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A Novel Approach to Multi-Attribute Group Decision-Making based on Interval-Valued Intuitionistic Fuzzy Power Muirhead Mean

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Abstract: This paper focuses on multi-attribute group decision-making (MAGDM) course in which attributes are evaluated in terms of interval-valued intuitionistic fuzzy (IVIF) information. More explicitly, this paper introduces new aggregation operators for IVIF information and further proposes a new IVIF MAGDM method. The power average (PA) operator and the Muirhead mean (MM) are two powerful and effective information aggregation technologies. The most attractive advantage of the PA operator is its power to combat the adverse effects of ultra-evaluation values on the information aggregation results. The prominent characteristic of the MM operator is that it is flexible to capture the interrelationship among any numbers of arguments, making it more powerful than Bonferroni mean (BM), Heronian mean (HM), and Maclaurin symmetric mean (MSM). To absorb the virtues of both PA and MM, it is necessary to combine them to aggregate IVIF information and propose IVIF power Muirhead mean (IVIFPMM) operator and the IVIF weighted power Muirhead mean (IVIFWPMM) operator. We investigate their properties to show the strongness and flexibility. Furthermore, a novel approach to MAGDM problems with IVIF decision-making information is introduced. Finally, a numerical example is provided to show the performance of the proposed method.

Keywords: multi-attribute group decision-making; interval-valued intuitionistic fuzzy sets; power average operator; Muirhead mean; interval-valued intuitionistic fuzzy power Muirhead mean

1. Introduction

There are quite a few decision-making (DM) activities in real life. For example, when buying a car, we usually have to comprehensively take into consideration the various indicators of the alternatives of potential candidates. When considering a good supplier globally, a company usually evaluates alternatives from multiple aspects. It is not difficult to find out that the essence of quite a few actual DM problems is multi-attribute decision-making or multi-attribute group decision-making (MAGDM) [1–5]. When using MAGDM theory framework to solve practical DM problems, we always need to consider four basic elements, all possible alternatives, multiple attributes, evaluation information, and best choice determining methods, among which the latter two are the most important and complicated. In other words, there are two fundamental issues in MAGDM, (1) how decision makers express their preference information in a proper way; (2) how the best candidate is determined. Thanks to Prof. Zadeh [6] who provided an efficient methodology, called fuzzy set theory (FST), to describe fuzzy information. Hence, FST-based MAGDM has sooner become a new hot research topic [7–10] and attracted attention from scholars and scientists all around the world. Although FST has achieved great success in MAGDM, many scholars have noticed its shortcomings and started to study new tools. A landmark discovery was the intuitionistic fuzzy sets (IFSs) proposed by Atanassov [11] in 1986. The IFSs are powerful for their membership grades (MGs) and non-membership grades (NMGs), which not only describe the degree of an element to a given fixed set but also contain the grade that the element does not belong to the fixed set. Due to this feature, IFSs received great attention from scholars in operational research and DM sciences. Xu [12] was the founding father of intuitionistic fuzzy aggregation operator theory. Afterward, quite a few intuitionistic fuzzy aggregation operators have been proposed based on Bonferroni mean (BM), Heronian mean (HM), and Maclaurin symmetric mean (MSM) [13–15], as scholars started to realize the relationship among attributes. Although IFSs have been proposed as early as 1986, they are still widely applied in the field of MAGDM until now. For example, Meng et al. [16] investigated group DM methods in which decision makers' evaluations are expressed by linguistic intuitionistic preference relations. Tao et al. [17] studied operations and aggregation operators for intuitionistic fuzzy numbers (IFNs) based on Archimedean copulas. Garg and Arora [18] focused on the combination of IFSs with soft sets and the corresponding DM methods. Cali and Balaman [19] proposed a new intuitionistic fuzzy MAGDM method by integrating ELECTRE and VIKOR. Garg and Rani [20] developed a new distinctive correlation coefficient measure of complex IFSs and illustrated its robustness.

Although the powerfulness of IFSs in dealing with MAGDM has been widely investigated, the limitations of IFS are also obvious. In IFSs, MG and NMG are denoted by two single certain values. Nevertheless, decision makers sometimes would like to use interval values rather than crisp numbers to express their preferences. Obviously, compared with crisp numbers, interval values contain more information and can express decision makers' evaluation information more comprehensively. Hence, Atanassov and Gargov [21] generalized the traditional IFSs and proposed the interval-valued intuitionistic fuzzy (IVIF) sets (IVIFSs). As known, aggregation operators are an efficient methodology in solving MAGDM problems. Hence, more and more scholars started to investigate aggregation operators for IVIF information. The most representative is the IVIF ordered weighted average operator developed by Xu [22]. Besides, more and more scientists began to notice that there is often strong interrelationship among attributes in MAGDM problems in reality. Hence, some aggregation operators, such as IVIF Bonferroni means (IVIFBMs) [23], IVIF Heronian means (IVIFHMs) [24], and IVIF Maclaurin symmetric means (IVIFMSMs) [25], were proposed to take such relationship into account. Recently, IVIF aggregation operator theory has achieved important development. To deal with complicated IVIF DM systems, some scholars introduced hybrid aggregation operators, such as the IVIF power Bonferroni mean (IVIFPBM) operator [26], the IVIF power Heronian mean (IVIFPHM) operator [27], and the IVIF power Maclaurin symmetric mean (IVIFPMSM) [28] operator. Take the IVIFPBM operator as an example; it is a combination of IVIF power average (IVIFPA) [29] operator and the IVIFBM operator. Hence, IVIFPBM operators have the advantages of both IVIFPA and IVIFBM operators. Similarly, IVIFPHM and IVIFPMSM have the merits as IVIFPBM. The recently proposed Muirhead mean (MM) [30] has similar advantages as BM, HM, and MSM, as all of them can capture the interrelationship among attributes. However, MM is believed to be more flexible due to its skill of considering the interrelationship among arbitrary numbers of attributes [31–34]. Hence, it is very necessary to compound power average (PA) [35] with MM to integrate IVIF information and propose IVIF power MM (IVIFPMM) operators. Furthermore, we utilize the proposed operators to propose a new method to handle IVIF MAGDM problems.

This paper is constructed as follows. Section 2 reviews basic knowledge that is used in the following sections. Section 3 introduces the IVIFPMM operator and its weighted form by taking the weight vector of attributes into account. Section 4 presents the main steps of a new algorithm of addressing MAGDM with IVIF information. Section 5 applies the new approach to real-life DM problems. Additionally, we also prove why our method is more powerful and flexible than others. Conclusions remarks can be found in Section 6.

2. Preliminaries

2.1. The Power Average and Muirhead Mean Operators

Considering that the unduly high and unduly low assessments provided by decision makers may have bad effects on the final results, Yager [35] introduced the PA operator in which the weight vector depends on the input data.

Definition 1 [35]. Let $a_i\left(i=1,2,\ldots,n\right)$ be a collection of positive real numbers, then the power average (PA) operator is defined as

$$PA\left(a_1,a_2,\ldots,a_n\right)=\frac{\sum_{i=1}^{n}\left(1+T\left(a_i\right)\right)a_i}{\sum_{i=1}^{n}\left(1+T\left(a_i\right)\right)}, \tag{1}$$

where $T\left(a_i\right)=\sum_{j=1,j\neq i}^{n}Sup\left(a_i,a_j\right)$ and $Sup\left(a_i,a_j\right)$ denotes the support degree for a_i from a_j, satisfying the following conditions: 1) $Sup\left(a_i,a_j\right)\in\left[0,1\right]$; 2) $Sup\left(a_i,a_j\right)=Sup\left(a_j,a_i\right)$; 3) $Sup\left(a,b\right)\geq Sup\left(c,d\right)$, if $\left|a-b\right|\leq\left|c-d\right|$.

The MM was introduced by Muirhead [30] for crisp numbers. Its flexibility is reflected in its ability to capture the interrelationship among arbitrary numbers of input variables.

Definition 2 [30]. Let $a_j\left(j=1,2,\ldots,n\right)$ be a set of real numbers and $S=\left(s_1,s_2,\ldots,s_n\right)$ be a collection of parameters, where $s_j\left(j=1,2,\ldots,n\right)$ is a non-negative real number. If

$$MM^S\left(a_1,a_2,\ldots,a_n\right)=\left(\frac{1}{n!}\sum_{\zeta\in T_n}\prod_{j=1}^{n}a_{\zeta(j)}^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}, \tag{2}$$

then MM^S is the Muirhead mean (MM) operator, where $\zeta\left(j\right)\left(j=1,2,\ldots,n\right)$ denotes any permutation of $\left(1,2,\ldots,n\right)$ and T_n represents all possible permutations of $\left(1,2,\ldots,n\right)$.

2.2. Interval-Valued Intuitionistic Fuzzy Sets

Definition 3 [21]. Let X be a universe of discourse, an interval-valued intuitionistic fuzzy set (IVIFS). A over X is defined as

$$A=\left\{\left\langle x,\mu_A\left(x\right),v_A\left(x\right)\right\rangle\middle|x\in X\right\}, \tag{3}$$

where $\mu_A\left(x\right),v_A\left(x\right)\subset\left[0,1\right]$ are two intervals numbers, representing the membership and non-membership degree, respectively, satisfying $0\leq\sup\left(\mu_A\left(x\right)\right)+\sup\left(v_A\left(x\right)\right)\leq1$ for all $x\in X$. For convenience, let $\mu_A\left(x\right)=\left[a,b\right]$ and $v_A\left(x\right)=\left[c,d\right]$, so that $\alpha=\left(\left[a,b\right],\left[c,d\right]\right)$, which can be called an interval-valued intuitionistic fuzzy number (IVIFN).

Then, Xu [22] introduced some operations of IVIFNs.

Definition 4 [22]. Let $\alpha_1=\left(\left[a_1,b_1\right],\left[c_1,d_1\right]\right)$, $\alpha_2=\left(\left[a_2,b_2\right],\left[c_2,d_2\right]\right)$, $\alpha=\left(\left[a,b\right],\left[c,d\right]\right)$ be any three IVIFNs and λ be a positive real number, then

1. $\alpha_1\oplus\alpha_2=\left(\left[a_1+a_2-a_1a_2,b_1+b_2-b_1b_2\right],\left[c_1c_2,d_1d_2\right]\right)$
2. $\alpha_1\otimes\alpha_2=\left(\left[a_1a_2,b_1b_2\right],\left[c_1+c_2-c_1c_2,d_1+d_2-d_1d_2\right]\right)$
3. $\lambda\alpha=\left(\left[1-\left(1-a\right)^{\lambda},1-\left(1-b\right)^{\lambda}\right],\left[c^{\lambda},d^{\lambda}\right]\right)$
4. $\alpha^{\lambda}=\left(\left[a^{\lambda},b^{\lambda}\right],\left[1-\left(1-c\right)^{\lambda},1-\left(1-d\right)^{\lambda}\right]\right)$

To compare two IVIFNs, Hung and Wu [36] gave the definitions of the score function and the accuracy function of IVIFNs.

Definition 5 [36]. Let $\alpha = ([a,b],[c,d])$ be an IVIFN, then a score function S and an accuracy function H can be defined as follows

$$S(\alpha) = (a-c+b-d)/2\,, \tag{4}$$

$$H(\alpha) = (a+b+c+d)/2\,, \tag{5}$$

Based on the score function and the accuracy function, Hung and Wu [36] introduced the comparison rule for two IVIFNs.

Definition 6 [36]. Let $\alpha_1 = ([a_1,b_1],[c_1,d_1])$ and $\alpha_2 = ([a_2,b_2],[c_2,d_2])$ be any two IVIFNs, $S(\alpha_1)$ and $S(\alpha_2)$ be the scores of α_1 and α_2, respectively; $H(\alpha_1)$ and $H(\alpha_2)$ be the accuracy of α_1 and α_2, respectively. Then, the comparison law of two IVIFNs can be defined as

1. *If* $S(\alpha_1) > S(\alpha_2)$, *then* $\alpha_1 > \alpha_2$;
2. *If* $S(\alpha_1) = S(\alpha_2)$, *then*

    if $H(\alpha_1) > H(\alpha_2)$, *then* $\alpha_1 < \alpha_2$;

    if $H(\alpha_1) = H(\alpha_2)$, *then* $\alpha_1 = \alpha_2$.

Xu [37] gave the definition of Hamming distance between any two IVIFNs.

Definition 7 [37]. Let $\alpha_1 = ([a_1,b_1],[c_1,d_1])$ and $\alpha_2 = ([a_2,b_2],[c_2,d_2])$ be any two IVIFNs, then the Hamming distance between α_1 and α_2 is defined as

$$d(\alpha_1,\alpha_2) = \frac{1}{4}\left(|a_1-a_2| + |b_1-b_2| + |c_1-c_2| + |d_1-d_2|\right), \tag{6}$$

3. Power Muirhead Mean Operators for Interval-Valued Intuitionistic Fuzzy Sets

In this section, we combine PA with MM within an IVIF environment and propose some interval-valued intuitionistic fuzzy power Muirhead mean operators.

3.1. The Interval-Valued Intuitionistic Fuzzy Power Muirhead Mean (IVIFPMM) Operator

Definition 8. Let $\alpha_j\,(i=1,2,\ldots,n)$ be a collection of IVIFNs and $S=(s_1,s_2,\ldots,s_n)$ be a collection of parameters, where $s_j\,(j=1,2,\ldots,n)$ is a non-negative real number. Then, the interval-valued intuitionistic fuzzy power Muirhead mean (IVIFPMM) operator is given as

$$IVIFPMM^S(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\frac{1}{n}\underset{\zeta\in T_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\frac{\left(1+T\left(\alpha_{\zeta(j)}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\alpha_j\right)\right)}\alpha_{\zeta(j)}\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}, \tag{7}$$

where

$$T(\alpha_j) = \sum_{i=1,i\neq j}^{n} Sup(\alpha_i,\alpha_j)\,, \tag{8}$$

and

$$Sup\left(\alpha_i,\alpha_j\right)=1-d\left(\alpha_i,\alpha_j\right), \tag{9}$$

where $\zeta(j)(j=1,2,...,n)$ denotes any permutation of $(1,2,...,n)$, T_n represents all possible permutations of $(1,2,...,n)$, and n is the balancing coefficient. $d\left(\alpha_i,\alpha_j\right)$ represents the Hamming distance between α_i and α_j, and $Sup\left(\alpha_i,\alpha_j\right)$ is the support for α_i from α_j, satisfying the following conditions:

(1) $Sup\left(\alpha_i,\alpha_j\right)\in[0,1]$;

(2) $Sup\left(\alpha_i,\alpha_j\right)=Sup\left(\alpha_j,\alpha_i\right)$;

(3) $Sup(\alpha,\beta)\geq Sup(\gamma,\rho)$, if $d(\alpha,\beta)\leq d(\gamma,\rho)$.

To simplify Eq. (7), let

$$\varpi_j=\frac{\left(1+T\left(\alpha_j\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\alpha_j\right)\right)}, \tag{10}$$

then, Eq. (7) can be written as

$$IVIFPMM^S\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\left(\frac{1}{n!}\underset{\zeta\in T_n}{\oplus}\overset{n}{\underset{j=1}{\otimes}}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}, \tag{11}$$

For convenience, we call $\varpi=\left(\varpi_1,\varpi_2,...,\varpi_n\right)^T$ the power weight vector, satisfying $\sum_{j=1}^{n}\varpi_j=1$ and $\varpi_j\in[0,1](j=1,2,...,n)$.

According to the operational rules of IVIFNs given in Definition 4, the following theorem can be obtained.

Theorem 1. Let $\alpha_j\left(j=1,2,...,n\right)$ be a collection of IVIFNs and $S=\left(s_1,s_2,...,s_n\right)$ be a collection of parameters, where $s_j\left(j=1,2,...,n\right)$ is a non-negative real number. The aggregated value by the IVIFPMM operator is also an IVIFN and

$IVIFPMM^S\left(\alpha_1,\alpha_2,...,\alpha_n\right)=$

$$\left(\begin{array}{l}\left[\left(1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-a_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}},\left(1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-b_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}\right],\\ \left[1-\left(1-\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-c_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}},1-\left(1-\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-d_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}\right]\end{array}\right) \tag{12}$$

Proof. According to Definition 4 and Eq. (11), we can obtain

$$n\varpi_{\zeta(j)}\alpha_{\zeta(j)}=\left(\left[1-\left(1-a_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}},1-\left(1-b_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right],\left[c_{\zeta(j)}^{n\varpi_{\zeta(j)}},d_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right]\right),$$

and

$$\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}=\left(\left[\left(1-\left(1-a_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j},\left(1-\left(1-b_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right],\left[1-\left(1-c_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j},1-\left(1-d_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right]\right).$$

Therefore,

$$\underset{j=1}{\overset{n}{\otimes}}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}=$$

$$\left(\left[\prod_{j=1}^{n}\left(1-\left(1-a_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j},\prod_{j=1}^{n}\left(1-\left(1-b_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right],\left[1-\prod_{j=1}^{n}\left(1-c_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j},1-\prod_{j=1}^{n}\left(1-d_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right]\right).$$

Further,

$$\underset{\zeta\in T_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}=$$

$$\left(\left[1-\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-a_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right),1-\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-b_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)\right],\right.$$

$$\left.\left[\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-c_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right),\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-d_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right]\right).$$

Then,

$$\frac{1}{n!}\underset{\zeta\in T_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}=$$

$$\left(\left[1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-a_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}},1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-b_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}},\right],\right.$$

$$\left.\left[\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-c_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}},\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-d_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}}\right]\right).$$

Thus,

$$\left(\frac{1}{n!}\underset{\zeta\in T_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}=$$

$$\left(\left[\left(1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-a_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}},\left(1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-b_{\zeta(j)}\right)^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}\right],\right.$$

$$\left.\left[1-\left(1-\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-c_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}},1-\left(1-\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-d_{\zeta(j)}^{n\varpi_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}\right]\right).$$

Example 1. There are three IVIFNs, that is, $\alpha_1=\left(\left[0.1,0.2\right],\left[0.3,0.4\right]\right)$, $\alpha_2=\left(\left[0.2,0.3\right],\left[0.4,0.5\right]\right)$, and $\alpha_3=\left(\left[0.3,0.5\right],\left[0.2,0.3\right]\right)$. We utilize the IVIFPMM operator to aggregate them.

Step 1. Calculate the support degrees $Sup(\alpha_i,\alpha_j)(i,j=1,2,3)$. We can obtain

$$Sup(\alpha_1,\alpha_2)=0.8323 \quad Sup(\alpha_1,\alpha_3)=0.8697 \quad Sup(\alpha_2,\alpha_1)=0.8323 \quad Sup(\alpha_2,\alpha_3)=0.9626$$
$$Sup(\alpha_3,\alpha_1)=0.8697 \quad Sup(\alpha_3,\alpha_2)=0.9626\text{;}$$

Step 2. Calculate the power weight vector. We have

$$T(\alpha_1)=Sup(\alpha_1,\alpha_2)+Sup(\alpha_1,\alpha_3)=0.8323+0.8697=1.7020\text{;}$$

$$T(\alpha_2)=Sup(\alpha_2,\alpha_1)+Sup(\alpha_2,\alpha_3)=0.8323+0.9626=1.7949\text{;}$$

$$T(\alpha_3)=Sup(\alpha_3,\alpha_1)+Sup(\alpha_3,\alpha_2)=0.8697+0.9626=1.8323\text{.}$$

Then,

$$\varpi_1=\frac{1+T(\alpha_1)}{(1+T(\alpha_1))+(1+T(\alpha_2))+(1+T(\alpha_3))}=\frac{1+1.7020}{(1+1.7020)+(1+1.7949)+(1+1.8323)}=0.3244\text{.}$$

Similarly, we can get

$$\varpi_2=0.3356,\ \varpi_3=0.3400\text{.}$$

Step 3. Calculate the overall value $\alpha=([a,b],[c,d])$ by the IVIFPMM operator. Suppose $S=(2,3,4)$, then we have

$$\alpha=IVIFPMM^{(2,3,4)}(\alpha_1,\alpha_2,\alpha_3)=\left(\frac{1}{3!}\sum_{\zeta\in T_n}\prod_{j=1}^{3}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}},$$

$$a=\left(1-\left(\begin{array}{l}\left(1-\left(1-(1-0.1)^{4\varpi_1}\right)^2\times\left(1-(1-0.2)^{4\varpi_2}\right)^3\times\left(1-(1-0.3)^{4\varpi_3}\right)^4\right)\times\\ \left(1-\left(1-(1-0.1)^{4\varpi_1}\right)^2\times\left(1-(1-0.3)^{4\varpi_2}\right)^3\times\left(1-(1-0.2)^{4\varpi_3}\right)^4\right)\times\\ \left(1-\left(1-(1-0.2)^{4\varpi_1}\right)^2\times\left(1-(1-0.1)^{4\varpi_2}\right)^3\times\left(1-(1-0.3)^{4\varpi_3}\right)^4\right)\times\\ \left(1-\left(1-(1-0.2)^{4\varpi_1}\right)^2\times\left(1-(1-0.3)^{4\varpi_2}\right)^3\times\left(1-(1-0.1)^{4\varpi_3}\right)^4\right)\times\\ \left(1-\left(1-(1-0.3)^{4\varpi_1}\right)^2\times\left(1-(1-0.2)^{4\varpi_2}\right)^3\times\left(1-(1-0.1)^{4\varpi_3}\right)^4\right)\times\\ \left(1-\left(1-(1-0.3)^{4\varpi_1}\right)^2\times\left(1-(1-0.1)^{4\varpi_2}\right)^3\times\left(1-(1-0.2)^{4\varpi_3}\right)^4\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{2+3+4}}=0.2417$$

Similarly, we can get $b=0.3957$, $c=0.3828$, and $d=0.4989$. Thus, we have

$$\alpha=IVIFPMM^{(2,3,4)}(\alpha_1,\alpha_2,\alpha_3)=([0.2417,0.3957],[0.3828,0.4989])$$

Moreover, the IVIFPMM operator has the following properties.

Theorem 2. (Idempotency) Let $\alpha_j\,(j=1,2,\ldots,n)$ be a collection of IVIFNs, if $\alpha_j=\alpha=([a,b],[c,d])$ holds for all j, then,

$$IVIFPMM^{S}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\alpha\text{.} \tag{13}$$

Proof. Since $\alpha_j = \alpha = ([a,b],[c,d])$ holds for $j = 1,2,...,n$, we can get $Sup(\alpha_i, \alpha_j) = 1$ for $i, j = 1,2,...,n$. Thus, $\varpi_j = 1/n (j = 1,2,...,n)$ holds for $j = 1,2,...,n$. Therefore,

$$IVIFPMM^S(\alpha_1,\alpha_2,...,\alpha_n) = \left(\frac{1}{n!}\sum_{\zeta\in T_n}\prod_{j=1}^{n}\left(n\frac{1}{n}\alpha_j\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}} = \left(\frac{1}{n!}\sum_{\zeta\in T_n}\prod_{j=1}^{n}(\alpha_j)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}$$

$$= \left(\frac{1}{n!}\sum_{\zeta\in T_n}(\alpha)^{\sum_{j=1}^{n}s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}} = \left(\frac{1}{n!}n!(\alpha)^{\sum_{j=1}^{n}s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}} = \alpha .$$

Theorem 3. (Boundedness). Let $\alpha_j (j = 1,2,...,n)$ be a collection of IVIFNs, $\alpha^- = \min(\alpha_1,\alpha_2,...,\alpha_n) = ([a,b],[c,d])$, and $\alpha^+ = \max(\alpha_1,\alpha_2,...,\alpha_n) = ([e,f],[p,q])$. Then,

$$x \le IVIFPMM^S(\alpha_1,\alpha_2,...,\alpha_n) \le y \tag{14}$$

where $x = \left(\frac{1}{n!}\sum_{\zeta\in T_n}\prod_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha^-\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}$ *and* $y = \left(\frac{1}{n!}\sum_{\zeta\in T_n}\prod_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha^+\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}$.

Proof. From Definition 4, we can obtain

$$n\varpi_{\zeta(j)}\alpha_{\zeta(j)} \ge n\varpi_{\zeta(j)}\alpha^- ,$$

and

$$\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j} \ge \left(n\varpi_{\zeta(j)}\alpha^-\right)^{s_j} .$$

Therefore,

$$\otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j} \ge \otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha^-\right)^{s_j} .$$

Further,

$$\oplus_{\zeta\in T_n}\otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j} \ge \oplus_{\zeta\in T_n}\otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha^-\right)^{s_j} ,$$

and

$$\frac{1}{n!}\oplus_{\zeta\in T_n}\otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j} \ge \frac{1}{n!}\oplus_{\zeta\in T_n}\otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha^-\right)^{s_j} .$$

Thus,

$$\left(\frac{1}{n!}\oplus_{\zeta\in T_n}\otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}} \ge \left(\frac{1}{n!}\oplus_{\zeta\in T_n}\otimes_{j=1}^{n}\left(n\varpi_{\zeta(j)}\alpha^-\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}} = x ,$$

which means that $x \le IVIFPMM^S(\alpha_1,\alpha_2,...,\alpha_n)$.

Similarly, we can also prove $IVIFPMM^S(\alpha_1,\alpha_2,...,\alpha_n) \le y$. Thus, the proof of Theorem 3 is completed.

The most important feature of IVIFPMM operator is that it not only reduces or eliminates the negative effects of decision makers' unreasonable evaluations on final decision results but also reflects the interrelationship among any aggregated IVIFNs. In addition, we can obtain some special operators of IVIFPMM with respect to the change of the parameters. In the following, we discuss some special cases of the IVIFPMM operator.

Case 1. If $S=(1,0,\ldots,0)$, then the IVIFPMM operator reduces to the IVIFPA [29] operator. That is,

$$IVIFPMM^{(1,0,\ldots,0)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\overset{n}{\underset{j=1}{\oplus}}\left(\frac{\left(1+T(\alpha_j)\right)\alpha_j}{\sum_{j=1}^{n}\left(1+T(\alpha_j)\right)}\right)=\overset{n}{\underset{j=1}{\oplus}}\left(\varpi_j\alpha_j\right)$$

$$=\left(\left[1-\prod_{j=1}^{n}\left(1-a_j\right)^{\varpi_j},1-\prod_{j=1}^{n}\left(1-b_j\right)^{\varpi_j}\right],\left[\prod_{j=1}^{n}c_j^{\varpi_j},\prod_{j=1}^{n}d_j^{\varpi_j}\right]\right). \quad (15)$$

In this case, if $Sup(\alpha_i,\alpha_j)=t$ for all $i\neq j$, then $n\varpi_j=1$, then the IVIFPMM operator reduces to the interval-valued intuitionistic fuzzy average (IVIFA) operator [22], that is,

$$IVIFPMM^{(1,0,\ldots,0)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\left[1-\left(\prod_{i=1}^{n}(1-a_i)\right)^{1/n},1-\left(\prod_{i=1}^{n}(1-b_i)\right)^{1/n}\right],\left[\prod_{i=1}^{n}c_i^{1/n},\prod_{i=1}^{n}d_i^{1/n}\right]\right)$$

$$=IVIFA(\alpha_1,\alpha_2,\ldots,\alpha_n). \quad (16)$$

Case 2. If $S=(1,1,0,0,\ldots,0)$, then the IVIFPMM operator reduces to the IVIFPBM operator proposed by Liu and Li [26], that is,

$$IVIFPMM^{(1,1,0,0,\ldots,0)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\frac{1}{n(n-1)}\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\otimes}}\left((\varpi_i\alpha_i)\oplus(\varpi_j\alpha_j)\right)\right)^{\frac{1}{2}}$$

$$\left(\left[\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-(1-a_i)^{\varpi_i}\right)\left(1-(1-a_j)^{\varpi_j}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-(1-b_i)^{\varpi_i}\right)\left(1-(1-b_j)^{\varpi_j}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right],\right.$$

$$\left.\left[1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-c_i^{\varpi_i}\right)\left(1-c_j^{\varpi_j}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-d_i^{\varpi_i}\right)\left(1-d_j^{\varpi_j}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right]\right). \quad (17)$$

In this case, if $Sup(\alpha_i,\alpha_j)=t$ for all $i\neq j$, then $n\varpi_j=1$, then the IVIFPMM operator reduces to the IVIFBM operator [23] (when $s=t=1$), that is,

$$IVIFPMM^{(1,1,0,0,\ldots,0)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-a_ia_j\right)^{\frac{1}{n(n-1)}}\right)^{1/2},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-b_ib_j\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right],\right.$$

$$\left.\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-c_i)(1-c_j)\right)^{\frac{1}{n(n-1)}}\right)^{1/2},1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-d_i)(1-d_j)\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right]\right). \quad (18)$$

Case 3. If $S=(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})$, then the IVIFPMM operator reduces to the IVIFPMSM operator proposed by Liu et al. [28], that is,

$$IVIFPMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\frac{1}{C_n^k}\underset{1\le i_1<i_2<\ldots<i_k}{\oplus}\overset{n}{\underset{j=1}{\otimes}}\left(\varpi_{i_j}\alpha_{i_j}\right)\right)^{1/k}$$

$$\left(\left[\left(1-\prod_{1\le i_1<i_2\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-a_{i_j}\right)^{\varpi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le i_1<i_2\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-b_{i_j}\right)^{\varpi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right],\right.$$

$$\left.\left[1-\left(1-\prod_{1\le i_1<i_2\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-c_{i_j}^{\varpi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},1-\left(1-\prod_{1\le i_1<i_2\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-d_{i_j}^{\varpi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right]\right). \quad (19)$$

In this case, if $Sup(\alpha_i,\alpha_j)=t$ for all $i\ne j$, then $n\varpi_j=1$, then the IVIFPMM operator reduces to the IVIFMSM operator [25], that is,

$$IVIFPMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(\alpha_1,\alpha_2,\ldots,\alpha_n)=$$

$$\left(\left[\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}a_{i_j}\right)\right)^{C_n^k}\right)^{1/k},\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}b_{i_j}\right)\right)^{C_n^k}\right)^{1/k}\right],\right.$$

$$\left.\left[1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-c_{i_j}\right)\right)\right)^{C_n^k}\right)^{1/k},1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-d_{i_j}\right)\right)\right)^{C_n^k}\right)^{1/k}\right]\right). \quad (20)$$

Case 4. If $S=(1,1,\ldots,1)$ or $S=(1/n,1/n,\ldots,1/n)$, then the IVIFPMM operator reduces to the following:

$$IVIFPMM^{(1,1,\ldots,1)\,or\,(1/n,1/n,\ldots,1/n)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\overset{n}{\underset{j=1}{\otimes}}\left(n\frac{\left(1+T(\alpha_j)\right)\alpha_j}{\sum_{j=1}^{n}\left(1+T(\alpha_j)\right)}\right)^{1/n}=\overset{n}{\underset{j=1}{\otimes}}\left(n\varpi_j\alpha_j\right)^{1/n}$$

$$=\left(\left[\prod_{j=1}^{n}\left(1-\left(1-a_j\right)^{n\varpi_j}\right)^{1/n},\prod_{j=1}^{n}\left(1-\left(1-b_j\right)^{n\varpi_j}\right)^{1/n}\right],\left[1-\prod_{j=1}^{n}\left(1-c_j^{n\varpi_j}\right)^{1/n},1-\prod_{j=1}^{n}\left(1-d_j^{n\varpi_j}\right)^{1/n}\right]\right). \quad (21)$$

In this case, if $Sup(\alpha_i,\alpha_j)=t$ for all $i\ne j$, then $n\varpi_j=1$, then the IVIFPMM operator reduces to the interval-valued intuitionistic fuzzy geometric (IVIFG) operator [22], that is,

$$IVIFPMM^{(1,1,\ldots,1)\,or\,(1/n,1/n,\ldots,1/n)}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\left[\prod_{i=1}^{n}a_i^{1/n},\prod_{i=1}^{n}b_i^{1/n}\right],\left[1-\left(\prod_{i=1}^{n}(1-c_i)\right)^{1/n},1-\left(\prod_{i=1}^{n}(1-d_i)\right)^{1/n}\right]\right).$$

(22)

3.2. The Interval-Valued Intuitionistic Fuzzy Weighted Power Muirhead Mean (IVIFWPMM) Operator

Definition 9. Let α_j $(j=1,2,\ldots,n)$ be a collection of IVIFNs and $S=(s_1,s_2,\ldots,s_n)$ be a collection of parameters, where s_j $(j=1,2,\ldots,n)$ is a non-negative real number. Let $w=(w_1,w_2,\ldots,w_n)^T$ be the weight vector of $\alpha_j\left(j=1,2,...,n\right)$, satisfying the condition that $\sum_{j=1}^{n} w_j = 1$ and $0 \le w_j \le 1\left(j=1,2,...,n\right)$. Then, the interval-valued intuitionistic fuzzy weighted power Muirhead mean (IVIFWPMM) operator is defined as,

$$IVIFWPMM^{S}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\left(\frac{1}{n!}\underset{\zeta\in T_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\frac{w_{\zeta(j)}\left(1+T\left(\alpha_{\zeta(j)}\right)\right)}{\sum_{j=1}^{n}w_j\left(1+T\left(\alpha_j\right)\right)}\alpha_{\zeta(j)}\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}, \tag{23}$$

Where

$$T\left(\alpha_j\right)=\sum_{i=1,i\ne j}^{n}Sup\left(\alpha_i,\alpha_j\right), \tag{24}$$

And

$$Sup\left(\alpha_i,\alpha_j\right)=1-d\left(\alpha_i,\alpha_j\right), \tag{25}$$

where $\zeta\left(j\right)\left(j=1,2,...,n\right)$ denotes any permutation of $\left(1,2,...,n\right)$, T_n represents all possible permutations of $\left(1,2,...,n\right)$, and n is the balancing coefficient. $d\left(\alpha_i,\alpha_j\right)$ represents the distance between α_i and α_j, and $Sup\left(\alpha_i,\alpha_j\right)$ is the support for α_i from α_j, satisfying the following conditions in Definition 8.

For convenience, let

$$\delta_j=\frac{w_j\left(1+T\left(\alpha_j\right)\right)}{\sum_{j=1}^{n}w_j\left(1+T\left(\alpha_j\right)\right)}, \tag{26}$$

then, we call $\delta=\left(\delta_1,\delta_2,...,\delta_n\right)^T$ the power weight vector, satisfying $\sum_{j=1}^{n}\delta_j=1$ and $\delta_j\in\left[0,1\right]\left(j=1,2,...,n\right)$. Hence, Eq. (23) can be simplified as,

$$IVIFWPMM^{S}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\left(\frac{1}{n!}\underset{\zeta\in T_n}{\oplus}\underset{j=1}{\overset{n}{\otimes}}\left(n\delta_{\zeta(j)}\alpha_{\zeta(j)}\right)^{s_j}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}, \tag{27}$$

Based on the operations shown in Definition 4, the following theorem can be derived.

Theorem 4. Let $\alpha_j\left(j=1,2,...,n\right)$ be a collection of IVIFNs and $S=\left(s_1,s_2,...,s_n\right)$ be a collection of parameters, where $s_j\left(j=1,2,...,n\right)$ is a non-negative real number, then the aggregated value by the IVIFWPMM operator is also an IVIFN and

$IVIFWPMM^{S}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=$

$$\left(\left[\left(1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-a_{\zeta(j)}\right)^{n\delta_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}},\left(1-\left(\prod_{\zeta\in T_n}\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-b_{\zeta(j)}\right)^{n\delta_{\zeta(j)}}\right)^{s_j}\right)\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}\right],\right.$$
$$\left.\left[1-\left(1-\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-c_{\zeta(j)}^{n\delta_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}},1-\left(1-\left(\prod_{\zeta\in T_n}\left(1-\prod_{j=1}^{n}\left(1-d_{\zeta(j)}^{n\delta_{\zeta(j)}}\right)^{s_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}s_j}}\right]\right), \quad (28)$$

The proof of Theorem 4 is similar to the poof of that of Theorem 1.

4. A Method to MAGDM in the Interval-Valued Intuitionistic Fuzzy Context

In the present section, we propose a novel approach to MAGDM based on proposed operators. A typical MAGDM problem in IVIF context can be described as follows. Let $X=\{x_1,x_2,...,x_m\}$ be m alternatives, $G=\{G_1,G_2,...,G_n\}$ be n attributes, $w=(w_1,w_2,...,w_n)^T$ be the weight vector, satisfying $\sum_{i=1}^{n}w_i=1$ and $w_i\geq 0, i=1,2,...,n$. Let $D=\{D_1,D_2,...,D_t\}$ be a set of decision makers with the weight vector being $\gamma=(\gamma_1,\gamma_2,...,\gamma_t)^T$ the weight vector, satisfying $\gamma_k, k=1,2,...,t$ and $\sum_{k=1}^{t}\gamma_k=1$. For a decision maker D_k, he/she is required to express his/her preference information by an IVIFN $\alpha_{ij}^k=\left(\left[a_{ij}^k,b_{ij}^k\right],\left[c_{ij}^k,d_{ij}^k\right]\right)$ for an alternative x_i with respect to attribute G_j. In the following steps, we propose a novel method to MAGDM in which attribute values take the form of IVIFNs based on the proposed operators.

Step 1. Standardize the decision matrices according to the following equation,

$$r_{ij}^k=\begin{cases}\left(\left[a_{ij}^k,b_{ij}^k\right],\left[c_{ij}^k,d_{ij}^k\right]\right) & \textit{where } G_j \textit{ is positive type}\\ \left(\left[c_{ij}^k,d_{ij}^k\right],\left[a_{ij}^k,b_{ij}^k\right]\right) & \textit{where } G_j \textit{ is negative type}\end{cases}, \quad (29)$$

Step 2. Calculate the supports $Sup\left(\alpha_{ij}^k,\alpha_{ij}^d\right)$ according to the following equation,

$$Sup\left(\alpha_{ij}^k,\alpha_{ij}^d\right)=1-d\left(\alpha_{ij}^k,\alpha_{ij}^d\right)\left(k,d=1,2,...,t;k\neq d;i=1,2...,m;j=1,2,...,n\right), \quad (30)$$

where $d\left(\alpha_{ij}^k,\alpha_{ij}^d\right)$ is the Hamming distance between α_{ij}^k and α_{ij}^k.

Step 3. Calculate $T\left(\alpha_{ij}^k\right)$ by

$$T\left(\alpha_{ij}^k\right)=\sum_{k=1,k\neq d}^{n}Sup\left(\alpha_{ij}^k,\alpha_{ij}^d\right)\left(k,d=1,2,...,t;i=1,2...,m;j=1,2,...,n\right), \quad (31)$$

Step 4. Compute the power weights δ_{ij}^k associated with the IVIFN α_{ij}^k by

$$\delta_{ij}^k=\frac{\gamma_k\left(1+T\left(\alpha_{ij}^k\right)\right)}{\sum_{k=1}^{t}\gamma_k\left(1+T\left(\alpha_{ij}^k\right)\right)}, \quad (32)$$

where $k=1,2,...,t, i=1,2...,m, j=1,2,...,n$, $\delta_{ij}^k>0$ and $\sum_{k=1}^{t}\delta_{ij}^k=1$.

Step 5. Utilized the interval-valued intuitionistic fuzzy power weighted average (IVIFPWA) operator proposed by He et al. [29] to aggregate individual decision matrix, that is,

$$\alpha_{ij} = IVIFPWA\left(\alpha_{ij}^1, \alpha_{ij}^2, ..., \alpha_{ij}^t\right) = \left(\left[1 - \prod_{k=1}^{t}\left(1 - a_{ij}^k\right)^{\delta_{ij}^k}, 1 - \prod_{k=1}^{t}\left(1 - b_{ij}^k\right)^{\delta_{ij}^k}\right], \left[\prod_{k=1}^{t}\left(c_{ij}^k\right)^{\delta_{ij}^k}, \prod_{k=1}^{t}\left(d_{ij}^k\right)^{\delta_{ij}^k}\right]\right), \quad (33)$$

Thus, a collective decision matrix can be obtained.

Step 6. Calculate the supports $Sup\left(\alpha_{il}, \alpha_{if}\right)$ by

$$Sup\left(\alpha_{il}, \alpha_{if}\right) = 1 - d\left(\alpha_{il}, \alpha_{if}\right), \quad (34)$$

where $i = 1, 2, ..., n; l, f = 1, 2, ..., n; l \neq f$ and $d\left(\alpha_{il}, \alpha_{if}\right)$ is the Hamming distance between α_{il} and a_{if}.

Step 7. Compute $T\left(\alpha_{ij}\right)$ by

$$T\left(\alpha_{ij}\right) = \sum\nolimits_{f,l=1, f \neq l}^{n} Sup\left(\alpha_{il}, \alpha_{if}\right), \quad (35)$$

where $i = 1, 2, ..., m; l, f = 1, 2, ..., n$.

Step 8. Calculate the power weight η_{ij} associated with the IVIFN α_{ij} according to the following formula,

$$\eta_{ij} = \frac{w_j\left(1 + T\left(\alpha_{ij}\right)\right)}{\sum_{j=1}^{n} w_j\left(1 + T\left(\alpha_{ij}\right)\right)}, \quad (36)$$

where $i = 1, 2, ..., m; j = 12, ..., n$.

Step 9. For alternative $x_i\left(i = 1, 2, ..., n\right)$, utilize the IVIFWPMM operator

$$\alpha_i = IVIFWPMM^S\left(\alpha_{i1}, \alpha_{i2}, ..., \alpha_{in}\right), \quad (37)$$

to aggregate attributes, and an overall evaluation value can be obtained.

Step 10. Rank the overall evaluation values $\alpha_i\left(i = 1, 2..., n\right)$ according to Definition 3.

Step 11. Rank alternatives according to the rank of the overall values, and choose the best alternative.

5. Case Analysis

In Section 3, we proposed the IVIFWPMM operator, which is a powerful and useful information aggregation tool for interval-valued intuitionistic fuzzy information. Additionally, Section 4 introduced a new approach for interval-valued intuitionistic fuzzy MAGDM. To validate the newly developed MAGDM method, this section applies it to a real decision-making problem. Talent strategy is a major, macroscopic, and overall conception and arrangement for the cultivation. In June 2010, the Central Committee of the Communist Party of China and the State Council issued the "National Medium and Long Term Talent Development Plan (2010–2020)" and issued a notice requesting all localities and departments to conscientiously implement the reality. In order to train more talents for the country and society, domestic universities generally increase the proportion of admissions. So, the number of college students is increasing. Considering that the size of the existing library is no longer sufficient for all teachers and students, a university is preparing to build a new library. After the initial bidding, the university decides to take the seat of the new library from the four listed builders ($x_i, i = 1, 2, 3, 4$). In order to choose the most suitable builder, the four alternatives are evaluated from four perspectives. They are, G_1: social influence, G_2: quality, G_3: reputation, and G_4: service attitude. Weight vector of the four attributes is $w_i = (0.3, 0.4, 0.1, 0.2)^T$. Three experts $D_t\left(t = 1, 2, 3\right)$ in civil engineering and project management are invited to evaluate the four builders. Weight vector of decision makers is $\gamma = \left(0.32, 0.45, 0.23\right)^T$. The three experts are required to utilize

IVIFNs to express their preference information over alternatives, and three interval-valued intuitionistic fuzzy decision matrices $R^t = (\alpha_{ij}^t)_{4\times 4} \left(t = 1, 2, 3\right)$ are constructed. (See Tables 1–3).

Table 1. The IVIF decision matrix R^1 given by D_1.

	G_1	G_2	G_3	G_4
x_1	([0.5, 0.6],[0.3, 0.4])	([0.4, 0.6],[0.2, 0.3])	([0.6, 0.8],[0.1, 0.2])	([0.6, 0.7],[0.1, 0.3])
x_2	([0.7, 0.8],[0.1, 0.2])	([0.5, 0.6],[0.1, 0.2])	([0.3, 0.5],[0.3, 0.4])	([0.7, 0.8],[0.1, 0.2])
x_3	([0.4, 0.6],[0.2, 0.3])	([0.5, 0.7],[0.1, 0.2])	([0.4, 0.6],[0.3, 0.4])	([0.5, 0.6],[0.2, 0.3])
x_4	([0.6, 0.7],[0.1, 0.3])	([0.5, 0.6],[0.2, 0.3])	([0.4, 0.5],[0.2, 0.4])	([0.4, 0.7],[0.2, 0.3])

Table 2. The IVIF decision matrix R^2 given by D_2.

	G_1	G_2	G_3	G_4
x_1	([0.7, 0.8],[0.1, 0.2])	([0.5, 0.6],[0.1, 0.3])	([0.4, 0.5],[0.2, 0.4])	([0.5, 0.8],[0.1, 0.2])
x_2	([0.5, 0.6],[0.2, 0.3])	([0.6, 0.7],[0.2, 0.3])	([0.5, 0.5],[0.2, 0.3])	([0.6, 0.7],[0.1, 0.2])
x_3	([0.4, 0.5],[0.1, 0.2])	([0.6, 0.8],[0.1, 0.2])	([0.5, 0.7],[0.2, 0.3])	([0.6, 0.7],[0.1, 0.3])
x_4	([0.5, 0.6],[0.2, 0.3])	([0.4, 0.5],[0.3, 0.4])	([0.6, 0.8],[0.1, 0.2])	([0.5, 0.8],[0.1, 0.2])

Table 3. The IVIF decision matrix R^3 given by D_3.

	G_1	G_2	G_3	G_4
x_1	([0.6, 0.6],[0.2, 0.3])	([0.5, 0.8],[0.1, 0.2])	([0.5, 0.7],[0.1, 0.2])	([0.6, 0.7],[0.2, 0.3])
x_2	([0.7, 0.8],[0.1, 0.2])	([0.4, 0.5],[0.3,0.4])	([0.6, 0.7],[0.1, 0.2])	([0.5, 0.6],[0.2, 0.3])
x_3	([0.6 ,0.6],[0.2, 0.3])	([0.5, 0.7],[0.2, 0.3])	([0.6, 0.8],[0.1, 0.2])	([0.5, 0.6],[0.3, 0.4])
x_4	([0.4, 0.5],[0.3, 0.4])	([0.6, 0.8],[0.1, 0.2])	([0.5, 0.6],[0.2, 0.3])	([0.7, 0.8],[0.1, 0.2])

Table 4. The collective IVIF decision matrix *R*.

	G_1	G_2
x_1	([0.6217, 0.7057],[0.1676, 0.2749])	([0.4701, 0.6581],[0.1247, 0.2737])
x_2	([0.6250, 0.7293],[0.1354, 0.2387])	([0.5289, 0.6305],[0.1755, 0.2811])
x_3	([0.4533, 0.5582],[0.1469, 0.2504])	([0.5475, 0.7498],[0.1172, 0.2194])
x_4	([0.5146, 0.6160],[0.1760, 0.3201])	([0.4843, 0.6221],[0.2049, 0.3112])
	G_1	G_2
x_1	([0.4961, 0.6708],[0.1356, 0.2711])	([0.5583, 0.7495],[0.1174, 0.2505])
x_2	([0.4712, 0.5542],[0.1945, 0.2999])	([0.6163, 0.7187],[0.1169, 0.2191])
x_3	([0.4965, 0.7004],[0.1942, 0.2996])	([0.5473, 0.6481],[0.1612, 0.3205])
x_4	([0.5198, 0.6838],[0.1472, 0.2749])	([0.5291, 0.7729],[0.1243, 0.2271])

5.1. The decision-making process

In this subsection, we use the method introduced in Section 4 to determine the optimal alternative. The decision-making steps are presented as follows.

Step 1. As all attributes are benefit type, the original decision matrices do not need to be normalized.

Step 2. Calculate the $Sup\left(\alpha_{ij}^k, \alpha_{ij}^d\right)$ according to Eq. (30). For convenience, we utilize the symbol S_d^k to represent the support between α_{ij}^k and α_{ij}^d ($i, j = 1, 2, 3, 4$; $k, d = 1, 2, 3$; $k \neq d$). Hence, we obtain the following results

$$S_2^1 = S_1^2 = \begin{bmatrix} 0.800 & 0.950 & 0.800 & 0.925 \\ 0.850 & 0.900 & 0.900 & 0.950 \\ 0.925 & 0.950 & 0.900 & 0.925 \\ 0.925 & 0.900 & 0.800 & 0.900 \end{bmatrix} \quad S_3^1 = S_1^3 = \begin{bmatrix} 0.925 & 0.875 & 0.950 & 0.975 \\ 1.000 & 0.850 & 0.775 & 0.850 \\ 0.950 & 0.950 & 0.800 & 0.950 \\ 0.825 & 0.875 & 0.925 & 0.850 \end{bmatrix}$$

$$S_3^2 = S_2^3 = \begin{bmatrix} 0.875 & 0.925 & 0.850 & 0.900 \\ 0.850 & 0.850 & 0.875 & 0.900 \\ 0.875 & 0.900 & 0.900 & 0.875 \\ 0.900 & 0.775 & 0.875 & 0.950 \end{bmatrix}.$$

Step 3. Calculate $T\left(\alpha_{ij}^k\right)$ according to Eq. (31). For convenience, we use the symbol T^k to represent the values $T\left(\alpha_{ij}^k\right)\left(i, j = 1, 2, 3, 4; k = 1, 2, 3\right)$

$$T^1 = \begin{bmatrix} 1.725 & 1.825 & 1.750 & 1.900 \\ 1.850 & 1.750 & 1.675 & 1.800 \\ 1.875 & 1.900 & 1.700 & 1.875 \\ 1.750 & 1.775 & 1.725 & 1.750 \end{bmatrix} T^2 = \begin{bmatrix} 1.675 & 1.875 & 1.650 & 1.825 \\ 1.700 & 1.750 & 1.775 & 1.850 \\ 1.800 & 1.850 & 1.800 & 1.800 \\ 1.825 & 1.675 & 1.675 & 1.850 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 1.800 & 1.800 & 1.800 & 1.875 \\ 1.850 & 1.700 & 1.650 & 1.750 \\ 1.825 & 1.850 & 1.700 & 1.825 \\ 1.725 & 1.650 & 1.800 & 1.800 \end{bmatrix}.$$

Step 4. For a decision maker D_k, calculate his/her power weight associated with the IVIFN α_{ij}^k on the basis of his/her weight γ_k, according to Eq. (32). For convenience, we use the symbol δ^k to represent the values $\delta_{ij}^k\left(i, j = 1, 2, 3, 4; k = 1, 2, 3\right)$. Therefore, we can obtain the following

$$\delta^1 = \begin{bmatrix} 0.3206 & 0.3181 & 0.3239 & 0.3244 \\ 0.3278 & 0.3213 & 0.3154 & 0.3187 \\ 0.3251 & 0.3238 & 0.3148 & 0.3251 \\ 0.3168 & 0.3287 & 0.3206 & 0.3136 \end{bmatrix} \quad \delta^2 = \begin{bmatrix} 0.4426 & 0.4553 & 0.4390 & 0.4444 \\ 0.4367 & 0.4519 & 0.4601 & 0.4562 \\ 0.4453 & 0.4475 & 0.4590 & 0.4453 \\ 0.4576 & 0.4456 & 0.4426 & 0.4570 \end{bmatrix}$$

$$\delta^3 = \begin{bmatrix} 0.2368 & 0.2266 & 0.2371 & 0.2312 \\ 0.2355 & 0.2268 & 0.2245 & 0.2251 \\ 0.2296 & 0.2287 & 0.2262 & 0.2296 \\ 0.2256 & 0.2257 & 0.2368 & 0.2294 \end{bmatrix}.$$

Step 5. Utilize the IVIF weighted PA (IVIFWPA) operator to aggregate individual decision matrices into a collective one, as shown in Table 4. The calculation process of the IVIFPWA operator can be found as Eq. (33).

Step 6. Calculate the support between α_{il} and α_{if}, that is, $Sup\left(\alpha_{il}, \alpha_{if}\right)$, according to Eq. (34). For convenience, we use the symbol S^{lf} to represent the value $Sup\left(\alpha_{il}, \alpha_{if}\right)\left(i, l, f = 1, 2, 3, 4; l \neq f\right)$. Hence, we can obtain the following results:

$$S^{12} = S^{21} = \left(0.9392, 0.9306, 0.9134, 0.9814\right) \quad S^{13} = S^{31} = \left(0.9509, 0.8877, 0.9295, 0.9632\right)$$

$$S^{14} = S^{41} = \left(0.9545, 0.9856, 0.9329, 0.9210\right) \quad S^{23} = S^{32} = \left(0.9870, 0.9570, 0.9356, 0.9522\right)$$

$$S^{24} = S^{42} = \left(0.9475, 0.9260, 0.9382, 0.9099\right) \quad S^{34} = S^{43} = \left(0.9551, 0.8830, 0.9608, 0.9577\right).$$

Step 7. Calculate the support $T\left(\alpha_{ij}\right)$ according to Eq. (35). Similarly, we use the symbol T_{ij} to denote the value $T\left(\alpha_{ij}\right)$ for simplicity, and we can obtain the following matrix:

$$T=\begin{bmatrix} 2.8446 & 2.8737 & 2.8930 & 2.8572 \\ 2.8040 & 2.8136 & 2.7277 & 2.7946 \\ 2.7759 & 2.7872 & 2.8259 & 2.8320 \\ 2.8656 & 2.8436 & 2.8732 & 2.7886 \end{bmatrix}.$$

Step 8. Calculate the power weight η_{ij} associated with the IVIFN α_{ij} according to Eq. (36), and we have

$$\eta=\begin{bmatrix} 0.2985 & 0.4010 & 0.1008 & 0.1997 \\ 0.3004 & 0.4016 & 0.0982 & 0.1998 \\ 0.2983 & 0.3990 & 0.1008 & 0.2019 \\ 0.3018 & 0.4001 & 0.1009 & 0.1972 \end{bmatrix}.$$

Step 9. For alternative $x_i\left(i=1,2,3,4\right)$, utilize the IVIFWPMM operator to calculate the overall evaluation $\alpha_i\left(i=1,2,3,4\right)$. Without the loss of generality, let $S=(1, 1, 1, 1)$, and the overall evaluation values are shown as follows:

$$\alpha_1=\left[\left(0.8819,0.9343\right),\left(0.0237,0.0563\right)\right] \quad \alpha_2=\left[\left(0.8917,0.9249\right),\left(0.0272,0.0531\right)\right]$$

$$\alpha_3=\left[\left(0.8712,0.9259\right),\left(0.0276,0.0570\right)\right] \quad \alpha_4=\left[\left(0.8726,0.9300\right),\left(0.0286,0.0590\right)\right].$$

Step 10. Calculate the score values $S\left(\alpha_i\right)\left(i=1,2,3,4\right)$ of the overall evaluation values, and we can get

$$S\left(\alpha_1\right)=0.8680 \quad S\left(\alpha_2\right)=0.8682 \; S\left(\alpha_3\right)=0.8563 \quad S\left(\alpha_4\right)=0.8575 .$$

Step 11. According to the score values $S\left(\alpha_i\right)\left(i=1,2,3,4\right)$, the ranking orders of the alternatives can be determined, that is, $A_2 \succ A_1 \succ A_4 \succ A_3$ .Therefore, A_2 is the best alternative.

5.2. Sensitivity analysis

As we know, the vector of parameter *S* has a significant role in the decision results. In the following section, we investigate the influence of the parameters on the score function and the final decision results. As shown above, the IVIFWPMM operators are used to calculate the comprehensive evaluation values in step 9. Therefore, we assign different vectors of parameters in the IVIFWPMM operator and present the scores and ranking orders in Table 5.

Table 5. Scores and ranking orders with different *S* in the IVIFWPMM operator.

S	**Score function** $S\left(\alpha_i\right)\left(i=1,2,3,4\right)$	**Ranking orders**
$S = (1, 0, 0, 0)$	$S\left(\alpha_1\right)=0.5340 \quad S\left(\alpha_2\right)=0.5318$ $S\left(\alpha_3\right)=0.4980 \quad S\left(\alpha_4\right)=0.5011$	$A_1 \succ A_2 \succ A_4 \succ A_3$
$S = (1, 1, 0, 0)$	$S\left(\alpha_1\right)=0.7471 \quad S\left(\alpha_2\right)=0.7468$ $S\left(\alpha_3\right)=0.7256 \quad S\left(\alpha_4\right)=0.7277$	$A_1 \succ A_2 \succ A_4 \succ A_3$
$S = (1, 1, 1, 0)$	$S\left(\alpha_1\right)=0.8265 \quad S\left(\alpha_2\right)=0.8266$ $S\left(\alpha_3\right)=0.8113 \quad S\left(\alpha_4\right)=0.8129$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$S = (1, 1, 1, 1)$	$S\left(\alpha_1\right)=0.8680 \; S\left(\alpha_2\right)=0.8682$	$A_2 \succ A_1 \succ A_4 \succ A_3$

$S(\alpha_3) = 0.8563$ $S(\alpha_4) = 0.8575$

From Table 5, we can find out that with different S in the IVIFWPMM, the scores of comprehensive evaluation values and ranking orders are also different. In addition, when the interrelationship among more numbers of attributes is taken into consideration, the scores of comprehensive evaluation values increase. Therefore, the parameter vector S can be viewed as decision makers' attitude to pessimism and optimism. Decision makers can choose proper S according to reality and actual needs.

5.3. Comparison analysis

In subsection 5.1, we utilized the proposed method to solve the above example successfully, which has proven the availability of the newly developed method. In addition, we also analyzed the impacts of the parameters on the decision results in subsection 5.2. The sensitivity analysis illustrates the high flexibility of the proposed method. To further demonstrate its great superiorities, the present subsection compares the proposed method with some existing MAGDM methods. More specifically, we compare the proposed method with that proposed by Xu [22] based on the IVIF weighted average (IVIFWA) operator, that introduced by He et al. [29] based on the IVIFPWA operator, that presented by Xu and Chen [23] on the basis of the IVIF weighted BM (IVIFWBM) operator, that developed by Yu and Wu [24] based on the generalized IVIF weighted HM (GIVIFWHM) operator, that put forward by Sun and Xia [25] based on the IVIF weighted MSM (IVIFWMSM) operator, that proposed by Liu and Li [26] based on the IVIF weighted power BM (IVIFWPBM) operator, that introduced by Liu [27] based on the IVIF weighted power HM (IVIFWPHM) operator, and that developed by Liu et al. [28] based on the IVIF weighted MSM (IVIFWPMSM) operator. We use the above methods to solve the following example and compare their ranking orders. The example is revised from reference [25].

Example 2: There are five high technological enterprises $A_i\left(i=1,2,3,4,5\right)$. In order to choose the enterprises with highest innovation capability, decision-makers are required to evaluate the five alternatives from four attributes, that is, innovation resources input ability (G_1), research and development ability (G_2), manufacturing capacity and marketing ability (G_3), and innovation output capacity (G_4). The weight vector of the four attributes is $w=\left(0.2,0.1,0.3,0.4\right)^T$. For the attribute $G_j\left(j=1,2,3,4\right)$ of alternative $A_i\left(i=1,2,3,4,5\right)$, decision makers use an IVIFN $\alpha_{ij}=\left(\left[a_{ij},b_{ij}\right],\left[c_{ij},d_{ij}\right]\right)$ to express their evaluation values, and an IVIF decision matrix can be obtained as shown in Table 6.

Table 6. The IVIF decision matrix.

	G_1	G_2	G_3	G_4
x_1	([0.4, 0.5],[0.3, 0.4])	([0.4, 0.6],[0.2, 0.4])	([0.1, 0.3],[0.5, 0.6])	([0.3, 0.4],[0.3, 0.5])
x_2	([0.6, 0.7],[0.2, 0.3])	([0.6, 0.7],[0.2, 0.3])	([0.4, 0.7],[0.1, 0.2])	([0.5, 0.6],[0.1, 0.3])
x_3	([0.3 ,0.6],[0.3, 0.4])	([0.5, 0.6],[0.3, 0.4])	([0.5, 0.6],[0.1, 0.3])	([0.4, 0.5],[0.2, 0.4])
x_4	([0.7, 0.8],[0.1, 0.2])	([0.6, 0.7],[0.1, 0.3])	([0.3, 0.4],[0.1, 0.2])	([0.3, 0.7],[0.1, 0.2])
x_5	([0.3, 0.4],[0.2, 0.3])	([0.3, 0.5],[0.1, 0.3])	([0.2, 0.5],[0.4, 0.5])	([0.3, 0.4],[0.5, 0.6])

In the following, we utilize the above-mentioned methods to solve the example and present their results in Table 7.

Table 7. Decision-making results by different methods.

Method	**Score function** $S(\alpha_i)(i=1,2,3,4)$	**Ranking orders**
Method introduced by Xu [22]	$S(\alpha_1)=-0.0661$ $S(\alpha_2)=0.3904$ $S(\alpha_3)=0.2185$ $S(\alpha_4)=0.3962$ $S(\alpha_5)=-0.0396$	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$

Method given by He et al. [29] ($\lambda = 1$)	$S(\alpha_1) = -0.5178$ $S(\alpha_2) = -0.2335$ $S(\alpha_3) = -0.3945$ $S(\alpha_4) = -0.1933$ $S(\alpha_5) = -0.6314$	$A_4 \succ A_2 \succ A_3 \succ A_1 \succ A_5$
Method presented by Xu and Chen [23] ($s = t = 1$)	$S(\alpha_1) = -0.7085$ $S(\alpha_2) = -0.4866$ $S(\alpha_3) = -0.5854$ $S(\alpha_4) = -0.4648$ $S(\alpha_5) = -0.6912$	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
Method put forward by Yu and Wu [24] ($p = q = 1$)	$S(\alpha_1) = -0.6965$ $S(\alpha_2) = -0.4602$ $S(\alpha_3) = -0.5606$ $S(\alpha_4) = -0.4382$ $S(\alpha_5) = -0.6846$	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
Method proposed by Sun and Xia [25] ($k = 2$)	$S(\alpha_1) = -0.7085$ $S(\alpha_2) = -0.4866$ $S(\alpha_3) = -0.5854$ $S(\alpha_4) = -0.4648$ $S(\alpha_5) = -0.6912$	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
Method developed by Liu and Li [26] ($s = t = 1$)	$S(\alpha_1) = -0.1075$ $S(\alpha_2) = 0.3037$ $S(\alpha_3) = 0.1401$ $S(\alpha_4) = 0.3156$ $S(\alpha_5) = -0.0614$	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
Method raised by Liu [27] ($s = t = 1$)	$S(\alpha_1) = -0.0763$ $S(\alpha_2) = 0.3407$ $S(\alpha_3) = 0.1866$ $S(\alpha_4) = 0.3604$ $S(\alpha_5) = -0.0443$	$A_4 \succ A_2 \succ A_3 \succ A_1 \succ A_5$
Method proposed by Liu et al. [28] ($k = 2$)	$S(\alpha_1) = -0.1075$ $S(\alpha_2) = 0.3037$ $S(\alpha_3) = 0.1401$ $S(\alpha_4) = 0.3156$ $S(\alpha_5) = -0.0614$	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
The proposed method based on the IVIFWPMM operator $S = (0.5,0.5,0.5,0.5)$	$S(\alpha_1) = 0.6069$ $S(\alpha_2) = 0.7685$ $S(\alpha_3) = 0.6977$ $S(\alpha_4) = 0.7758$ $S(\alpha_5) = 0.6164$	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$

From Table 7, we find that the decision result derived by the proposed method and those obtained by others are the same, which proves the effectiveness and validity of the proposed method. However, the shortcomings and irrationalities of existing decision-making methods are obvious. Xu's [22] method is based on the IVIFWA operator, which does not consider the interrelationship among attributes. Additionally, Xu's [22] method does not consider to wipe off the bad influence of decision-makers' unreasonable evaluation values on the final decision results. In other words, if decision makers make unreasonable evaluations, the decision-making results are also unreasonable via Xu's [22] method. Compared with Xu's [22] method, our method is more flexible and robust. The advantages of the proposed approach are reflected in its ability to capture the interrelationship among attributes, and its efficiency in eliminating the bad effects of decision makers' unreasonable assessments on the results. Analogously, He et al.'s [29] method considers the power weighting vectors but fail to reflect the interrelationship among attributes. Thus, our method is more powerful than He et al.'s [29] method. Similarly, Xu and Chen's [23], Yu and Wu's [24], and Sun and Xia's [25] methods are based on BM, HM, and MSM, respectively. Thus, all of them have the capacity of reflecting the interrelationship among attributes. Nevertheless, they neglect the power weighting vectors. Our method takes not only the interrelationship among attributes but also the power weighting vectors into consideration. Thus, the newly introduced method has advantages over the methods proposed by Xu and Chen's [23], Yu and Wu's [24], and Sun and Xia's [25]. The methods developed by Liu and Li [26], Liu [27], and Liu et al. [28] are on the basis of the IVIFWPBM, IVIFWPHM, and IVIFWPMSM operators, respectively. Thus, all of the three methods not only focus on the power weighting vectors but also capture the interrelationship among attributes. More specifically, Liu and Li's [26] and Liu's [27] methods consider the interrelationship between any two attributes, and Liu et al.'s method [28] can capture the interrelationship among multiple attributes. Therefore, Liu et al.'s method [28] is better and more flexible than Liu and Li's [26] and Liu's [27] methods to some extent. However, all of them fail to consider the interrelationship among any attributes, which is precisely the most prominent advantage of the newly proposed method. In addition, as mentioned in Section 3, Liu and Li's [26] and Liu et al.'s [28] methods are special cases of the proposed method, which demonstrates the flexibility and generality of the proposed method. Hence, our method is of higher flexibility, powerfulness, and generality over existing interval-valued intuitionistic fuzzy MAGDM methods [22–29]. To sum up, the reasons why decision-makers should use the proposed MAGDM method are as follows. Firstly, decision-makers may provide extreme evaluation values due to the high complexity of real-life MAGDM problems. Our proposed method can reduce the bad influence of unreasonable evolution information, making the decision results more reliable. Secondly, there is usually an interrelationship among attributes, and our method can effectively deal with such kind of interrelationship. Additionally, our method is very flexible as there is a parameter vector in the proposed IVIFWPMM operator. Therefore, decision-makers should choose the proposed method to determine the best alternatives in real MAGDM procedure.

6. Conclusion remarks

The PA and MM operators have good performance in the process of information aggregation. PA makes the aggregation results more reasonable as it can eliminate the bad influence of unduly high or low aggregated arguments. MM is a powerful information technique, which reflects the interrelationship among any numbers of input variables. IVIFSs are a good tool to describe decision makers' preferential information in MAGDM. In order to fully exploit the advantages of PA, MM, and IVIFSs, this paper developed the IVIFPMM and IVIFWPMM operators. Further, a new MAGDM method with interval-valued intuitionistic fuzzy information was introduced. A builder selection problem was presented to demonstrate validity. Through comparison analysis, the superiorities and advantages can be found. To sum up, the contributions of this paper are two-fold. Firstly, we developed some new aggregation operators for IVIFSs. These newly proposed operators exhibit higher flexibility and powerfulness over most existing interval-valued intuitionistic fuzzy aggregation operators. Secondly, a new interval-valued intuitionistic fuzzy MAGDM method was proposed. In further works, we shall investigate more aggregation operators for IVIFSs.

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