \quad (\text{B2b})$$

517

$$\frac{\partial}{\partial \tilde{y}_p} \left( \overline{v^{(3)} \rho^{(0)}{}^{s,p,z}} \right) = 0, \quad (\text{B2c})$$

518

$$\frac{\partial \overline{\rho^{(2)}{}^{s,p,z}}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v_p^{(1)} \rho^{(3)}{}^{s,p,z}} + \overline{v^{(2)} \rho^{(2)}{}^{s,p,z}} + \overline{v^{(4)} \rho^{(0)}{}^{s,p,z}} \right) = 0. \quad (\text{B2d})$$

519 Thermodynamic equations at  $\mathcal{O}(\varepsilon^6)$ ,  $\mathcal{O}(\varepsilon^7)$ :

$$\overline{w^{(4)}{}^{s,p,z}} = \frac{\overline{S_\theta^{(6)}{}^{s,p,z}}}{\partial \theta^{(2)} / \partial z}, \quad (\text{B3a})$$

520

$$\frac{\partial \overline{\rho^{(0)} \theta^{(2)}{}^{s,p,z}}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v_p^{(1)} \rho^{(0)} \theta^{(3)}{}^{s,p,z}} + \overline{v^{(2)} \rho^{(0)} \theta^{(2)}{}^{s,p,z}} \right) = \overline{S_\theta^{(7)} \rho^{(0)}{}^{s,p,z}}. \quad (\text{B3b})$$

521 Hydrostatic balance at  $\mathcal{O}(\varepsilon^2)$

$$\overline{\rho^{(2)}{}^{s,p,z}} = -\overline{\rho^{(0)} \theta^{(2)}{}^{s,p,z}} + \overline{p^{(2)}{}^{s,p,z}}. \quad (\text{B4})$$

522 Equations (B1a,B1b) show that  $\overline{v^{(2)}{}^{s,p,z}}$  and  $\overline{v^{(3)}{}^{s,p,z}}$  are related to source terms, thus in the equa-  
 523 tions below they will be replaced by them. Note that  $\rho^{(3)} \partial \pi^{(3)} / \partial x_p = f \rho^{(3)} v_p^{(1)}$  (via (A5)). Taking  
 524 the hydrostatic balance equation (B4), using it to substitute  $\rho^{(2)}$  in the continuity equation (B2d)  
 525 and matching the  $\partial \overline{\rho^{(0)} \theta^{(2)}{}^{s,p,z}} / \partial t_m$  term in the thermodynamic equation (B3b) yields

$$\begin{aligned} & \frac{\partial \overline{p^{(2)}{}^{s,p,z}}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v_p^{(1)} \rho^{(0)} \theta^{(3)}{}^{s,p,z}} + \overline{v_p^{(1)} \rho^{(3)}{}^{s,p,z}} + \overline{v^{(4)} \rho^{(0)}{}^{s,p,z}} \right) \\ & = \overline{\rho^{(0)} S_\theta^{(7)}{}^{s,p,z}} + \frac{\partial}{\partial \tilde{y}_p} \left( \overline{(\rho^{(2)} + \rho^{(0)} \theta^{(2)}) \frac{S_u^{(3)}{}^{s,p,z}}{f}} \right). \end{aligned} \quad (\text{B5})$$



526 Rewriting the momentum equation then gives:

$$\begin{aligned}
& \frac{1}{f} \frac{\partial \overline{u^{(0)} \rho^{(0)}}^{s,p,z}}{\partial t_m} + \frac{1}{f} \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v^{(1)} u^{(1)} \rho^{(0)}}^{s,p,z} \right) - \frac{1}{f} \frac{\tan \phi_p}{a} \left( \overline{v^{(1)} u^{(1)} \rho^{(0)}}^{s,p,z} \right) \\
& - \overline{\rho^{(0)} v^{(4)}}^{s,p,z} - \overline{\rho^{(3)} v_p^{(1)}}^{s,p,z} = \frac{1}{f} \overline{\rho^{(0)} S_u^{(5)}}^{s,p,z} + \frac{1}{f} \frac{\partial}{\partial \tilde{y}_p} \left( \overline{\frac{S_u^{(3)}}{f} u^{(0)} \rho^{(0)}}^{s,p,z} \right) \\
& - \frac{1}{f} \frac{\tan \phi_p}{a} \left( \overline{\frac{S_u^{(3)}}{f} u^{(0)} \rho^{(0)}}^{s,p,z} \right) - \frac{\rho^{(0)} \overline{S_\theta^{(6)}}^{s,p,z} \cos \phi_p}{f \partial \theta^{(2)} / \partial z}. \tag{B6}
\end{aligned}$$

527 The latter two equations are then used in section 4b to derive the barotropic pressure equation (16)

528 or (17).

### 529 *b. Baroclinic equation*

530 This section shows the steps in deriving the baroclinic mean flow equation, which is derived

531 through the  $\mathcal{O}(\varepsilon^7)$  thermodynamic equation (A12) using the continuity and momentum equations

532 averaged over  $t_s, \lambda_s, t_p, \lambda_p$  (denoted with an overbar). The averaged equations are:

533 Thermodynamic equations at  $\mathcal{O}(\varepsilon^6), \mathcal{O}(\varepsilon^7)$ :

$$\overline{w^{(4)}}^{t_s, \lambda_s, p} = \frac{\overline{S_\theta^{(6)}}^{t_s, \lambda_s, p}}{\partial \theta^{(2)} / \partial z}, \tag{B7a}$$

534

$$\begin{aligned}
& \frac{\partial \overline{\rho^{(0)} \theta^{(2)}}^{t_s, \lambda_s, p}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p} \left( \overline{v_p^{(1)} \rho^{(0)} \theta^{(3)}}^{t_s, \lambda_s, p} + \overline{v^{(2)} \rho^{(0)} \theta^{(2)}}^{t_s, \lambda_s, p} \right) \\
& + \frac{\partial}{\partial \tilde{y}_s} \left( \overline{v_s^{(1)} \rho^{(0)} \theta^{(4)}}^{t_s, \lambda_s, p} + \overline{v^{(2)} \rho^{(0)} \theta^{(3)}}^{t_s, \lambda_s, p} - \overline{v^{(2)} \frac{z}{a}}^{t_s, \lambda_s, p} \right) \\
& + \frac{\partial}{\partial z} \left( \overline{w^{(4)} \rho^{(0)} \theta^{(3)}}^{t_s, \lambda_s, p} - \overline{w^{(4)} \frac{z}{a}}^{t_s, \lambda_s, p} \right) + \overline{\rho^{(0)} w^{(5)}}^{t_s, \lambda_s, p} \frac{\partial \theta^{(2)}}{\partial z} = \overline{S_\theta^{(7)} \rho^{(0)}}^{t_s, \lambda_s, p}, \tag{B7b}
\end{aligned}$$

535 where terms with  $z/a$  come from corrections to the shallow-atmosphere approximation.

536 Continuity equations at  $\mathcal{O}(\varepsilon^4), \mathcal{O}(\varepsilon^5)$ :

$$\frac{\partial}{\partial \tilde{y}_p} \left( \overline{v^{(1)} \rho^{(0)}}^{t_s, \lambda_s, p} \right) + \frac{\partial}{\partial \tilde{y}_s} \left( \overline{v^{(2)} \rho^{(0)}}^{t_s, \lambda_s, p} \right) + \frac{\partial}{\partial z} \left( \overline{w^{(4)} \rho^{(0)}}^{t_s, \lambda_s, p} \right) = 0, \tag{B8a}$$

$$\frac{\partial}{\partial \tilde{y}_p} \left( \overline{v^{(2)} \rho^{(0) t_s, \lambda_s, p}} \right) + \frac{\partial}{\partial \tilde{y}_s} \left( \overline{v^{(3)} \rho^{(0) t_s, \lambda_s, p}} \right) + \frac{\partial}{\partial z} \left( \overline{w^{(5)} \rho^{(0) t_s, \lambda_s, p}} \right) = 0. \quad (\text{B8b})$$

Momentum equations at  $\mathcal{O}(\varepsilon^3)$ ,  $\mathcal{O}(\varepsilon^4)$ :

$$\overline{v^{(2) t_s, \lambda_s, p}} = - \frac{\overline{S_u^{(3) t_s, \lambda_s, p}}}{f}, \quad (\text{B9a})$$

$$\overline{v^{(3) t_s, \lambda_s, p}} = - \frac{\overline{S_u^{(4) t_s, \lambda_s, p}}}{f} + \frac{\partial}{\partial \tilde{y}_s} \left( \frac{\overline{u_s^{(1)} v_s^{(1) t_s, \lambda_s, p}}}{f} \right) + \frac{\overline{w^{(4) t_s, \lambda_s, p}}}{f} \frac{\partial u^{(0)}}{\partial z}. \quad (\text{B9b})$$

Here note that terms with  $v_p^{(1)} \theta^{(3)}$  or  $w^{(4)} \theta^{(3)}$ ,  $v_p^{(1)}$  and  $w^{(4)}$  cannot simply be averaged over  $\lambda_p$  and  $t_p$  - we need to average  $v_p^{(1)} \theta^{(3)}$  or  $w^{(4)} \theta^{(3)}$  together as also  $\theta^{(3)}$  depends on planetary scales. This means that, in order to replace the  $w^{(4)}$  and  $v_p^{(1)}$  terms in equation (B7b), the  $\mathcal{O}(\varepsilon^6)$  thermodynamic equation and  $\mathcal{O}(\varepsilon^3)$  momentum equation have to first be multiplied by  $\theta^{(3)}$  and then averaged over  $\lambda_s, t_s, \lambda_p, t_p$ . For the  $\mathcal{O}(\varepsilon^3)$  momentum equation this gives

$$\overline{\theta^{(3)} v^{(2) t_s, \lambda_s, p}} = - \frac{\overline{\theta^{(3)} S_u^{(3) t_s, \lambda_s, p}}}{f} + \frac{\overline{\theta^{(3)} \partial \pi^{(4) t_s, \lambda_s, p}}}{f \partial x_p}. \quad (\text{B10})$$

Multiplying equation (B10) by  $\rho^{(0)}$  and taking  $\partial / \partial \tilde{y}_s$  of it yields

$$\frac{\partial}{\partial \tilde{y}_s} \left( \overline{\rho^{(0)} \theta^{(3)} v^{(2) t_s, \lambda_s, p}} \right) = - \frac{\partial}{\partial \tilde{y}_s} \left( \frac{\overline{\rho^{(0)} \theta^{(3)} S_u^{(3) t_s, \lambda_s, p}}}{f} \right) + \overline{\rho^{(0)} u_s^{(1)} \frac{\partial \theta^{(3) t_s, \lambda_s, p}}{\partial x_p}} \quad (\text{B11})$$

where  $u_s^{(1)} = -f^{-1} \partial \pi^{(4)} / \partial y_s$  was used. However, it is more complicated for the thermodynamic equation - here is a short derivation: First multiply the equation by  $\theta^{(3)}$

$$\begin{aligned} \frac{1}{2} \frac{\partial \theta^{(3)^2}}{\partial t_p} + \frac{\partial \theta^{(3)} \theta^{(4)}}{\partial t_s} + \frac{1}{2} \frac{\partial}{\partial \tilde{x}_p} \left( u^{(0)} \theta^{(3)^2} \right) + \frac{\partial}{\partial \tilde{x}_s} \left( \theta^{(3)} u^{(0)} \theta^{(4)} \right) \\ + \theta^{(3)} v^{(1)} \frac{\partial \theta^{(2)}}{\partial y_p} + \theta^{(3)} w^{(4)} \frac{\partial \theta^{(2)}}{\partial z} = \theta^{(3)} S_\theta^{(6)}, \end{aligned} \quad (\text{B12})$$

then average it over  $\lambda_s, t_s, \lambda_p, t_p$ :

$$\overline{\theta^{(3)} w^{(4) t_s, \lambda_s, p}} = - \overline{\theta^{(3)} v^{(1) t_s, \lambda_s, p}} \frac{\partial \theta^{(2)} / \partial y_p}{\partial \theta^{(2)} / \partial z} + \frac{\overline{\theta^{(3)} S_\theta^{(6) t_s, \lambda_s, p}}}{\partial \theta^{(2)} / \partial z}. \quad (\text{B13})$$

549 We can derive an equation for  $\overline{w^{(5)}\rho^{(0)}}^{t_s, \lambda_s, p}$  by integrating (B8b) over  $z$  and using (B9a) and (B9b).

550 This yields:

$$\overline{w^{(5)}\rho^{(0)}}^{t_s, \lambda_s, p} = - \int_0^{z_{max}} \rho^{(0)} \frac{\partial}{\partial \tilde{y}_s} \left( \frac{\partial}{\partial \tilde{y}_s} \left( \frac{\overline{v_s^{(1)} u_s^{(1)}}^{t_s, \lambda_s, p}}{f} \right) \right) dz + S_{w5} \quad (\text{B14})$$

551 with

$$S_{w5} = - \int_0^{z_{max}} \left[ \frac{\partial}{\partial \tilde{y}_s} \left( \rho^{(0)} \left\{ \frac{\overline{S_\theta^{(6)}}^{t_s, \lambda_s, p}}{f} \frac{\partial u^{(0)}/\partial z}{\partial \theta^{(2)}/\partial z} - \frac{\overline{S_u^{(4)}}^{t_s, \lambda_s, p}}{f} \right\} \right) - \frac{\partial}{\partial \tilde{y}_p} \left( \rho^{(0)} \frac{\overline{S_u^{(3)}}^{t_s, \lambda_s, p}}{f} \right) \right] dz.$$

552 These equations are then used in section 4c to derive the final baroclinic equation for the mean  
553 flow (18, 19).

## 554 APPENDIX C

### 555 Derivation of the Angular Momentum Equation

556 This Appendix shows the derivation of angular momentum conservation for the zonal mean flow  
557 ( $u^{(0)}$ ) equation, following from the  $\mathcal{O}(\varepsilon^5)$  momentum equation (A8). Note that similar systems  
558 can be derived for higher order velocities as well and at all asymptotic orders, but are omitted for  
559 brevity.

560 Deriving an angular momentum equation for the mean flow means that something that corre-  
561 sponds to the fifth order momentum equation (A8) must be used. This means that, for example,  
562  $Du/Dt$  has to be of fifth order, which overall makes the angular momentum equation (23) a second  
563 order equation, thus the rest of the terms in the equation must follow that pattern.

564 Using these statements and noting that  $\phi = \phi_p$ , the angular momentum equation (23) becomes

$$\begin{aligned} \varepsilon^{-3} \varepsilon^5 \frac{DM}{Dt_m} = \varepsilon^{-3} \varepsilon^5 a \cos \phi_p \frac{Du^{(0)}}{Dt_m} - (u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots)(\varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \dots) \sin \phi_p \\ - \varepsilon^{-2} f (v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \dots) a \cos \phi_p, \quad (\text{C1}) \end{aligned}$$

565 where  $v^{(0)} = 0$  because the zonal mean flow is geostrophic to leading order (A4). In this form,  
 566 angular momentum is not conserved. To get a conservative form of this equation, multiply (C1)  
 567 by  $\rho = \rho^{(0)} + \varepsilon^2 \rho^{(2)} + \dots$

$$\begin{aligned} \varepsilon^2 \rho \frac{DM}{Dt_m} &= \varepsilon^2 a \cos \phi_p (\rho^{(0)} + \varepsilon^2 \rho^{(2)} + \dots) \frac{Du^{(0)}}{Dt_m} \\ &\quad - (\rho^{(0)} + \varepsilon^2 \rho^{(2)} + \dots)(u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots)(\varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \dots) \sin \phi_p \\ &\quad - \varepsilon^{-2} f(\rho^{(0)} + \varepsilon^2 \rho^{(2)} + \dots)(\varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \dots) a \cos \phi_p \end{aligned} \quad (C2)$$

568 and taking the same orders together, yields the second order angular momentum equation (omit  $\varepsilon$   
 569 everywhere)

$$\begin{aligned} \rho \frac{DM}{Dt_m} &= a \cos \phi_p \rho^{(0)} \frac{Du^{(0)}}{Dt_m} - (\rho^{(0)} u^{(1)} v^{(1)} + \rho^{(0)} u^{(0)} v^{(2)}) \sin \phi_p \\ &\quad - f(\rho^{(0)} v^{(4)} + \rho^{(2)} v^{(2)} + \rho^{(3)} v^{(1)}) a \cos \phi_p. \end{aligned} \quad (C3)$$

570 Note that since an equation for the mean flow is derived, (24) can be averaged over synoptic  
 571 scales  $(t_s, \lambda_s, \phi_s)$  and planetary time scale  $(t_p)$ , which simplifies it. To get the angular conservation  
 572 equation, the continuity equations (A14-A16) are needed, which can be written together as

$$\nabla_p \cdot (\overline{(\rho^{(0)} \mathbf{u}^{(i)})^{s,t_p}}) + \frac{\partial (\overline{(\rho^{(0)} w^{(i+3)})^{s,t_p}})}{\partial z} = 0 \quad (C4)$$

573 where overline denotes average over  $(t_s, t_p, \lambda_s, \phi_s)$ , and  $i = 0, 1, 2$  (where for  $i = 0$ ,  $w^{(3)} = 0$ ). This  
 574 equation can then be written in a shorter form as

$$\nabla_p^{3D} \cdot (\overline{(\rho^{(0)} \mathbf{u}_{3D}^{(i)})^{s,t_p}}) = 0 \quad (C5)$$

575 where

$$\nabla_p^{3D} \cdot = \left( \frac{1}{a \cos \phi_p} \frac{\partial}{\partial \lambda_p}, \frac{1}{a \cos \phi_p} \frac{\partial \cos \phi_p}{\partial \phi_p}, \frac{\partial}{\partial z} \right)$$

576 now includes the vertical derivative and  $\mathbf{u}_{3D}^{(i)} = (u^{(i)}, v^{(i)}, w^{(i+3)})$  is the three-dimensional velocity  
 577 field. Note that in general the continuity equation can be used to simplify expression (24), using

$$\begin{aligned}\rho \frac{DB}{Dt} &= \frac{D\rho B}{Dt} - B \frac{D\rho}{Dt} \\ &= \frac{\partial(\rho B)}{\partial t} + \nabla^{3D} \cdot (B\rho \mathbf{u}_{3D})\end{aligned}\quad (\text{C6})$$

578 where  $B$  is an arbitrary scalar, and  $\mathbf{u}_{3D}$  is three-dimensional velocity; noting that mass is conserved  
 579 for every order, the continuity equation for each order in general takes the form  $D\rho/Dt = -\rho \nabla_{3D} \cdot$   
 580  $\mathbf{u}$ , where  $\partial\rho/\partial t$  is mainly zero as  $\rho^{(0)}$  only depends on the vertical coordinate.

581 Using (C6) for  $\rho DM/Dt_m$  and (C5, A8) for  $\rho^{(0)} Du^{(0)}/Dt_m$  gives

$$\begin{aligned}\frac{\partial(\overline{\rho M})^{s,t_p}}{\partial t_m} + \nabla_p^{3D} \cdot (\overline{M\rho \mathbf{u}_{3D}})^{s,t_p} &= a \cos \phi_p \frac{\partial(\overline{\rho^{(0)} u^{(0)}})^{s,t_p}}{\partial t_m} \\ &+ a \cos \phi_p \nabla_p^{3D} \cdot \left( \overline{u^{(2)} \rho^{(0)} \mathbf{u}_{3D}^{(0)}}^{s,t_p} + \overline{u^{(1)} \rho^{(0)} \mathbf{u}_{3D}^{(1)}}^{s,t_p} + \overline{u^{(0)} \rho^{(0)} \mathbf{u}_{3D}^{(2)}}^{s,t_p} \right) \\ &- (\overline{\rho^{(0)} u^{(1)} v^{(1)}}^{s,t_p} + \overline{\rho^{(0)} u^{(0)} v^{(2)}}^{s,t_p}) \sin \phi_p - f (\overline{\rho^{(0)} v^{(4)}}^{s,t_p} + \overline{\rho^{(2)} v^{(2)}}^{s,t_p} + \overline{\rho^{(3)} v^{(1)}}^{s,t_p}) a \cos \phi_p.\end{aligned}\quad (\text{C7})$$

582 Note that the orders of separate terms on the right hand side are not given as they do not play  
 583 an important role in the further derivation (for simplicity), however note that overall  $\overline{\rho M}^{s,t_p}$  and  
 584  $\overline{M\rho \mathbf{u}_{3D}}^{s,t_p}$  are of the second order.

585 From (A8) multiplied by  $\rho^{(0)}$  it follows that

$$\begin{aligned}\rho^{(0)} \frac{Du^{(0)}}{Dt_m} &= f (\overline{v^{(4)} \rho^{(0)}}^{s,t_p} + \overline{v^{(1)} \rho^{(3)}}^{s,t_p} + \overline{v^{(2)} \rho^{(2)}}^{s,t_p}) \\ &+ \frac{\tan \phi_p}{a} \left( \overline{v^{(2)} u^{(0)} \rho^{(0)}}^{s,t_p} + \overline{v^{(1)} u^{(1)} \rho^{(0)}}^{s,t_p} \right) + \overline{\rho^{(0)} S_u^{(5)}}^{s,t_p} - \frac{\partial}{\partial x_p} \left( \overline{\pi^{(6)} \rho^{(0)}}^{s,t_p} \right) \\ &- \left[ \frac{\cos \phi_p}{\partial \theta^{(2)}/\partial z} \overline{S_\theta^{(6)}}^{s,t_p} + \overline{\rho^{(2)} S_u^{(3)}}^{s,t_p} + \frac{\partial}{\partial x_p} \left[ \frac{\cos \phi_p}{\partial \theta^{(2)}/\partial z} \left( \overline{u^{(0)} \theta^{(3)} \rho^{(0)}}^{s,t_p} + \frac{\overline{\rho^{(0)} \pi^{(3)} \partial \theta^{(2)}}^{s,t_p}}{f} \right) \right] \right]\end{aligned}\quad (\text{C8})$$

586 where the last two terms come from the  $w^{(4)} \cos \phi_p$  term using the thermodynamic equation (A11)  
587 averaged over synoptic scales and  $t_p$ ,  $f v^{(1)} \rho^{(3)} = \rho^{(3)} \partial \pi^{(3)} / \partial x_p$  (via (A5)), and  $\overline{f v^{(2)} \rho^{(2)}}^{s,t_p} =$   
588  $\overline{\pi^{(4)} \rho^{(2)}}^{s,t_p} + \overline{\rho^{(2)} S_u^{(3)}}^{s,t_p}$  (via (A6)). Notice that the first two terms on the right-hand-side of (C8)  
589 resemble the terms involving  $\sin \phi_p$  and  $f a \cos \phi_p$  in (C7), and lead to a cancellation after combin-  
590 ing (C7) and (C8). The terms that remain in the equation can all be integrated over a volume  $V_p$   
591  $(\lambda_p, \phi_p, z)$ . Following Gauss' theorem<sup>5</sup>, assuming no source-sink terms and assuming there is no  
592 orography (for simplicity) yields angular momentum conservation

$$\iiint_{V_p} \frac{\partial \overline{(\rho M)}^{s,t_p}}{\partial t_m} dV_p = 0. \quad (\text{C9})$$

593 The angular momentum equation can be linked to the barotropic pressure equation (17) using  
594 (C7), dividing it first by  $a \cos \phi_p$ , then integrating it over a longitude-height slice (over area  $A_p$ ,  
595 which effectively gives additional averaging over  $\lambda_p$  and  $z$ ) and using the divergence theorem again  
596 which gives

$$\begin{aligned} & \frac{1}{a \cos \phi_p} \left[ \frac{\partial \overline{(\rho M)}^{s,p,z}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p} \overline{(M \rho v)}^{s,p,z} \right] = \frac{\partial \overline{\rho^{(0)} u^{(0)}}^{s,p,z}}{\partial t_m} \\ & + \frac{\partial}{\partial \tilde{y}_p} \left( \overline{u^{(1)} \rho^{(0)} v^{(1)}}^{s,p,z} + \overline{u^{(0)} \rho^{(0)} v^{(2)}}^{s,p,z} \right) - \left( \overline{\rho^{(0)} u^{(1)} v^{(1)}}^{s,p,z} + \overline{\rho^{(0)} u^{(0)} v^{(2)}}^{s,p,z} \right) \frac{\tan \phi_p}{a} \\ & - f \left( \overline{\rho^{(0)} v^{(4)}}^{s,p,z} + \overline{\rho^{(2)} v^{(2)}}^{s,p,z} + \overline{\rho^{(3)} v^{(1)}}^{s,p,z} \right). \quad (\text{C10}) \end{aligned}$$

597 Here the overbar denotes an average over  $t_s, t_p, \lambda_s, \lambda_p, \phi_s, z$  and note that  $v^{(2)}$  is proportional to a  
598 source term under such an average (B1a). Now divide (C10) by  $f$ , take  $\partial / \partial \tilde{y}_p$  of it, and finally

---

<sup>5</sup>Gauss' theorem generally states  $\iiint_V \nabla \cdot \mathbf{F} dV = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS$ , where  $\mathbf{F}$  is a three-dimensional vector,  $\mathbf{n}$  is a normal vector on surface  $S$ , and  $\partial V$  is the surface around the volume  $V$  of interest. Note that in the case of  $\mathbf{F} = \rho \mathbf{M} \mathbf{u}$  the  $\iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = 0$  as  $\mathbf{u} \cdot \mathbf{n} = 0$  at the lower boundary and  $\rho \rightarrow 0$  at the upper boundary.

599 multiply it by  $f$ . This yields

$$\begin{aligned}
\mathcal{L} \left\{ \frac{1}{a \cos \phi_p} \left[ \frac{\partial \overline{\rho M}^{s,p,z}}{\partial t_m} + \frac{\partial}{\partial \tilde{y}_p} (\overline{M \rho v}^{s,p,z}) \right] \right\} &= \mathcal{L} \left\{ \frac{\partial \overline{\rho^{(0)} u^{(0)}}^{s,p,z}}{\partial t_m} \right\} \\
&+ \mathcal{L} \left\{ \frac{\partial}{\partial \tilde{y}_p} \left( \overline{u^{(1)} \rho^{(0)} v^{(1)}}^{s,p,z} \right) - \overline{(\rho^{(0)} u^{(1)} v^{(1)})}^{s,p,z} \frac{\tan \phi_p}{a} \right\} \\
&- f \frac{\partial}{\partial \tilde{y}_p} \left( \overline{\rho^{(0)} v^{(4)}}^{s,p,z} + \overline{\rho^{(2)} v^{(2)}}^{s,p,z} + \overline{\rho^{(3)} v^{(1)}}^{s,p,z} \right), \quad (C11)
\end{aligned}$$

600 where source terms were omitted for simplicity, the left-hand-side can be simplified to

$$\mathcal{L} \left\{ \frac{\rho}{a \cos \phi_p} \frac{DM^{s,p,z}}{Dt_m} \right\}$$

601 with

$$\mathcal{L} = \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f},$$

602 and the last term in the equation can be simplified to  $+f \partial \rho^{(2)} / \partial t_m$  via (B2d). Notice how all but

603 the last term on the right-hand-side resemble terms in the barotropic pressure equation (17). This

604 means that (17) can be rewritten using the angular momentum equation as

$$\mathcal{L} \left\{ \frac{\rho}{a \cos \phi_p} \frac{DM^{s,p,z}}{Dt_m} \right\} - f \frac{\partial \overline{\rho^{(2)}}^{s,p,z}}{\partial t_m} = -f \frac{\partial \overline{p^{(2)}}^{s,p,z}}{\partial t_m} - f \frac{\partial}{\partial \tilde{y}_p} \left( \overline{\rho^{(0)} v_p^{(1)} \theta^{(3)}}^{s,p,z} \right) \quad (C12)$$

605 where  $\rho^{(2)} = p^{(2)} - \rho^{(0)} \theta^{(2)}$  via (B4), which further simplifies it. This now gives a clear link

606 between the barotropic equation for the mean flow and the angular momentum.

## 607 APPENDIX D

### 608 The Non-acceleration Theorem

609 This Appendix shows the derivation of the non-acceleration theorem for the given asymptotic set

610 of equations. To derive this, a Transformed Eulerian Mean (TEM) (Andrews and McIntyre 1976;

611 Edmon et al. 1980) version of the zonal mean (averaged over  $\lambda_p, \lambda_s$ , denoted by  $[\cdot]$ ) momentum

612 and thermodynamic equations is necessary. From the zonal mean continuity ( $\mathcal{O}(\varepsilon^4, \varepsilon^5)$ ), thermo-  
613 dynamic ( $\mathcal{O}(\varepsilon^6, \varepsilon^7)$ ) and momentum equations ( $\mathcal{O}(\varepsilon^3, \varepsilon^4, \varepsilon^5)$ ) at different asymptotic orders, we  
614 can identify the residual meridional circulation ( $v_r^{(i)}, w_r^{(i)}$  with subscript  $r$  representing residual  
615 velocity and  $i$  represents its order)

$$[\rho^{(0)}v_r^{(2)}] = [\rho^{(0)}v^{(2)}] - \frac{\partial}{\partial z} \left[ \frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \right] \quad (\text{D1})$$

$$[\rho^{(0)}w_r^{(4)}] = [\rho^{(0)}w^{(4)}] + \frac{\partial}{\partial \tilde{y}_s} \left[ \frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \right] = [\rho^{(0)}w^{(4)}] \quad (\text{D2})$$

$$[\rho^{(0)}v_r^{(3)}] = [\rho^{(0)}v^{(3)}] - \frac{\partial}{\partial z} \left[ \frac{v_s^{(1)}\theta^{(4)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \right] \quad (\text{D3})$$

$$[\rho^{(0)}w_r^{(5)}] = [\rho^{(0)}w^{(5)}] + \frac{\partial}{\partial \tilde{y}_p} \left[ \frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \right] + \frac{\partial}{\partial \tilde{y}_s} \left[ \frac{v_s^{(1)}\theta^{(4)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \right], \quad (\text{D4})$$

616 which satisfy continuity equations at different orders.

617 Using the residual velocities (D1-D4), the zonal mean momentum equations at  $\mathcal{O}(\varepsilon^3, \varepsilon^4)$  (A6,  
618 A7) become

$$\frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_s} - f[\rho^{(0)}v_r^{(2)}] = [\rho^{(0)}S_u^{(3)}] + \frac{\partial}{\partial z} \left[ \frac{v_p^{(1)}\theta^{(3)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \right], \quad (\text{D5})$$

$$\begin{aligned} & \frac{\partial[\rho^{(0)}u^{(2)}]}{\partial t_s} + \frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_p} + [\rho^{(0)}w_r^{(4)}] \frac{\partial u^{(0)}}{\partial z} - f[\rho^{(0)}v_r^{(3)}] \\ & = [\rho^{(0)}S_u^{(4)}] - \frac{\partial}{\partial \tilde{y}_s} [\rho^{(0)}u_s^{(1)}v_s^{(1)}] + \frac{\partial}{\partial z} \left[ \frac{v_s^{(1)}\theta^{(4)}\rho^{(0)}}{\partial\theta^{(2)}/\partial z} \right], \end{aligned} \quad (\text{D6})$$

619 which can both be linked to the zonal mean wave activity equations on planetary (13) and synoptic  
620 (12) scales, respectively, through their respective zonal mean EP flux divergences ( $[\nabla_p^{3D} \cdot \mathbf{F}_p]$ ,  
621  $[\nabla_s^{3D} \cdot \mathbf{F}_s]$ ) that appear on the right-hand-side of (D5, D6). Thus, (D5, D6) can be rewritten in  
622 terms of wave activities as

$$\frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_s} + \frac{\partial[\mathcal{A}_p]}{\partial t_p} = f[\rho^{(0)}v_r^{(2)}] + [\rho^{(0)}S_u^{(3)}] + [S_p^{wa}], \quad (\text{D7})$$

$$\frac{\partial[\rho^{(0)}u^{(2)}]}{\partial t_s} + \frac{\partial[\rho^{(0)}u^{(1)}]}{\partial t_p} + \frac{\partial[\mathcal{A}_s]}{\partial t_s} = f[\rho^{(0)}v_r^{(3)}] - [\rho^{(0)}w_r^{(4)}] \frac{\partial u^{(0)}}{\partial z} + [\rho^{(0)}S_u^{(4)}] + [S_s^{wa}], \quad (\text{D8})$$



623 which, under synoptic scale averaging ( $\phi_s, t_s$ ), for steady eddies (wave activity tendencies vanish),  
 624 and in the absence of source-sink terms, satisfy the non-acceleration theorem, i.e. the tendencies  
 625 of the zonal mean velocities vanish. These equations also show that planetary wave activity af-  
 626 fects the zonal mean flow evolution on synoptic timescales, and that the synoptic wave activity  
 627 (linked to synoptic heat and momentum fluxes) affects the zonal mean flow evolution on plane-  
 628 tary timescales. However, the latter relationship vanishes under synoptic scale averaging, leaving  
 629 only the residual circulation terms and source-sink terms affecting the evolution of  $u_p^{(1)}$  in (D8).  
 630 This means that an evolution equation for  $p^{(3)}$  (related to  $u_p^{(1)}$ ), which can be derived in a similar  
 631 manner as the barotropic equation (evolution equation for  $p^{(2)}$ ) using  $\mathcal{O}(\varepsilon^4)$  u-momentum equa-  
 632 tion,  $\mathcal{O}(\varepsilon^6)$  thermodynamic equation,  $\mathcal{O}(\varepsilon^6)$  continuity equation, and hydrostatic balance for  $p^{(3)}$   
 633 averaged over synoptic scales and vertically, is only affected by the source-sink terms

$$\left( \frac{\partial}{\partial \tilde{y}_p} \frac{1}{f} \frac{\partial}{\partial y_p} - \frac{\beta}{f^2} \frac{\partial}{\partial y_p} - f \right) \frac{\overline{\partial p^{(3)} \lambda_{p,s,z}}}{\partial t_p} = - \overline{f \rho^{(0)} S_\theta^{(6)} \lambda_{p,s,z}} - \left( \frac{\partial}{\partial \tilde{y}_p} - \frac{\beta}{f} \right) \left( \overline{\rho^{(0)} S_u^{(4)} \lambda_{p,s,z}} \right). \quad (\text{D9})$$

634 This evolution equation suggests that a higher order momentum equation is needed to find the  
 635 dynamic influences on the mean flow on planetary spatial scales (averaged over synoptic scales)  
 636 and longer time scales ( $t_m$ ) - see barotropic pressure equation (16).

637 Note that (D7,D8) provide equations for zonal mean flow variations on shorter timescales (syn-  
 638 optic and planetary), which have dynamical importance for higher frequency atmospheric flow  
 639 (e.g. baroclinic life cycles or barotropic annular modes with timescales of 10 days or less). Upon  
 640 averaging over these scales, the slower variations in the mean flow ( $t_m$ ) emerge (as in the barotropic  
 641 equation for the mean flow).

642 The TEM version of the  $\mathcal{O}(\varepsilon^5)$  zonal momentum equation can also be derived using the same  
 643 residual velocities (with the same procedure), however, here we only show an equation averaged  
 644 over  $t_s, t_p, \lambda_s, \lambda_p, \phi_s, z$  as this was the averaging performed to derive the barotropic equation for the

645 mean flow (17). This yields

$$\begin{aligned} \frac{\overline{\partial \rho^{(0)} u^{(0)} }^{p,s,z}}{\partial t_m} + \overline{\rho^{(0)} v_r^{(2)} \frac{\partial u^{(0)}}{\partial \tilde{y}_p}}^{p,s,z} + \overline{\rho^{(0)} w_r^{(5)} \frac{\partial u^{(0)}}{\partial z}}^{p,s,z} + \overline{\rho^{(0)} w_r^{(4)} }^{p,s,z} \cos \phi_p \\ - \overline{f \rho^{(0)} v^{(4)} }^{p,s,z} - \overline{f \rho^{(3)} v_p^{(1)} }^{p,s,z} = \overline{\rho^{(0)} S_u^{(5)} }^{p,s,z} + \frac{\partial \overline{F^y}^{p,s,z}}{\partial \tilde{y}_p} \end{aligned} \quad (\text{D10})$$

646 with

$$F^y = -\overline{\rho^{(0)} u^{(1)} v^{(1)} \cos \phi_p} + \frac{\partial u^{(0)}}{\partial z} \frac{v_p^{(1)} \theta^{(3)} \rho^{(0)}}{\partial \theta^{(2)} / \partial z} \quad (\text{D11})$$

647 where  $a^{-1} \tan \phi_p \overline{\rho^{(0)} u^{(1)} v^{(1)}}$  was absorbed into  $F^y$  through  $\cos \phi_p$ . As in section 4b, many  
 648 terms in (D10) can be related to source-sink terms,  $v^{(4)}$  can be eliminated via the continuity and  
 649 thermodynamic equations, and  $f \rho^{(3)} v_p^{(1)}$  is related to meridional heat flux on planetary scales. To  
 650 link (D10) to the wave activity tendency, a higher order wave activity approximation would be  
 651 needed, and due to the planetary scale heat fluxes in (D10), also a boundary wave activity may  
 652 be needed, which are not the subject of this paper (only the leading order approximations are of  
 653 interest). Hence a non-acceleration theorem for this order of the momentum equation is yet to be  
 654 determined, but is expected to hold as is the case at lower orders.

655 The  $\mathcal{O}(\varepsilon^7)$  thermodynamic equation within the TEM framework (under a  $t_s, t_p, \lambda_s, \lambda_p, \phi_s$  aver-  
 656 age) is

$$\frac{\overline{\partial \rho^{(0)} \theta^{(2)} }^{s,p}}{\partial t_m} + \overline{\rho^{(0)} v_r^{(2)} }^{s,p} \frac{\partial \theta^{(2)}}{\partial y_p} + \overline{\rho^{(0)} w_r^{(5)} }^{s,p} \frac{\partial \theta^{(2)}}{\partial z} = \overline{\rho^{(0)} S_\theta^{(7)} }^{s,p} - \frac{\partial}{\partial z} \left( \frac{\overline{S_\theta^{(6)} \theta^{(3)} \rho^{(0)} }^{s,p}}{\partial \theta^{(2)} / \partial z} \right), \quad (\text{D12})$$

657 which completes the TEM version of the equations. Note that the  $\mathcal{O}(\varepsilon^6)$  thermodynamic equation  
 658 remains unchanged within the TEM framework and is hence not repeated here.

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724 **LIST OF TABLES**

725 **Table 1.** The assumptions for the scale separations between planetary ( $p$ ), synoptic ( $s$ )  
726 and zonal mean flow ( $m$ ). . . . . 47

727 TABLE 1. The assumptions for the scale separations between planetary ( $p$ ), synoptic ( $s$ ) and zonal mean flow  
 728 ( $m$ ).

	longitude	latitude	height	time
planetary	$\lambda_p = \lambda$	$\phi_p = \phi$	$z_p = z_s = z$	$t_p = \varepsilon^3 t$
synoptic	$\lambda_s = \varepsilon^{-1} \lambda_p$	$\phi_s = \varepsilon^{-1} \phi_p$	$z_p = z_s = z$	$t_s = \varepsilon^2 t = \varepsilon^{-1} t_p$
mean		$\phi_m = \phi_p$	$z_m = z_p = z$	$t_m = \varepsilon^5 t = \varepsilon^2 t_p$