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Arithmetic of Rational Points and Zero-cycles on Products of Kummer Varieties and K3 Surfaces

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Let k be a number field. In the spirit of a result by Yongqi Liang, we relate the arithmetic of rational points over finite extensions of k to that of zero-cycles over k for Kummer varieties over k. For example, for any Kummer variety X over k, we show that if the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points on X over all finite extensions of k, then the (2-primary) Brauer–Manin obstruction is the only obstruction to the Hasse principle for zero-cycles of any given odd degree on X over k. We also obtain similar results for products of Kummer varieties, K3 surfaces, and rationally connected varieties.

1 Introduction

Let X be a smooth, proper, geometrically integral variety over a number field k with a fixed choice of algebraic closure $\overline{k}$, let $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and let $\operatorname{Br} X := \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$ be the Brauer group of X. In [15], Manin described a pairing $\langle\ ,\ \rangle_{\mathrm{BM}} : X(\mathbf{A}_k) \times \operatorname{Br} X \to \mathbf{Q}/\mathbf{Z}$ such that the set of rational points $X(k)$ is contained in the subset $X(\mathbf{A}_k)^{\operatorname{Br} X}$ of adelic points that are orthogonal to $\operatorname{Br} X$ under the pairing. It can (and sometimes does) happen that $X(\mathbf{A}_k)^{\operatorname{Br} X} = \emptyset$ while $X(\mathbf{A}_k) \neq \emptyset$, yielding an obstruction to the Hasse principle on X.

This obstruction is known as the Brauer–Manin obstruction (for more details, we refer the reader to, e.g., [24]). Variations of the Brauer–Manin obstruction can be obtained by considering subsets $B \subset \mathrm{Br}\, X$ instead of the full Brauer group. For example, one can use the algebraic Brauer group $\mathrm{Br}_1 X := \ker(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$ or the d-primary torsion subgroup $\mathrm{Br}\, X\{d\}$. Although the Brauer–Manin obstruction may not always suffice to explain the failure of the Hasse principle (see, e.g., [23] and [21]), it is nonetheless known to be sufficient for some classes of varieties and is conjectured to be sufficient for geometrically rationally connected varieties (see [2]) and K3 surfaces (see [25]). For a survey of recent progress and open problems in the field of rational points and obstructions to their existence, we refer the reader to [32].

In general, it is interesting to also consider 0-cycles on varieties rather than just rational points. Recall that a 0-cycle on X is a formal sum of closed points of X, $z = \sum n_i P_i$, with degree $\deg(z) = \sum n_i[k(P_i) : k]$. A rational point of X over k is thus a 0-cycle of degree 1. Conjecturally, the Brauer–Manin obstruction to the Hasse principle for 0-cycles of degree 1 is the only one for smooth, proper, geometrically integral varieties over k (see [13] and [1]). In this paper, we focus on the relationship between the arithmetic of the rational points of a variety X over extension fields and the arithmetic of 0-cycles on X. We consider the set $\overline{\mathscr{K}}_{k,g}$ of smooth, projective, geometrically integral varieties over k of dimension $g \geq 2$ that become Kummer varieties upon base change to $\overline{k}$ and its subset $\mathscr{K}_{k,g} \subset \overline{\mathscr{K}}_{k,g}$ of Kummer varieties over k. We let $\overline{\mathscr{K}}_k := \cup_{g \geq 2} \overline{\mathscr{K}}_{k,g}$ and $\mathscr{K}_k := \cup_{g \geq 2} \mathscr{K}_{k,g}$.

Theorem 1.1.   Let k be a number field. Let $W := \prod_{i=1}^{r} X_i$, where $X_i \in \mathscr{K}_k$ for all $i \in \{1, \ldots, r\}$, and let δ be an odd integer. Suppose that, for all $i \in \{1, \ldots, r\}$ and all finite extensions l/k, the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points on X_i over l. Then the 2-primary Brauer–Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree δ on W.

In [25], Skorobogatov conjectured that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points on K3 surfaces over number fields. This leads to the following conditional result for Kummer surfaces.

Corollary 1.2.   Let δ be an odd integer. Then Skorobogatov's conjecture that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points on K3 surfaces over a number field k implies that the 2-primary Brauer–Manin obstruction

to the Hasse principle is the only one for 0-cycles of degree δ on products of Kummer surfaces over k.

Remark 1.3.   Note that we use the phrase "Kummer variety" in a slightly more general way than it is used in some texts. See Section 2 for the definition of a Kummer variety; a Kummer surface is a Kummer variety of dimension 2. Note that Kummer surfaces are K3 surfaces (see [29, p. 2]).

This paper was inspired by Liang's work in [14, Thm. 3.2.1], in which he takes X to be a geometrically rationally connected variety over a number field k. He shows that if the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points on X over every finite extension of k, then the Brauer–Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree 1 on X over k. See also [14, Prop. 3.4.1] for a result about 0-cycles of any degree. Liang also proved the analogue for weak approximation, in the sense defined in Definition 6.3. Our adaptation of Liang's strategy to the case of (geometric) Kummer varieties also yields an analogous result for this notion of weak approximation:

Theorem 1.4.   Let $W := \prod_{i=1}^{r} X_i$, where $X_i \in \overline{\mathscr{K}}_k$ for all $i \in \{1, \ldots, r\}$, and let $d \in \mathbf{Z}_{>0}$ and $\delta \in \mathbf{Z}$ with $\gcd(d, \delta) = 1$. Suppose that, for all $i \in \{1, \ldots, r\}$ and all finite extensions l/k, the d-primary Brauer–Manin obstruction is the only obstruction to weak approximation for rational points on X_i over l. Then the d-primary Brauer–Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree δ on W.

Our key observation, which allows us to employ Liang's method, is that while it may not be possible to control the growth of the whole of $\mathrm{Br}\, X / \mathrm{Br}_0 X$ under an extension of the base field, in the case of a variety $X \in \overline{\mathscr{K}}_k$ the d-primary part of $\mathrm{Br}\, X / \mathrm{Br}_0 X$ is unchanged so long as the base field extension has degree coprime to d and is linearly disjoint from a certain field extension depending on X. Control of the growth of the algebraic part $\mathrm{Br}_1 X / \mathrm{Br}_0 X$ of the Brauer group is easily achieved. We remark that, since the geometric Brauer group of a geometrically rationally connected variety is finite, the growth of the whole of $\mathrm{Br}\, X / \mathrm{Br}_0 X$ under extensions of the base field is easily controlled in Liang's paper. In the case of Kummer varieties, however, $\mathrm{Br}\, \overline{X}$ is infinite. This makes controlling the growth of the transcendental part of the Brauer group under extensions of the base field much more difficult. In order to achieve this, we use the close relationship between the Brauer group of a Kummer variety and the Brauer

group of the underlying abelian variety. The exploitation of the rich structure of abelian varieties is a common theme underlying much of the recent rapid progress in the study of rational points on the related K3 surfaces and Kummer varieties (see, e.g., [5], [7], [8], [11], [12], [18], [27], [29] and [30]). Our work in Section 4 on the transcendental part of the Brauer group of geometrically abelian varieties and geometrically Kummer varieties may be of independent interest.

After obtaining our results for Kummer varieties, we were made aware of a recent preprint of Ieronymou [10] in which he uses Liang's approach together with work of Orr and Skorobogatov in [20] to prove analogues of Liang's results in [14] for K3 surfaces. Since Kummer surfaces are both K3 surfaces and Kummer varieties of dimension 2, there is an overlap between our work and that of Ieronymou (note that Ieronymou's results do not require that the degree of the 0-cycle is odd). We remark, however, that our results are not implied by those of Ieronymou since we work with Kummer varieties of arbitrary dimension. Combining our results for Kummer varieties with Ieronymou's results for K3 surfaces and Liang's work on rationally connected varieties gives Theorem 1.5 below, which is a generalisation of Theorem 1.4 for products of these types of variety. Denote by $\overline{\mathscr{R}}_k$ the set of smooth, projective, geometrically integral, geometrically rationally connected varieties over k. Denote by $\mathsf{K3}_k$ the set of K3 surfaces over k.

Theorem 1.5.   Let k be a number field. Let $W := \prod_{i=1}^{r} X_i$, where $X_i \in \overline{\mathscr{R}}_k \cup \mathsf{K3}_k \cup \overline{\mathscr{K}}_k$ for all $i \in \{1, \dots, r\}$. Let $d \in \mathbf{Z}_{>0}$, let $\delta \in \mathbf{Z}$, and assume that $\gcd(d, \delta) = 1$ if at least one of the X_i is in $\overline{\mathscr{K}}_{k,g}$ with $g \geq 3$. For $X_i \in \overline{\mathscr{R}}_k \cup \mathsf{K3}_k$, let $a_i = \#(\mathrm{Br}\, X_i / \mathrm{Br}_0 X_i)$ and let c be the least common multiple of all the a_i and d. Suppose that, for all finite extensions l/k,

1. if $X_i \in \overline{\mathscr{R}}_k \cup \mathsf{K3}_k$ then the Brauer–Manin obstruction is the only obstruction to the Hasse principle (respectively, to weak approximation) for rational points on X_i over l;
2. if $X_i \in \overline{\mathscr{K}}_k$ then the d-primary Brauer–Manin obstruction is the only obstruction to the Hasse principle (respectively, to weak approximation) for rational points on X_i over l.

Then the c-primary Brauer–Manin obstruction is the only obstruction to the Hasse principle (respectively, to weak approximation) for 0-cycles of degree δ on W.

All our results thus far prove sufficiency of the Brauer–Manin obstruction for 0-cycles, asssuming the sufficiency of the Brauer–Manin obstruction for rational points over finite extensions of the base field. This leads one to ask in what circumstances

it is enough to simply assume sufficiency of the Brauer–Manin obstruction to rational points over the base field. Our final results give some insight into when this may be possible.

Theorem 1.6.   Let k be a number field and let X be a smooth, projective, geometrically integral variety over k. Suppose that $X(\mathbf{A}_k) \neq \emptyset$ and let $[\alpha] \in (\operatorname{Br} X / \operatorname{Br}_0 X)[2]$. Then there exist uncountably many finite extensions l/k such that

$$X(\mathbf{A}_k)^\alpha = \emptyset \implies X(\mathbf{A}_l)^{\operatorname{Res}_{l/k}\alpha} = \emptyset.$$

We have the following corollary for Kummer varieties:

Corollary 1.7.   Let X be a Kummer variety over a number field k. Suppose that $(\operatorname{Br} X / \operatorname{Br}_0 X)\{2\} \cong \mathbf{Z}/2\mathbf{Z}$ and that $X(\mathbf{A}_k) \neq \emptyset$. Then there exist uncountably many finite extensions l/k such that

$$X(\mathbf{A}_k)^{\operatorname{Br} X} = \emptyset \implies X(\mathbf{A}_l)^{\operatorname{Br} X_l} = \emptyset.$$

In particular, if the Brauer–Manin obstruction to the Hasse principle is the only one for k-rational points, then it is also the only one for l-rational points, for all field extensions l/k as above.

1.1   Structure of the paper

In Section 2 we define the families of varieties to be studied in this paper. In Section 3 we state some preliminary results concerning Brauer groups and Brauer–Manin sets of products of varieties. In Section 4 we show that the d-primary part of the transcendental Brauer group of a geometrically abelian or Kummer variety remains unchanged when the base field is extended, given certain conditions on the extension. In Section 5 we use the results of Section 4 to show that for Kummer varieties, the d-primary part of the quotient $\operatorname{Br} X / \operatorname{Br}_0 X$ remains unchanged when the base field is extended, given certain conditions on the extension. In Section 6 we gather various definitions concerning Brauer–Manin obstructions to local–global principles for 0-cycles. In Section 7 we prove our main result, which is Theorem 1.5, and deduce from it various consequences including Theorem 1.1. Section 8 contains the proofs of Theorem 1.6 and Corollary 1.7.

1.2   Notation and terminology

Throughout this paper, we use the following notation:

$$
\begin{aligned}
k \quad & \text{is a field of characteristic 0,}\\
\overline{k} \quad & \text{is a fixed algebraic closure of } k,\\
\Gamma_k \quad & \text{denotes the absolute Galois group } \mathrm{Gal}(\overline{k}/k),\\
\Omega_k \quad & \text{is the set of all nontrivial places of } k, \text{ when } k \text{ is a number field,}\\
X \quad & \text{is a variety over } k,\\
X_l \quad & \text{is the base-change } X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,l \text{ of } X \text{ to a field extension } l/k,\\
\overline{X} \quad & \text{denotes } X_{\overline{k}},\\
\mathrm{Br}_1 X \quad & \text{denotes } \ker(\mathrm{Br}\,X \to \mathrm{Br}\,\overline{X}),\\
\mathrm{Br}_0 X \quad & \text{denotes } \mathrm{im}(\mathrm{Br}\,k \to \mathrm{Br}\,X),\\
\mathrm{Br}_1 X/\mathrm{Br}_0 X \quad & \text{is the \textit{algebraic} part of the Brauer group of } X,\\
\mathrm{Br}\,X/\mathrm{Br}_1 X \quad & \text{is the \textit{transcendental} part of the Brauer group of } X.
\end{aligned}
$$

For an abelian group A and an integer $d \in \mathbf{Z}_{>0}$, we use the following notation:

$$
\begin{aligned}
A[d] \quad & \text{denotes the } d\text{-torsion subgroup of } A,\\
A\{d\} \quad & \text{denotes the } d\text{-primary part } \varinjlim_{n} A[d^n] \text{ of } A.
\end{aligned}
$$

2   Some Sets of Varieties

Let k be a field of characteristic 0. We define here our main objects of study.

Definition 2.1.   Denote by $\overline{\mathscr{R}}_k$ the set of smooth, projective, geometrically integral, geometrically rationally connected varieties over k. Denote by $\mathsf{K3}_k$ the set of K3 surfaces over k. (Note that being a K3 surface is a geometric property.)

Definition 2.2 ([29, Defn. 2.1]).   Let A be an abelian variety over k of dimension ≥ 2. Any k-torsor T under $A[2]$ gives rise to a 2-covering $\rho : V \to A$, where V is the quotient of $A \times_k T$ by the diagonal action of $A[2]$ and ρ comes from the projection onto the 1st factor. Then $T = \rho^{-1}(O_A)$ and V has the structure of a k-torsor under A. The class of T maps to the class of V under the map $\mathrm{H}^1_{\text{ét}}(k, A[2]) \to \mathrm{H}^1_{\text{ét}}(k, A)$ induced by the inclusion of group schemes $A[2] \to A$ and, in particular, the period of V divides 2. Let $\sigma : \tilde{V} \to V$ be the blow-up of V at $T \subset V$. The involution $[-1] : A \to A$ fixes $A[2]$ and induces involutions ι on V and $\tilde{\iota}$ on $\tilde{V}$ whose fixed point sets are T and the exceptional divisor, respectively. We call $\mathrm{Kum}\,V := \tilde{V}/\tilde{\iota}$ the Kummer variety associated to V (or T). We remark that the quotient

Kum V is geometrically isomorphic to Kum A, so in particular it is smooth. A Kummer variety of dimension 2 is called a Kummer surface.

Definition 2.3.   We denote by $\overline{\mathscr{K}}_{k,g}$ the set of smooth, projective, geometrically integral varieties X over k of dimension g such that $\overline{X}$ is a Kummer variety over $\overline{k}$, and by $\mathscr{K}_{k,g}$ the subset of Kummer varieties over k of dimension g. We let $\overline{\mathscr{K}}_k = \cup_{g \geq 2} \overline{\mathscr{K}}_{k,g}$ and, likewise, $\mathscr{K}_k = \cup_{g \geq 2} \mathscr{K}_{k,g}$.

Remark 2.4.   By [19, Thm. 3], the set $\overline{\mathscr{K}}_{k,2}$ consists precisely of the varieties X in $\mathsf{K3}_k$ such that $\mathrm{NS}\,\overline{X}$ contains a primitive sublattice isomorphic to the Kummer lattice. Moreover, by [30, Prop. 2.1], there exists some absolute $M \in \mathbf{Z}_{\geq 1}$ (not depending on k) such that for each $X \in \overline{\mathscr{K}}_{k,2}$, there is an extension k_0/k of degree at most M for which $X_{k_0} \in \mathscr{K}_{k_0,2}$. In fact, the proof of [30, Prop. 2.1] shows that one can take $M = 2 \cdot \# \,\mathrm{GL}_{20}(\mathbf{F}_3)$.

We will use properties of the underlying abelian varieties to deduce results for Kummer varieties. Therefore, it is convenient to define the following set.

Definition 2.5.   Denote by $\overline{\mathscr{A}}_{k,g}$ the set of smooth, projective, geometrically integral varieties X over k of dimension g such that $\overline{X}$ is an abelian variety over $\overline{k}$. Let $\overline{\mathscr{A}}_k = \cup_{g \geq 1} \overline{\mathscr{A}}_{k,g}$.

3   Preliminaries on Brauer Groups of Products of Varieties

In order to prove our main results, we need a finiteness result for the Brauer groups of products of varieties.

Proposition 3.1.   Let k be a number field and let X and Y be smooth, projective, geometrically integral varieties over k. If $\mathrm{Br}\,\overline{X}^{\Gamma_k}$, $\mathrm{Br}_1 X / \mathrm{Br}_0 X$, $\mathrm{Br}\,\overline{Y}^{\Gamma_k}$, and $\mathrm{Br}_1 Y / \mathrm{Br}_0 Y$ are all finite, then so are $\mathrm{Br}(\overline{X} \times \overline{Y})^{\Gamma_k}$, $\mathrm{Br}_1(X \times Y) / \mathrm{Br}_0(X \times Y)$, and, consequently, $\mathrm{Br}(X \times Y) / \mathrm{Br}_0(X \times Y)$.

Proof.   By [28, Cor. 3.2(i)], $\mathrm{Br}(\overline{X} \times \overline{Y})^{\Gamma_k}$ is finite since both $\mathrm{Br}\,\overline{X}^{\Gamma_k}$ and $\mathrm{Br}\,\overline{Y}^{\Gamma_k}$ are (by hypothesis). Using the natural injection

$$\mathrm{Br}(X \times Y) / \mathrm{Br}_1(X \times Y) \hookrightarrow \mathrm{Br}(\overline{X} \times \overline{Y})^{\Gamma_k},$$

it follows that $\mathrm{Br}(X \times Y) / \mathrm{Br}_1(X \times Y)$ is finite. Moreover, by [28, Remark 1.9], the cokernel of the natural map

$$\mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}\,\overline{X}) \oplus \mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}\,\overline{Y}) \to \mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}(\overline{X} \times \overline{Y}))$$

is finite. Since k is a number field, the Hochschild–Serre spectral sequence

$$\mathrm{H}^p_{\mathrm{cts}}(\mathrm{Gal}(\overline{k}/k), \mathrm{H}^q_{\mathrm{\acute{e}t}}(\overline{X}, \mathbf{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{\acute{e}t}}(X, \mathbf{G}_m)$$

yields $\mathrm{H}^1_{\mathrm{\acute{e}t}}(k, \mathrm{Pic}\,\overline{X}) = \mathrm{Br}_1 X / \mathrm{Br}_0 X$, and similarly $\mathrm{H}^1_{\mathrm{\acute{e}t}}(k, \mathrm{Pic}\,\overline{Y}) = \mathrm{Br}_1 Y / \mathrm{Br}_0 Y$. By hypothesis, $\mathrm{Br}_1 X / \mathrm{Br}_0 X$ and $\mathrm{Br}_1 Y / \mathrm{Br}_0 Y$ are finite; hence, $\mathrm{H}^1_{\mathrm{\acute{e}t}}(k, \mathrm{Pic}(\overline{X} \times \overline{Y})) = \mathrm{Br}_1(X \times Y) / \mathrm{Br}_0(X \times Y)$ is also finite. Putting everything together, it follows that $\mathrm{Br}(X \times Y) / \mathrm{Br}_0(X \times Y)$ is finite, as required. ∎

Remark 3.2.   By induction on n, Proposition 3.1 holds for $\prod_{i=1}^{n} X_i$, where the X_i are smooth, projective, geometrically integral varieties over k.

Remark 3.3.   Let l/k be an extension of number fields. By [29, Cor. 2.8] and [26, Thm. 1.2], respectively, if $X \in \overline{\mathscr{K}}_k$ or $X \in \mathsf{K3}_k$, then both $\mathrm{Br}\,\overline{X}^{\Gamma_l}$ and $\mathrm{Br}_1 X_l / \mathrm{Br}_0 X_l$ are finite. If X/k is geometrically rationally connected, then $\mathrm{Br}\,\overline{X}$ is finite and the finiteness of $\mathrm{Br}\,X_l / \mathrm{Br}_0 X_l$ is an easy consequence of the Hochschild–Serre spectral sequence—see [14, Prop. 3.1.1], for example.

Lemma 3.4.   Let k be a number field and let X and Y be smooth, projective, geometrically integral varieties over k. Let l/k be a finite extension. Then the identification $(X \times Y)(\mathbf{A}_l) = X(\mathbf{A}_l) \times Y(\mathbf{A}_l)$ induces natural inclusions

$$(X \times Y)(\mathbf{A}_l)^{\mathrm{Br}(X \times Y)} \subset X(\mathbf{A}_l)^{\mathrm{Br}\,X} \times Y(\mathbf{A}_l)^{\mathrm{Br}\,Y}$$

and

$$(X \times Y)(\mathbf{A}_l)^{\mathrm{Br}(X \times Y)\{d\}} \subset X(\mathbf{A}_l)^{\mathrm{Br}\,X\{d\}} \times Y(\mathbf{A}_l)^{\mathrm{Br}\,Y\{d\}}$$

for all $d \in \mathbf{Z}_{>0}$.

Proof.   Note that both the assignments $V \mapsto V(\mathbf{A}_l)^{\mathrm{Br}\,V\{d\}}$ and $V \mapsto V(\mathbf{A}_l)^{\mathrm{Br}\,V}$ from the category of varieties over k to the category of sets are functorial. Hence, the result follows immediately by functoriality applied to the projections $X \times Y \to X$ and $X \times Y \to Y$. ∎

Remark 3.5.   By induction on n, Lemma 3.4 holds for $\prod_{i=1}^{n} X_i$, where the X_i are smooth, projective, geometrically integral varieties over k.

Remark 3.6.   Note that in Lemma 3.4 when we write $X(\mathbf{A}_l)^{\mathrm{Br}\,X}$ we are implicitly using the restriction map $\mathrm{Res}_{l/k} : \mathrm{Br}\,X \to \mathrm{Br}\,X_l$ to obtain a Brauer–Manin pairing $X(\mathbf{A}_l) \times \mathrm{Br}\,X \to \mathbf{Q}/\mathbf{Z}$. The image of $\mathrm{Res}_{l/k}$ may not be equal to $\mathrm{Br}\,X_l$, so Lemma 3.4 does not follow from [28, Thm. C].

4   Controlling the Transcendental Brauer Groups of Geometrically Abelian Varieties and Geometrically Kummer Varieties Over Field Extensions

Let k be a field of characteristic 0. In this section we show that the d-primary part of the transcendental Brauer group of a geometrically abelian or Kummer variety does not grow when the base field is extended, so long as the field extension satisfies certain conditions.

Proposition 4.1.   Let $d, n \in \mathbf{Z}_{>0}$. Let X be a smooth, proper, geometrically integral variety over k. Let l/k be a finite extension with degree coprime to d such that $\mathrm{Br}\,\overline{X}[d^n]^{\Gamma_l} = \mathrm{Br}\,\overline{X}[d^n]^{\Gamma_k}$. Then the restriction map gives a canonical isomorphism of abelian groups

$$\mathrm{Res}_{l/k} : (\mathrm{Br}\,X)[d^n]/(\mathrm{Br}_1\,X)[d^n] \xrightarrow{\sim} (\mathrm{Br}\,X_l)[d^n]/(\mathrm{Br}_1\,X_l)[d^n].$$

Proof.   *Step 1:* We will show that the corestriction map induces an injection

$$\mathrm{Cor}_{l/k} : (\mathrm{Br}\,X_l)[d^n]/(\mathrm{Br}_1\,X_l)[d^n] \hookrightarrow (\mathrm{Br}\,X)[d^n]/(\mathrm{Br}_1\,X)[d^n].$$

Let $B \in (\mathrm{Br}\,X_l)[d^n]$ and suppose that $\mathrm{Cor}_{l/k}\,B \in \mathrm{Br}_1\,X$. By [3, Lemma 1.4],

$$\mathrm{Res}_{\overline{k}/k} \circ \mathrm{Cor}_{l/k}\,B = \sum_{\sigma \in \Gamma_k/\Gamma_l} \sigma(\mathrm{Res}_{\overline{k}/l}\,B), \tag{4.1}$$

where the sum is over a set of coset representatives of Γ_l in Γ_k. Since $\mathrm{Cor}_{l/k}\,B \in \mathrm{Br}_1\,X$, the left-hand side equals zero. Moreover, $\mathrm{Res}_{\overline{k}/l}\,B \in \mathrm{Br}\,\overline{X}[d^n]^{\Gamma_l}$. By our hypothesis, this implies that $\mathrm{Res}_{\overline{k}/l}\,B \in \mathrm{Br}\,\overline{X}[d^n]^{\Gamma_k}$. Therefore, (4.1) becomes

$$0 = [l : k]\,\mathrm{Res}_{\overline{k}/l}\,B. \tag{4.2}$$

Now recall that $[l : k]$ is coprime to d. Therefore, (4.2) shows that $\mathrm{Res}_{\overline{k}/l}\,B = 0$, since $B \in (\mathrm{Br}\,X_l)[d^n]$. Hence, $B \in (\mathrm{Br}_1\,X_l)[d^n]$, as required.

Step 2: Since $\mathrm{Cor}_{l/k} \circ \mathrm{Res}_{l/k} = [l : k]$ and $[l : k]$ is coprime to d, the corestriction map

$$\mathrm{Cor}_{l/k} : (\mathrm{Br}\,X_l)[d^n]/(\mathrm{Br}_1\,X_l)[d^n] \to (\mathrm{Br}\,X)[d^n]/(\mathrm{Br}_1\,X)[d^n]$$

is surjective. Step 1 shows that it is an isomorphism and consequently the restriction map

$$\mathrm{Res}_{l/k} : (\mathrm{Br}\,X)[d^n]/(\mathrm{Br}_1\,X)[d^n] \to (\mathrm{Br}\,X_l)[d^n]/(\mathrm{Br}_1\,X_l)[d^n]$$

is also an isomorphism.   ∎

Lemma 4.2.   Let $d \in \mathbf{Z}_{>0}$. Let $X \in \overline{\mathscr{A}}_k \cup \overline{\mathscr{K}}_k$. Let l/k be a finite extension of degree coprime to d such that the image of Γ_l in $\mathrm{Aut}(\mathrm{Br}\,\overline{X}[d])$ is the same as that of Γ_k. Then for all $n \in \mathbf{Z}_{>0}$, the image of Γ_l in $\mathrm{Aut}(\mathrm{Br}\,\overline{X}[d^n])$ is the same as that of Γ_k.

Proof.   Factorise d into primes, writing $d = p_1^{e_1} \ldots p_r^{e_r}$. Then

$$\mathrm{Br}\,\overline{X}[d^n] = \mathrm{Br}\,\overline{X}[p_1^{e_1 n}] \times \cdots \times \mathrm{Br}\,\overline{X}[p_r^{e_r n}]$$

and

$$\mathrm{Aut}(\mathrm{Br}\,\overline{X}[d^n]) = \mathrm{Aut}(\mathrm{Br}\,\overline{X}[p_1^{e_1 n}]) \times \cdots \times \mathrm{Aut}(\mathrm{Br}\,\overline{X}[p_r^{e_r n}]).$$

Thus, it suffices to prove the lemma when d is a prime p. Let $g := \dim X$.

Step 1: First we show that there exists $m \in \mathbf{Z}_{\geq 0}$ such that for all $n \in \mathbf{Z}_{>0}$, $\mathrm{Br}\,\overline{X}[p^n] \cong (\mathbf{Z}/p^n\mathbf{Z})^m$ as abelian groups. We begin by proving this in the case where $\overline{X}$ is an abelian variety, denoted by $\overline{A}$.

The Kummer exact sequence for $\overline{A}$ gives the following exact sequence of Galois modules:

$$0 \to \mathrm{NS}\,\overline{A}/p^n \to \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_{p^n}) \to \mathrm{Br}\,\overline{A}[p^n] \to 0.$$

As an abelian group with no Galois action,

$$\mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_{p^n}) = \mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbf{Z}/p^n\mathbf{Z}) = \wedge^2 \mathrm{H}^1_{\text{ét}}(\overline{A}, \mathbf{Z}/p^n\mathbf{Z}).$$

It is well known that $\mathrm{H}^1_{\text{ét}}(\overline{A}, \mathbf{Z}/p^n\mathbf{Z})$ is isomorphic to $\mathrm{Hom}(\overline{A}[p^n], \mathbf{Z}/p^n\mathbf{Z})$, see [27, Lemma 2.1], for example. As an abelian group, $\overline{A}[p^n] = (\mathbf{Z}/p^n\mathbf{Z})^{2g}$ and hence

$$\wedge^2 \mathrm{Hom}(\overline{A}[p^n], \mathbf{Z}/p^n\mathbf{Z}) = \wedge^2 (\mathbf{Z}/p^n\mathbf{Z})^{2g} = (\mathbf{Z}/p^n\mathbf{Z})^{g(2g-1)}.$$

Furthermore, $\mathrm{NS}\,\overline{A}$ is finitely generated and torsion-free of rank ρ with $1 \leq \rho \leq g^2$. Therefore, as an abelian group, $\mathrm{Br}\,\overline{A}[p^n] \cong (\mathbf{Z}/p^n\mathbf{Z})^m$, where $m = g(2g-1) - \rho$.

Now if $X \in \overline{\mathscr{K}}_{k,g}$ then $\overline{X} = \mathrm{Kum}\,\overline{A}$ for some abelian variety A. Hence, by [29, Prop. 2.7], $\mathrm{Br}\,\overline{X} = \mathrm{Br}\,\overline{A}$. This completes Step 1.

Step 2: Since $\mathrm{Br}\,\overline{X}[p^n] \cong (\mathbf{Z}/p^n\mathbf{Z})^m$ as abelian groups, we have $\mathrm{Aut}(\mathrm{Br}\,\overline{X}[p^n]) \cong \mathrm{GL}_m(\mathbf{Z}/p^n\mathbf{Z})$. The inclusion $\mathrm{Br}\,\overline{X}[p] \subset \mathrm{Br}\,\overline{X}[p^n]$ induces a surjective map

$$\alpha : \mathrm{Aut}(\mathrm{Br}\,\overline{X}[p^n]) \to \mathrm{Aut}(\mathrm{Br}\,\overline{X}[p]),$$

which can be identified with the map $\mathrm{GL}_m(\mathbf{Z}/p^n\mathbf{Z}) \to \mathrm{GL}_m(\mathbf{Z}/p\mathbf{Z})$ given by reduction modulo p. Therefore, $\ker\alpha$ is a p-group.

Let $\phi : \Gamma_k \to \mathrm{Aut}(\mathrm{Br}\,\overline{X}[p^n])$ be the map giving the Galois action. The first isomorphism theorem gives

$$\#\phi(\Gamma_k) = \#(\phi(\Gamma_k) \cap \ker\alpha) \cdot \#\alpha(\phi(\Gamma_k))$$

and

$$\#\phi(\Gamma_l) = \#(\phi(\Gamma_l) \cap \ker\alpha) \cdot \#\alpha(\phi(\Gamma_l)).$$

Note that $\alpha \circ \phi : \Gamma_k \to \mathrm{Aut}(\mathrm{Br}\,\overline{X}[p])$ is the map giving the Galois action. Hence, $\alpha(\phi(\Gamma_k)) = \alpha(\phi(\Gamma_l))$ by assumption. Therefore,

$$[\phi(\Gamma_k) : \phi(\Gamma_l)] = \frac{\#(\phi(\Gamma_k))}{\#(\phi(\Gamma_l))} = \frac{\#(\phi(\Gamma_k) \cap \ker\alpha)}{\#(\phi(\Gamma_l) \cap \ker\alpha)}.$$

This is a power of p, since $\ker\alpha$ is a p-group. On the other hand, $[\phi(\Gamma_k) : \phi(\Gamma_l)]$ divides $[\Gamma_k : \Gamma_l] = [l : k]$, which is coprime to p by assumption. Therefore, $[\phi(\Gamma_k) : \phi(\Gamma_l)] = 1$, as required. ∎

Corollary 4.3.   Let $d \in \mathbf{Z}_{>0}$. Let $X \in \overline{\mathscr{A}}_k \cup \overline{\mathscr{K}}_k$. Let F/k be a finite extension such that Γ_F acts trivially on $\mathrm{Br}\,\overline{X}[d]$. Let l/k be a finite extension of degree coprime to d that is linearly disjoint from F/k. Then for all $n \in \mathbf{Z}_{>0}$,

$$\mathrm{Res}_{l/k} : (\mathrm{Br}\,X)[d^n]/(\mathrm{Br}_1\,X)[d^n] \to (\mathrm{Br}\,X_l)[d^n]/(\mathrm{Br}_1\,X_l)[d^n]$$

is an isomorphism and consequently

$$\mathrm{Res}_{l/k} : (\mathrm{Br}\,X/\mathrm{Br}_1\,X)\{d\} \to (\mathrm{Br}\,X_l/\mathrm{Br}_1\,X_l)\{d\}$$

is an isomorphism.

Proof.   Let $\phi : \Gamma_k \to \operatorname{Aut}(\operatorname{Br}\overline{X}[d])$ be the map giving the Galois action. Since l/k is linearly disjoint from F/k, we have $\Gamma_k = \Gamma_l \Gamma_F$ and hence

$$\phi(\Gamma_k) = \phi(\Gamma_l \Gamma_F) = \phi(\Gamma_l),$$

since Γ_F acts trivially on $\operatorname{Br}\overline{X}[d]$ by assumption.

Now by Lemma 4.2, the image of Γ_l in $\operatorname{Aut}(\operatorname{Br}\overline{X}[d^n])$ is the same as that of Γ_k, for all $n \in \mathbf{Z}_{>0}$. Now the 1st statement follows from Proposition 4.1. Since X is smooth, $\operatorname{Br}X$ is a torsion abelian group by [6]. Hence, the natural inclusion $\operatorname{Br}X\{d\}/\operatorname{Br}_1 X\{d\} \hookrightarrow (\operatorname{Br}X/\operatorname{Br}_1 X)\{d\}$ is an equality, and similarly for X_l. Since

$$\frac{\operatorname{Br}X\{d\}}{\operatorname{Br}_1 X\{d\}} = \frac{\varinjlim_n \operatorname{Br}X[d^n]}{\varinjlim_n \operatorname{Br}_1 X[d^n]} = \varinjlim_n \frac{\operatorname{Br}X[d^n]}{\operatorname{Br}_1 X[d^n]},$$

the 2nd statement follows from the 1st one.   ∎

Next, we give a uniform version of Corollary 4.3, which applies to the whole of $\overline{\mathscr{A}}_{k,g} \cup \overline{\mathscr{K}}_{k,g}$ and every $n \in \mathbf{Z}_{>0}$. We will need a couple of auxiliary lemmas.

Definition 4.4.   Let $d, m \in \mathbf{Z}_{>0}$ and let d have prime factorisation $d = \prod_{i=1}^r p_i^{\epsilon_i}$. We define $Q_{d,m} := \{q \in \mathbf{Z} \text{ prime} : q \mid \prod_{i=1}^r p_i(p_i^m - 1)\}$ and $N_{d,m} := \prod_{q \in Q_{d,m}} q$.

Lemma 4.5.   Let M be a finite group such that $\operatorname{Aut}M$ has exponent dividing e. Let l/k be a finite extension of degree coprime to e. Let $\phi : \Gamma_k \to \operatorname{Aut}M$ be a homomorphism making M into a Γ_k-module. Then $\phi(\Gamma_k) = \phi(\Gamma_l)$.

Proof.   We have $\phi(\Gamma_l) \leq \phi(\Gamma_k) \leq \operatorname{Aut}M$ with $[\phi(\Gamma_k) : \phi(\Gamma_l)]$ dividing $[\Gamma_k : \Gamma_l] = [l : k]$ and with $\gcd([l : k], e) = \gcd([l : k], \exp(\operatorname{Aut}M)) = \gcd([l : k], \#\operatorname{Aut}M) = 1$. Since M is finite, Lagrange's theorem gives

$$\#\operatorname{Aut}M = [\operatorname{Aut}M : \phi(\Gamma_l)] \cdot \#\phi(\Gamma_l) = [\operatorname{Aut}M : \phi(\Gamma_k)] \cdot \#\phi(\Gamma_k),$$

implying that $[\operatorname{Aut}M : \phi(\Gamma_l)] = [\operatorname{Aut}M : \phi(\Gamma_k)][\phi(\Gamma_k) : \phi(\Gamma_l)]$. Hence, $[\phi(\Gamma_k) : \phi(\Gamma_l)]$ divides both $\#\operatorname{Aut}M$ and $[l : k]$. Since $\gcd([l : k], \#\operatorname{Aut}M) = 1$, it follows that $[\phi(\Gamma_k) : \phi(\Gamma_l)] = 1$ and thus that $\phi(\Gamma_k) = \phi(\Gamma_l)$, as required.   ∎

Lemma 4.6.   Let $d \in \mathbf{Z}_{>0}$ and let $g \in \mathbf{Z}_{>1}$. Let l/k be a finite extension of degree coprime to $N_{d,g(2g-1)-1}$. Then for all $X \in \overline{\mathscr{A}}_{k,g} \cup \overline{\mathscr{K}}_{k,g}$, and all $n \in \mathbf{N}$, the image of Γ_l in $\operatorname{Aut}(\operatorname{Br}\overline{X}[d^n])$ is the same as the image of Γ_k.

Proof.   The proof of Lemma 4.2 shows that, as an abelian group with no Galois action, $\operatorname{Br}\overline{X}[d^n] \cong (\mathbf{Z}/d^n\mathbf{Z})^{g(2g-1)-\rho}$, where $1 \le \rho \le g^2$. By [9, Thm. 4.1], the primes dividing the exponent of $\operatorname{Aut}((\mathbf{Z}/d^n\mathbf{Z})^m)$ belong to $Q_{d,m}$. Now apply Lemma 4.5 with e a suitable power of $N_{d,g(2g-1)-1}$ to see that the image of Γ_k in $\operatorname{Aut}(\operatorname{Br}\overline{X}[d^n])$ is the same as the image of Γ_l. ∎

Corollary 4.7.   Let $d \in \mathbf{Z}_{>0}$ and let $g \in \mathbf{Z}_{>1}$. Then for all $X \in \overline{\mathscr{A}}_{k,g} \cup \overline{\mathscr{K}}_{k,g}$, all $n \in \mathbf{Z}_{>0}$, and all finite extensions l/k of degree coprime to $N_{d,g(2g-1)-1}$,

$$\operatorname{Res}_{l/k} : (\operatorname{Br}X)[d^n]/(\operatorname{Br}_1 X)[d^n] \to (\operatorname{Br}X_l)[d^n]/(\operatorname{Br}_1 X_l)[d^n]$$

is an isomorphism and consequently

$$\operatorname{Res}_{l/k} : (\operatorname{Br}X / \operatorname{Br}_1 X)\{d\} \to (\operatorname{Br}X_l / \operatorname{Br}_1 X_l)\{d\}$$

is an isomorphism.

Proof.   Clearly, if $[l : k]$ is coprime to $N_{d,g(2g-1)-1}$, then it is coprime to d. Now Lemma 4.6 allows us to apply Proposition 4.1. ∎

5   Controlling the Brauer Groups of Geometrically Kummer Varieties Over Field Extensions

Throughout this section, k will be a number field. Here we give sufficient criteria for the d-primary part of the algebraic part of the Brauer group of a Kummer variety X to be unchanged under an extension of the base field. We then combine this with the results of Section 4 to give sufficient criteria for the d-primary part of the quotient of the Brauer group of a Kummer variety X by the subgroup $\operatorname{Br}_0 X$ of constant Brauer elements to be unchanged under an extension of the base field.

The following well-known lemma tells us how to control the algebraic part of the Brauer group over finite extensions of the base field.

Lemma 5.1.   Let X be a smooth, proper, geometrically integral variety over k with $\operatorname{Pic}\overline{X}$ free of rank r. Let k'/k be a finite Galois extension such that $\operatorname{Pic}X_{k'} = \operatorname{Pic}\overline{X}$. If a finite

extension l/k is linearly disjoint from k'/k, then the natural homomorphism

$$\mathrm{Res}_{l/k} : \mathrm{Br}_1 X / \mathrm{Br}_0 X \to \mathrm{Br}_1 X_l / \mathrm{Br}_0 X_l$$

is an isomorphism.

Proof.   We have $\mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}\,\overline{X}) = \mathrm{H}^1_{\text{cts}}(\mathrm{Gal}(k'/k), \mathrm{Pic}\,X_{k'})$ and, writing $l' := k'l$ for the compositum field, $\mathrm{H}^1_{\text{cts}}(\mathrm{Gal}(l'/l), \mathrm{Pic}\,X_{l'}) = \mathrm{H}^1_{\text{ét}}(l, \mathrm{Pic}\,\overline{X})$. Moreover, the restriction map induces an isomorphism

$$\mathrm{Res}_{l/k} : \mathrm{H}^1_{\text{cts}}(\mathrm{Gal}(k'/k), \mathrm{Pic}\,X_{k'}) \xrightarrow{\sim} \mathrm{H}^1_{\text{cts}}(\mathrm{Gal}(l'/l), \mathrm{Pic}\,X_{l'}).$$

The Hochschild–Serre spectral sequence

$$\mathrm{H}^p_{\text{cts}}(\mathrm{Gal}(\overline{k}/k), \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbf{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbf{G}_m)$$

yields $\mathrm{Br}_1 X / \mathrm{Br}_0 X = \mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}\,\overline{X})$, and similarly $\mathrm{Br}_1 X_l / \mathrm{Br}_0 X_l = \mathrm{H}^1_{\text{ét}}(l, \mathrm{Pic}\,\overline{X})$, whence the result.   ∎

We also have a uniform version of Lemma 5.1:

Lemma 5.2.   Let $\mathfrak{F}_{k,r}$ be the set of all smooth, proper, geometrically integral varieties over k with $\mathrm{Pic}\,\overline{X}$ free of rank r. If a finite extension l/k has degree coprime to $\#\,\mathrm{GL}_r(\mathbf{F}_3)$, then the natural homomorphism

$$\mathrm{Res}_{l/k} : \mathrm{Br}_1 X / \mathrm{Br}_0 X \to \mathrm{Br}_1 X_l / \mathrm{Br}_0 X_l$$

is an isomorphism for all $X \in \mathfrak{F}_{k,r}$.

Proof.   Let $X \in \mathfrak{F}_{k,r}$. Since Γ_k acts on $\mathrm{Pic}\,\overline{X}$, we have a map $\phi_X : \Gamma_k \to \mathrm{Aut}(\mathrm{Pic}\,\overline{X})$. Let k'_X be the fixed field of $\ker \phi_X$. Then $[k'_X : k] = \#\,\mathrm{im}\,\phi_X$. Since $\mathrm{Pic}\,\overline{X}$ is isomorphic to $\mathbf{Z}^r$ as an abelian group, $\mathrm{im}\,\phi_X$ is isomorphic to a finite subgroup of $\mathrm{GL}_r(\mathbf{Z})$. By a result of Minkowski (see [17, p. 197]), any such subgroup is isomorphic to a subgroup of $\mathrm{GL}_r(\mathbf{F}_3)$. Therefore, $[k'_X : k]$ divides $\#\,\mathrm{GL}_r(\mathbf{F}_3)$. Since l/k is coprime to $\#\,\mathrm{GL}_r(\mathbf{F}_3)$, it follows that $[l : k]$ is coprime to $[k'_X : k]$ and hence that l/k and k'_X/k are linearly disjoint. Therefore, we can apply Lemma 5.1 to deduce that $\mathrm{Res}_{l/k} : \mathrm{Br}_1 X / \mathrm{Br}_0 X \to \mathrm{Br}_1 X_l / \mathrm{Br}_0 X_l$ is an isomorphism.   ∎

Remark 5.3. If $X \in \overline{\mathscr{K}}_{k,g}$, then $\operatorname{Pic}\overline{X}$ is a finitely generated free $\mathbf{Z}$-module of rank $\leq 2^{2g} + g^2$ (see [29, Cor. 2.4]).

Theorem 5.4. Let $d, n \in \mathbf{Z}_{>0}$, let $g \in \mathbf{Z}_{>1}$, and let $X \in \overline{\mathscr{K}}_{k,g}$. Fix a finite Galois extension k'/k such that $\operatorname{Pic}X_{k'} = \operatorname{Pic}\overline{X}$. Let F/k be a finite extension such that Γ_F acts trivially on $\operatorname{Br}\overline{X}[d]$. Let l/k be a finite extension of degree coprime to d, which is linearly disjoint from Fk'/k. Then

$$\operatorname{Res}_{l/k} : (\operatorname{Br}X)[d^n]/(\operatorname{Br}_0 X)[d^n] \to (\operatorname{Br}X_l)[d^n]/(\operatorname{Br}_0 X_l)[d^n]$$

is an isomorphism and consequently

$$\operatorname{Res}_{l/k} : (\operatorname{Br}X/\operatorname{Br}_0 X)\{d\} \to (\operatorname{Br}X_l/\operatorname{Br}_0 X_l)\{d\}$$

is an isomorphism.

Proof. We have the following commutative diagram with exact rows

where the 1st and 3rd vertical arrows are isomorphisms by Lemma 5.1 and by Corollary 4.3, respectively. It follows from the commutativity of the diagram that the middle vertical arrow is also an isomorphism. ∎

We also have the following uniform version of Theorem 5.4:

Theorem 5.5. Fix $d \in \mathbf{Z}_{>0}$ and $g \in \mathbf{Z}_{>1}$. Then, for all $X \in \overline{\mathscr{K}}_{k,g}$, all $n \in \mathbf{Z}_{>0}$, and all finite extensions l/k of degree coprime to $\#\operatorname{GL}_{2^{2g}+g^2}(\mathbf{F}_3) \cdot N_{d,g(2g-1)-1}$, the natural homomorphism

$$\operatorname{Res}_{l/k} : (\operatorname{Br}X)[d^n]/(\operatorname{Br}_0 X)[d^n] \to (\operatorname{Br}X_l)[d^n]/(\operatorname{Br}_0 X_l)[d^n]$$

is an isomorphism and consequently

$$\operatorname{Res}_{l/k} : (\operatorname{Br}X/\operatorname{Br}_0 X)\{d\} \to (\operatorname{Br}X_l/\operatorname{Br}_0 X_l)\{d\}$$

is an isomorphism.

Proof. This follows from Lemma 5.2 together with Remark 5.3 and Corollary 4.7. ∎

6 Preliminaries on Zero-cycles

In this section k will always be a number field. Recall the construction of the Brauer–Manin set for 0-cycles. Let X be a smooth, geometrically integral variety over k. For any field extension l/k, we denote by $\underline{X}_l$ the set of closed points on X_l. Let $Z_0(X_l)$ denote the set of 0-cycles of X over a field extension l/k, that is, the set of formal sums $\sum_{x \in \underline{X}_l} n_x x$ where the integers n_x are zero for all but finitely many $x \in \underline{X}_l$. If $\delta \in \mathbf{Z}_{\geq 0}$, we denote by $Z_0^\delta(X_l) \subset Z_0(X_l)$ the subset of 0-cycles of degree δ, that is, 0-cycles $z := \sum_{x \in \underline{X}_l} n_x x$ such that $\deg(z) := \sum_{x \in \underline{X}_l} [l(x) : l] n_x = \delta$. We can extend the Brauer–Manin pairing to 0-cycles of degree δ by defining, for any subset $B \subset \operatorname{Br} X$,

$$
\langle \, , \, \rangle_{\mathrm{BM}} : \quad \prod_{v \in \Omega_k} Z_0^\delta(X_{k_v}) \times B \quad \to \mathbf{Q}/\mathbf{Z}
$$
$$
\left(\left(\sum_{x_v \in \underline{X}_{k_v}} n_{x_v} x_v \right)_v, \alpha \right) \mapsto \sum_v \operatorname{inv}_v \left(\sum_{x_v \in \underline{X}_{k_v}} n_{x_v} \operatorname{Cor}_{k_v(x_v)/k_v} \alpha(x_v) \right).
$$

Definition 6.1.   Let X be a smooth, geometrically integral variety over k. Let $B \subset \operatorname{Br} X$ be a subset of the Brauer group. We define the Brauer–Manin set associated to B for 0-cycles of degree δ, denoted by $Z_0^\delta(X_{\mathbf{A}_k})^B$, to be the left kernel of the Brauer–Manin pairing above.

Definition 6.2.   Let $\{X_\omega\}_\omega$ be a family of smooth, geometrically integral varieties over k. For each X_ω, let $B_\omega \subset \operatorname{Br} X_\omega$ be a subset of the Brauer group. We say that the $\{B_\omega\}_\omega$-obstruction to the Hasse principle for 0-cycles of degree δ is the only one for the family $\{X_\omega\}_\omega$ if $Z_0^\delta(X_{\omega,\mathbf{A}_k})^{B_\omega} \neq \emptyset$ implies $Z_0^\delta(X_\omega) \neq \emptyset$, for all X_ω.

The following definition is taken from [14] and slightly differs from the definition given in, for example, [4].

Definition 6.3 ([14]).   Let $\{X_\omega\}_\omega$ be a family of smooth, geometrically integral varieties over k. For each X_ω, let $B_\omega \subset \operatorname{Br} X_\omega$ be a subset of the Brauer group. We say that the $\{B_\omega\}_\omega$-obstruction to weak approximation for 0-cycles of degree δ is the only one if for any $n \in \mathbf{Z}_{>0}$, for any finite subset $S \subset \Omega_k$, and for any $(z_v)_{v \in \Omega_k} \in Z_0^\delta(X_{\omega,\mathbf{A}_k})^{B_\omega}$, there exists some $z_{n,S} \in Z_0^\delta(X_\omega)$ such that $z_{n,S}$ and z_v have the same image in $\operatorname{CH}_0(X_{\omega,k_v})/n$ for all $v \in S$, for all X_ω, where CH_0 denotes the usual Chow group of 0-cycles.

Remark 6.4.   In Definitions 6.2 and 6.3, when for all X_ω we take $B_\omega := \operatorname{Br} X_\omega$ or $B_\omega := \operatorname{Br} X_\omega\{d\}$ for some $d \in \mathbf{Z}_{>0}$, we say, respectively, that the Brauer–Manin obstruction or the d-primary Brauer–Manin obstruction is the only one for 0-cycles of degree δ for the family $\{X_\omega\}_\omega$.

The main conjecture governing the arithmetic properties of 0-cycles of degree 1 is the following:

Conjecture 6.5 (Colliot-Thélène). Let $\{X_\omega\}_\omega$ be the family of *all* smooth, proper, geometrically integral varieties over k. Then the Brauer–Manin obstruction to weak approximation for 0-cycles of degree 1 is the only one for $\{X_\omega\}_\omega$.

7 From Rational Points to Zero-cycles

In this section, we prove our main results, which are various analogues of [14, Thm. 3.2.1] wherein we relate the Brauer–Manin obstruction for rational points to that for 0-cycles on products of (geometrically) Kummer varieties, K3 surfaces and rationally connected varieties. We begin with a lemma that allows us to extend the base field without affecting the Brauer–Manin obstruction. Throughout this section, k will always be a number field.

Lemma 7.1.

(1) If $X \in \overline{\mathscr{R}}_k$, then there exists a finite extension F_X/k such that for all finite extensions ℓ/k that are linearly disjoint from F_X/k,

$$\operatorname{Res}_{\ell/k} : \operatorname{Br} X / \operatorname{Br}_0 X \to \operatorname{Br} X_\ell / \operatorname{Br}_0 X_\ell$$

is an isomorphism.

(2) If $X \in \mathsf{K3}_k$, then for all $n \in \mathbf{Z}_{>0}$, there exists a finite extension $F_{X,n}$ such that for all finite extensions ℓ/k with $[\ell : k] \leq n$ that are linearly disjoint from $F_{X,n}/k$,

$$\operatorname{Res}_{\ell/k} : \operatorname{Br} X / \operatorname{Br}_0 X \to \operatorname{Br} X_\ell / \operatorname{Br}_0 X_\ell$$

is an isomorphism.

(3) If $X \in \overline{\mathscr{K}}_k$, then there exists a finite extension F_X/k such that for any $d \in \mathbf{Z}_{>0}$ and all finite extensions ℓ/k of degree coprime to d that are linearly disjoint from F_X/k,

$$\operatorname{Res}_{\ell/k} : (\operatorname{Br} X / \operatorname{Br}_0 X)\{d\} \to (\operatorname{Br} X_\ell / \operatorname{Br}_0 X_\ell)\{d\}$$

is an isomorphism.

Proof. For $X \in \overline{\mathscr{R}}_k$, this is [14, Prop. 3.1.1]. For $X \in \mathsf{K3}_k$, see the proof of [10, Thm. 1.1]. For $X \in \overline{\mathscr{K}}_k$, noting that $\operatorname{Br} \overline{X}[d]$ is finite, let F/k be a finite extension such that Γ_F acts

trivially on $\operatorname{Br} \overline{X}[d]$. Let k'/k be a finite Galois extension such that $\operatorname{Pic} X_{k'} = \operatorname{Pic} \overline{X}$. Now Theorem 5.4 shows that we can take $F_X = Fk'$.   ∎

We are now ready to prove our main result, which is Theorem 1.5.

Remark 7.2.   In the proof of Theorem 1.5, we use a notion of two 0-cycles on a variety X over k_v being "sufficiently close" as described by Liang in [14, Section 1.1]. In particular, by [31, Lemma 1.8], if two 0-cycles on X are sufficiently close then they have the same image in $\operatorname{CH}_0(X)/n$. Furthermore, since the evaluation maps are locally constant, if $b \in \operatorname{Br}(X)$ and two 0-cycles z and z' on X are sufficiently close, then b has the same evaluation at z and z'.

Proof of Theorem 1.5   We follow closely the strategy of the proof of [14, Thm. 3.2.1] and give here just a sketch of the proof; for more details, we refer the reader to Liang's paper [14]. We prove the result for weak approximation; the argument for the Hasse principle is similar.

(1)   Let $(\zeta_v)_v \in Z_0^\delta(W_{\mathbf{A}_k})$ be orthogonal to $\operatorname{Br}(W)\{c\}$. Consider the projection $\operatorname{pr} : W \times \mathbf{P}_k^1 \to W$ with section $\sigma : w \mapsto (w, u_0)$, where $u_0 \in \mathbf{P}_k^1(k)$ is some fixed rational point. We note that $\operatorname{pr}^* : \operatorname{Br} W \to \operatorname{Br}(W \times \mathbf{P}_k^1)$ is an isomorphism. Let $\sigma_* : Z_0^\delta(W) \to Z_0^\delta(W \times \mathbf{P}_k^1)$ be the induced map on zero-cycles. Then $(\sigma_*(\zeta_v))_v$ on $W \times \mathbf{P}_k^1$ is orthogonal to $\operatorname{Br}(W \times \mathbf{P}_k^1)\{c\}$. To ease notation, we let $z_v := \sigma_*(\zeta_v)$ for all $v \in \Omega_k$. Let $\{b_i\}_{i=1}^s$ be a complete set of representatives for $(\operatorname{Br}(W \times \mathbf{P}_k^1)/\operatorname{Br}_0(W \times \mathbf{P}_k^1))\{c\}$, which is finite by Proposition 3.1. Let $m := \#(\operatorname{Br} W/\operatorname{Br}_0 W)\{c\}$. We fix $n \in \mathbf{Z}_{>0}$ and a finite subset $S \subset \Omega_k$. We also fix a closed point $P := (w_0, u_0) \in W \times \mathbf{P}_k^1$ and let $\delta_P := [k(P) : k]$.

(2)   Let $S_0 \subset \Omega_k$ be a finite subset containing S and all the archimedean places of k. By enlarging S_0 if necessary, we can assume that, for any $v \notin S_0$,

- $\langle z'_v, b_i \rangle_{\mathrm{BM}} = 0$ for any $i \in \{1, ..., s\}$ and for any $z'_v \in Z_0(W_{k_v} \times \mathbf{P}_{k_v}^1)$, and
- W_{k_v} has a k_v-point w_v (this follows from the Lang–Weil estimates and Hensel's lifting).

(3)   For each $v \in S_0$, write $z_v = z_v^+ - z_v^-$, where z_v^+ and z_v^- are effective 0-cycles with disjoint supports, and let $z_v^1 := z_v + dmn\delta_P z_v^-$. Note that $\deg(z_v^1) = \delta + dmn\delta_P \deg(z_v^-)$. By adding to each z_v^1 a suitable multiple of the 0-cycle $dmnP_v$, where $P_v := P \otimes_k k_v$, we obtain 0-cycles z_v^2 of the same degree $\Delta > 0$

for all $v \in S_0$, where we can take Δ to satisfy $\Delta \equiv \delta \mod dmn\delta_P$. Using the natural projection $\pi : W_{k_v} \times \mathbf{P}^1_{k_v} \to \mathbf{P}^1_{k_v}$ and a moving lemma by Liang [14, Lemma 1.3.1], for each $v \in S_0$ we can find an effective 0-cycle z_v^3 of degree Δ such that $\pi_*(z_v^3)$ is separable and z_v^3 is sufficiently close to z_v^2. By Remark 7.2, z_v^3 has the same image in $\mathrm{CH}_0(W_{k_v})/n$ as z_v^2, z_v^1, and z_v.

(4) For each $i = 1, \ldots, r$, let F_{X_i} be as in Lemma 7.1, with $F_{X_i} := F_{X_i, \Delta}$ if $X_i \in \mathsf{K3}_k$, and let $F := F_{X_1} \ldots F_{X_r}$ be the compositum.

(5) By [14, Prop. 3.3.3] and the discussion in the proof of [14, Prop. 3.4.1], we can find a closed point $\theta \in \mathbf{P}^1_k$ sufficiently close, for $v \in S_0$, to $\pi_*(z_v^3)$ such that $[k(\theta) : k] = \Delta$ and $k(\theta)/k$ is linearly disjoint from F/k. Following the proof of [14, Thm. 3.2.1], making the identification $\pi^{-1}(\theta) \cong W_{k(\theta)}$ we also obtain an adelic point $(\mathcal{M}_w)_w \in W_{k(\theta)}(\mathbf{A}_{k(\theta)})$ such that $\sum_{w|v} \mathcal{M}_w$ is sufficiently close to z_v^3 (and hence to z_v^2) for all $v \in S_0$. Hence, $\sum_{w|v} \mathcal{M}_w$ has the same image as z_v in $\mathrm{CH}_0(W_{k_v})/n$ for all $v \in S$, and also $(\mathcal{M}_w)_w \in W_{k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, W\{c\}}$.

(6) Via Lemma 3.4 and Remark 3.5, we view $(\mathcal{M}_w)_w$ as an adelic point $((\mathcal{M}_{1,w})_w, \ldots, (\mathcal{M}_{r,w})_w)$ in $X_{1,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_1\{c\}} \times \cdots \times X_{r,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_r\{c\}}$. Hence, we have an adelic point $(\mathcal{M}_{i,w})_w$ in $X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_i\{c\}}$ for each $i = 1, \ldots, r$.

(7) Recall that $k(\theta)/k$ is linearly disjoint from F_{X_i} for all $i = 1, \ldots, r$ and also $[k(\theta) : k] = \Delta$. For $X_i \in \overline{\mathscr{R}}_k \cup \mathsf{K3}_k$, Lemma 7.1 shows that $\mathrm{Res}_{k(\theta)/k} : \mathrm{Br}\, X_i / \mathrm{Br}_0 X_i \to \mathrm{Br}\, X_{i,k(\theta)} / \mathrm{Br}_0 X_{i,k(\theta)}$ is an isomorphism. Furthermore, since $a_i \mid c$, we have $\mathrm{Br}\, X_i / \mathrm{Br}_0 X_i = (\mathrm{Br}\, X_i / \mathrm{Br}_0 X_i)\{c\}$. Therefore, $(\mathcal{M}_{i,w})_w \in X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_i\{c\}} = X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_{i,k(\theta)}}$. By assumption, the Brauer–Manin obstruction is the only obstruction to weak approximation for rational points on $X_{i,k(\theta)}$. Hence, there exists a global point $\mathcal{M}_i \in X_{i,k(\theta)}(k(\theta))$ sufficiently close to $\mathcal{M}_{i,w}$ for all places w of $k(\theta)$ lying above places in S.

(8) Now suppose that for some i we have $X_i \in \overline{\mathscr{K}}_k$. Since $\overline{\mathscr{K}}_{k,2} \subset \mathsf{K3}_k$, if $X_i \in \overline{\mathscr{K}}_{k,2}$ then we are in the previous step. If $X_i \in \overline{\mathscr{K}}_{k,g}$ for some $g \geq 3$ then, by assumption, δ is coprime to d. Since $\Delta \equiv \delta \mod dmn\delta_P$, this implies that Δ is coprime to d and Lemma 7.1 gives $X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_i\{d\}} = X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_{i,k(\theta)}\{d\}}$. Since $(\mathcal{M}_{i,w})_w \in X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_i\{c\}} \subset X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_i\{d\}}$, we obtain $(\mathcal{M}_{i,w})_w \in X_{i,k(\theta)}(\mathbf{A}_{k(\theta)})^{\mathrm{Br}\, X_{i,k(\theta)}\{d\}}$.

By assumption, the d-primary Brauer–Manin obstruction is the only obstruction to weak approximation for rational points on $X_{i,k(\theta)}$. Hence, there exists a global point $\mathcal{M}_i \in X_{i,k(\theta)}(k(\theta))$ sufficiently close to $\mathcal{M}_{i,w}$ for all places w of $k(\theta)$ lying above places in S.

(9)   It follows that the $k(\theta)$-point $\mathcal{M} := (\mathcal{M}_1, \dots, \mathcal{M}_r) \in W_{k(\theta)}(k(\theta))$ is sufficiently close to $\mathcal{M}_w$ for all places w of $k(\theta)$ lying above places in S, implying that $\mathcal{M}$ and z_v have the same image in $\mathrm{CH}_0(W_{k_v})/n$ for all $v \in S$.

(10)   When viewed as a 0-cycle on W, $\mathcal{M}$ has degree Δ. Since $\Delta \equiv \delta \mod dmn\delta_P$, adding a suitable multiple of the degree δ_P closed point $w_0 = \mathrm{pr}(P)$ to $\mathcal{M}$ yields a global 0-cycle of degree δ on W with the same image as z_v in $\mathrm{CH}_0(W_{k_v})/n$ for all $v \in S$, as required.   ■

Remark 7.3.   Let K/k be a finite extension and let $N \in \mathbf{Z}_{>0}$ be coprime to δ. The proof above shows that Theorem 1.5 still holds with the weaker assumptions obtained by replacing the phrase "for all finite extensions l/k" with "for all finite extensions l/k of degree coprime to N that are linearly disjoint from K/k". The same goes for Theorems 1.1 and 7.7 below.

Remark 7.4.   The product of two or more Kummer varieties is not, in general, a Kummer variety. Similarly, the product of two or more K3 surfaces is not a K3 surface. On the other hand, the product of two or more geometrically rationally connected varieties is again geometrically rationally connected.

Remark 7.5.   It is clear that the strategy of the proof of Theorem 1.5 could be used to deal with other (products of) varieties, provided that enough information on the Brauer groups and on how to control them over certain field extensions is known.

Remark 7.6.   Let X be a Kummer variety over k. By [5, Thm. 1.7], it follows that $X_l(\mathbf{A}_l)^{\mathrm{Br} X_l} \neq \emptyset$ if and only if $X_l(\mathbf{A}_l)^{\mathrm{Br} X_l\{2\}} \neq \emptyset$, for any finite extension l/k. Hence, for any such X, saying "the 2-primary Brauer–Manin obstruction to the Hasse principle is the only one for rational points on X_l" is equivalent to saying "the Brauer–Manin obstruction to the Hasse principle is the only one for rational points on X_l".

As a special case of Theorem 1.5, we obtain the following result for the Hasse principle:

Theorem 7.7.   Let $W := \prod_{i=1}^{r} X_i$, where $X_i \in \mathscr{K}_k \cup \mathsf{K3}_k \cup \overline{\mathscr{R}}_k$ for all $i \in \{1, \dots, r\}$. Let $\delta \in \mathbf{Z}$ and assume that δ is odd if at least one of the X_i is in $\mathscr{K}_{k,g}$ with $g \geq 3$. Suppose that, for all $i \in \{1, \dots, r\}$ and all finite extensions l/k, the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points on $X_{i,l}$. Then the Brauer–Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree δ on W.

Proof.   Take $d = 2$ in Theorem 1.5. By [5, Thm. 1.7], the assumption that the Brauer–Manin obstruction to the Hasse principle is the only one for rational points on $X_l \in \mathscr{K}_l$ implies that the 2-primary Brauer–Manin obstruction to the Hasse principle is the only one for rational points on X_l.   ∎

Remark 7.8.   In Theorem 7.7, we cannot include geometrically Kummer varieties of dimension greater than 2 unless they are Kummer varieties over k because [5, Thm. 1.7] only applies to Kummer varieties, not to the larger class of geometrically Kummer varieties. However, geometrically Kummer surfaces are included because they are K3 surfaces.

Proof of Theorem 1.1   This is an immediate consequence of [5, Thm. 1.7] and the proof of Theorem 1.5 applied with $d = 2$.   ∎

In [25], Skorobogatov made the following conjecture:

Conjecture 7.9 (Skorobogatov).   The Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on K3 surfaces over number fields.

Proof of Corollary 1.2   Run the proof of Theorem 1.5 for $d = 2$, taking into account Remark 7.6.   ∎

It is natural to ask the following question:

Question 7.10.   Is the Brauer–Manin obstruction the only obstruction to the Hasse principle for the class $\mathscr{K}_k$ of Kummer varieties over a number field k? Or indeed for the class $\overline{\mathscr{K}}_k$ of geometrically Kummer varieties over a number field k? Is the Brauer–Manin obstruction the only obstruction to weak approximation for either of these classes?

By [5, Thm. 1.7], the 1st part of Question 7.10 is equivalent to asking whether the 2-primary Brauer–Manin obstruction is the only obstruction to the Hasse principle for the class $\mathscr{K}_k$ of Kummer varieties over a number field k.

As a corollary of Theorem 7.7, we obtain a result that is conditional on Conjecture 7.9, a positive answer to the 1st part of our Question 7.10, and the following conjecture of Colliot-Thélène (stated in full generality in [2, p. 174]):

Conjecture 7.11 (Colliot-Thélène).   The Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points for the class $\overline{\mathscr{R}}_k$ of all smooth, projective, geometrically integral, geometrically rationally connected varieties over a number field k.

Corollary 7.12.   Let k be a number field. Let $X_i \in \mathscr{K}_k \cup \mathsf{K3}_k \cup \overline{\mathscr{R}}_k$ for all $i = 1, \ldots, r$ and write $W := \prod_{i=1}^{r} X_i$. Let $\delta \in \mathbf{Z}$ and assume that δ is odd if at least one of the X_i is in $\mathscr{K}_{k,g}$ with $g \geq 3$. If Conjectures 7.9 and 7.11 are true and if the 1st part of our Question 7.10 has a positive answer, then the Brauer–Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree δ on W.

8   Transferring Emptiness of the Brauer–Manin Set Over Field Extensions

Let k be a number field. In the previous section, we proved that the Brauer–Manin obstruction to the Hasse principle is the only one for 0-cycles on certain varieties X/k, assuming that the Brauer–Manin obstruction to the Hasse principle on X_l is the only one for rational points for sufficiently many finite extensions l/k. One wonders whether it is enough to simply assume that the Brauer–Manin obstruction to the Hasse principle is the only one for rational points on X. This leads one to ask when sufficiency of the Brauer–Manin obstruction can be transferred over field extensions, which is the topic of Theorem 1.6, proved below. Similar ideas have appeared in [22].

Proof of Theorem 1.6   Let $\alpha \in \operatorname{Br} X\{2\}$ be a representative for $[\alpha] \in \operatorname{Br} X / \operatorname{Br}_0 X$. Let $\mathcal{X}$ be an integral model for X over $\operatorname{Spec} \mathcal{O}_S$, where $S \subset \Omega_k$ is some finite subset containing the archimedean places and such that α comes from an element in $\operatorname{Br} \mathcal{X}\{2\}$. Since $\operatorname{Br} \mathcal{O}_v = 0$ for all $v \notin S$, we have $\langle x_v, \alpha \rangle_{\mathrm{BM}} = 0$ for any $x_v \in X(k_v)$ and $v \notin S$.

Let l/k be a finite extension of odd degree that is completely split at all places in S. Suppose that $X(\mathbf{A}_l)^{\alpha_l} \neq \emptyset$, where $\alpha_l := \operatorname{Res}_{l/k} \alpha$. Let $(x_w)_w \in X(\mathbf{A}_l)^{\alpha_l}$. Since $\operatorname{inv}_w(\alpha_l(x_w)) = 0$ for any place w of l lying above a place v of k with $v \notin S$, we have

$$0 = \sum_{w \in \Omega_l} \operatorname{inv}_w(\alpha_l(x_w)) = \sum_{v \in S} \sum_{w|v} \operatorname{inv}_w(\alpha_l(x_w)).$$

Since l/k is completely split at all places in S, for each place $w|v$ with $v \in S$ we have $[l_w : k_v] = 1$ and hence there exists some $u_v^{(w)} \in X(k_v)$ with $\operatorname{inv}_w(\alpha_l(x_w)) = \operatorname{inv}_v(\alpha(u_v^{(w)}))$. It follows that

$$\sum_{v \in S} \sum_{w|v} \operatorname{inv}_w(\alpha_l(x_w)) = \sum_{v \in S} \sum_{w|v} \operatorname{inv}_v(\alpha(u_v^{(w)})) = 0. \tag{8.1}$$

We now construct an adelic point $(x'_v)_v \in X(\mathbf{A}_k)^\alpha$. By assumption, $X(\mathbf{A}_k) \neq \emptyset$, so let $(x''_v)_v \in X(\mathbf{A}_k)$. Let $x'_v := x''_v$ for all $v \notin S$. For $v \in S$, we proceed as follows. Define

$$A_v := \sum_{w|v} \operatorname{inv}_v(\alpha(u_v^{(w)})).$$

If $A_v = 0$, then there exists some $w|v$ such that $\mathrm{inv}_v(\alpha(u_v^{(w)})) = 0$, since α has order 2 in $\mathrm{Br}\, X / \mathrm{Br}_0\, X$ and $\#\{w|v\} = [l : k]$ is odd. In this case, let $x'_v := u_v^{(w)}$. Note that, from 8.1, there is an even number (possibly zero) of places $v \in S$ such that $A_v = 1/2$; let $\mathcal{V}$ be the set of all such $v \in S$. If $\mathcal{V} = \emptyset$, then $A_v = 0$ for all $v \in S$. This means that we have already constructed an adelic point $(x'_v)_v \in X(\mathbf{A}_k)$ with the property that $(x'_v)_v \in X(\mathbf{A}_k)^\alpha$. So let us suppose that $\mathcal{V} \neq \emptyset$ and pair the places in $\mathcal{V}$, writing $\mathcal{V} = \{v_1, v'_1, v_2, v'_2, ..., v_r, v'_r\}$. For $i = 1, ..., r$, we do the following. Since $A_{v_i} = 1/2$ and $A_{v'_i} = 1/2$, we can pick some $u_{v_i}^w \in X(k_{v_i})$ and some $u_{v'_i}^{w'} \in X(k_{v'_i})$ such that $\mathrm{inv}_{v_i}(\alpha(u_{v_i}^w)) = \mathrm{inv}_{v'_i}(\alpha(u_{v'_i}^{w'})) = 1/2$. We then set $x'_{v_i} := u_{v_i}^w$ and $x'_{v'_i} := u_{v'_i}^{w'}$. At the end of this process, we obtain an adelic point $(x'_v)_v \in X(\mathbf{A}_k)$ that, by construction, has the property that $(x'_v)_v \in X(\mathbf{A}_k)^\alpha$.

Hence, for any finite extension l/k of odd degree that is completely split at all places in S, we have shown that $X(\mathbf{A}_l)^{\alpha l} \neq \emptyset$ implies that $X(\mathbf{A}_k)^\alpha \neq \emptyset$. The fact that there exist uncountably many such field extensions l/k follows from, for example, the proof of [16, Thm. 4.1]. ∎

Proof of Corollary 1.7   This is a special case of Theorem 1.6 in conjunction with [5, Thm. 1.7]. ∎

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