

Vectorial variational principles in L^∞ and their characterisation through PDE systems

Article

Accepted Version

Ayanbayev, B. and Katzourakis, N. (2021) Vectorial variational principles in L^∞ and their characterisation through PDE systems. *Applied Mathematics & Optimization*, 83 (2). pp. 833-848. ISSN 1432-0606 doi: <https://doi.org/10.1007/s00245-019-09569-y> Available at <https://centaur.reading.ac.uk/83793/>

It is advisable to refer to the publisher's version if you intend to cite from the work. See [Guidance on citing](#).

To link to this article DOI: <http://dx.doi.org/10.1007/s00245-019-09569-y>

Publisher: Springer

All outputs in CentAUR are protected by Intellectual Property Rights law, including copyright law. Copyright and IPR is retained by the creators or other copyright holders. Terms and conditions for use of this material are defined in the [End User Agreement](#).

www.reading.ac.uk/centaur

CentAUR

Central Archive at the University of Reading

Reading's research outputs online

VECTORIAL VARIATIONAL PRINCIPLES IN L^∞ AND THEIR CHARACTERISATION THROUGH PDE SYSTEMS

BIRZHAN AYANBAYEV AND NIKOS KATZOURAKIS

ABSTRACT. We discuss two distinct minimality principles for general supremal first order functionals for maps and characterise them through solvability of associated second order PDE systems. Specifically, we consider Aronsson's standard notion of absolute minimisers and the concept of ∞ -minimal maps introduced more recently by the second author. We prove that C^1 absolute minimisers characterise a divergence system with parameters probability measures and that C^2 ∞ -minimal maps characterise Aronsson's PDE system. Since in the scalar case these different variational concepts coincide, it follows that the non-divergence Aronsson's equation has an equivalent divergence counterpart.

1. INTRODUCTION

Let $n, N \in \mathbb{N}$ and $H \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n})$ with $\Omega \subseteq \mathbb{R}^n$ an open set. In this paper we consider the supremal functional

$$(1.1) \quad E_\infty(u, \mathcal{O}) := \operatorname{ess\,sup}_{\mathcal{O}} H(\cdot, u, Du), \quad u \in W_{\operatorname{loc}}^{1,\infty}(\Omega; \mathbb{R}^N), \quad \mathcal{O} \Subset \Omega,$$

defined on maps $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$. In (1.1) and subsequently, we see the gradient as a matrix map $Du = (D_i u_\alpha)_{i=1 \dots n}^{\alpha=1 \dots N} : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^{N \times n}$. Variational problems for (1.1) have been pioneered by Aronsson in the 1960s in the scalar case $N = 1$ ([2]-[6]). Nowadays the study of such functionals (and of their associated PDEs describing critical points) form a fairly well-developed area of vivid interest, called Calculus of Variations in L^∞ . For pedagogical general introductions to the theme we refer to [10, 19, 32].

One of the main difficulties in the study of (1.1) which prevents us from utilising the standard machinery of Calculus of Variations for conventional (integral) functionals as e.g. in [24] is that it is non-local, in the sense that a global minimiser u of $E_\infty(\cdot, \Omega)$ in $W_g^{1,\infty}(\Omega; \mathbb{R}^N)$ for some fixed boundary data g may not minimise $E_\infty(\cdot, \mathcal{O})$ in $W_u^{1,\infty}(\mathcal{O}; \mathbb{R}^N)$. Namely, global minimisers are not generally local minimisers, a property which is automatic for integral functionals. The remedy proposed by Aronsson (adapted) to the vector case is to build locality into the minimality notion:

Definition 1. Let $u \in W_{\operatorname{loc}}^{1,\infty}(\Omega; \mathbb{R}^N)$. We say that u is an *absolute minimiser* of (1.1) on Ω if

$$(1.2) \quad \left. \begin{array}{l} \forall \mathcal{O} \Subset \Omega, \\ \forall \phi \in W_0^{1,\infty}(\mathcal{O}; \mathbb{R}^N) \end{array} \right\} \implies E_\infty(u, \mathcal{O}) \leq E_\infty(u + \phi, \mathcal{O}).$$

2010 *Mathematics Subject Classification.* Primary 35J47, 35J62, 53C24; Secondary 49J99.

Key words and phrases. Calculus of Variations in L^∞ ; L^∞ variational principle; Aronsson system; ∞ -Laplacian; absolute minimisers; ∞ -minimal maps.

N.K. has been partially financially supported by the EPSRC grant EP/N017412/1.

In the scalar case of $N = 1$, Aronsson's concept of absolute minimisers turns out to be the appropriate substitute of mere minimisers. Indeed, absolute minimisers possess the desired uniqueness properties subject to boundary conditions and, most importantly, the possibility to characterise them through a necessary (and sufficient) condition of satisfaction of a certain nonlinear nondivergence second order PDE, known as the Aronsson equation ([9, 10, 12, 11, 13, 14, 17, 18, 20, 21, 27, 39, 44]). The latter can be written for functions $u \in C^2(\Omega)$ as

$$(1.3) \quad \mathbb{H}_P(\cdot, u, Du) \cdot D(\mathbb{H}(\cdot, u, Du)) = 0.$$

The Aronsson equation, being degenerate elliptic and non-divergence when formally expanded, is typically studied in the framework of viscosity solutions. In the above, $\mathbb{H}_P, \mathbb{H}_\eta, \mathbb{H}_x$ denotes the derivatives of $\mathbb{H}(x, \eta, P)$ with respect to the respective arguments and “ \cdot ” is the Euclidean inner product.

In this paper we are interested in characterising appropriately defined minimisers of (1.1) in the general vectorial case of $N \geq 2$ through solvability of associated PDE systems which generalise the Aronsson equation (1.3). As the wording suggests and we explain below, when $N \geq 2$ Aronsson's notion of Definition 1 is no longer the unique possible L^∞ variational concept. In any case, the extension of Aronsson's equation to the vectorial case reads

$$(1.4) \quad \begin{aligned} & \mathbb{H}_P(\cdot, u, Du) D(\mathbb{H}(\cdot, u, Du)) \\ & + \mathbb{H}(\cdot, u, Du) [\mathbb{H}_P(\cdot, u, Du)]^\perp \left(\text{Div}(\mathbb{H}_P(\cdot, u, Du)) - \mathbb{H}_\eta(\cdot, u, Du) \right) = 0. \end{aligned}$$

In the above, for any linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^N$, $[A]^\perp$ symbolises the orthogonal projection $\text{Proj}_{\mathbb{R}(A)^\perp}$ on the orthogonal complement of its range $\mathbb{R}(A) \subseteq \mathbb{R}^N$. We will refer to the PDE system (1.4) as the “Aronsson system”, in spite of the fact it was actually derived by the second author in [28], wherein the connections between general vectorial variational problems and their associated PDEs were first studied, namely those playing the role of Euler-Lagrange equations in L^∞ . The Aronsson system was derived through the well-known method of L^p -approximations and is being studied quite systematically since its discovery, see e.g. [28]-[31], [34, 37]. The additional normal term which is not present in the scalar case imposes an extra layer of complexity, as it might be discontinuous even for smooth solutions (see [29, 31]).

For simplicity and in order to illustrate the main ideas in a manner which minimises technical complications, *in this paper we restrict our attention exclusively to regular minimisers and solutions*. In general, solutions to (1.4) are nonsmooth and the lack of divergence structure combined with its vectorial nature renders its study beyond the reach of viscosity solutions. To this end, the theory of \mathcal{D} -solutions introduced in [34] and subsequently utilised in several works (see e.g. [1, 22, 34, 35]) offers a viable alternative for the study of general locally Lipschitz solutions to (1.4), and in fact it works far beyond the realm of Calculus of Variations in L^∞ . We therefore leave the generalisation of the results herein to a lower regularity setting for future work.

Additionally to absolute minimisers, for reasons to be explained later, in the paper [30] a special case of the next L^∞ variational concept was introduced (therein for $\mathbb{H}(x, \eta, P) = |P|^2$):

Definition 2. Let $u \in C^1(\Omega; \mathbb{R}^N)$. We say that u is an ∞ -minimal map for (1.1) on Ω if (i) and (ii) below hold true:

(i) u is a *rank-one absolute minimiser*, namely it minimises with respect to essentially scalar variations vanishing on the boundary along fixed unit directions:

$$(1.5) \quad \left. \begin{array}{l} \forall \mathcal{O} \Subset \Omega, \forall \xi \in \mathbb{R}^N \\ \forall \phi \in C_0^1(\overline{\mathcal{O}}; \text{span}[\xi]) \end{array} \right\} \implies E_\infty(u, \mathcal{O}) \leq E_\infty(u + \phi, \mathcal{O}).$$

(ii) u has *∞ -minimal area*, namely it minimises with respect to variations which are normal to the range of the matrix field $H_P(\cdot, u, Du)$ and free on the boundary:

$$(1.6) \quad \left. \begin{array}{l} \forall \mathcal{O} \Subset \Omega, \forall \phi \in C^1(\mathbb{R}^n; \mathbb{R}^N) \\ \text{with } \phi^\top H_P(\cdot, u, Du) = 0 \text{ on } \mathcal{O} \end{array} \right\} \implies E_\infty(u, \mathcal{O}) \leq E_\infty(u + \phi, \mathcal{O}).$$

In the above,

$$C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N) := \{\psi \in C^1(\mathbb{R}^n; \mathbb{R}^N) : \psi = 0 \text{ on } \partial\mathcal{O}\}.$$

Note also that *when $N = 1$ absolute minimisers and ∞ -minimal maps coincide*, at least when $\{H_P = 0\} \subseteq \{H = 0\}$. Further, in the event that $H_P(\cdot, u, Du)$ has discontinuous rank on \mathcal{O} , the only continuous normal vector fields ϕ may be only those vanishing on the set of discontinuities.

In [30] it was proved that C^2 ∞ -minimal maps of full rank (namely immersions or submersions) are ∞ -Harmonic, that is solutions to the so-called ∞ -Laplace system. The latter is a special case of (1.4), corresponding to the choice $H(x, \eta, P) = |P|^2$:

$$(1.7) \quad Du D(|Du|^2) + |Du|^2 [Du]^\perp \Delta u = 0.$$

The fullness of rank was assumed because of the possible discontinuity of the coefficient $[Du]^\perp$, which may well happen even for smooth solutions (for explicit examples see [29]). In this paper we bypass this difficulty by replacing the orthogonal projection $[\cdot]^\perp$ by the projection on the subspace of those normal vectors which have local normal C^1 extensions in a open neighbourhood:

Definition 3. Let $V : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N \times n}$ be a matrix field and note that

$$R(V(x))^\perp = N(V(x)^\top),$$

where for any $x \in \Omega$, $N(V(x)^\top)$ is the nullspace of the transpose $V(x)^\top \in \mathbb{R}^{n \times N}$. We define the orthogonal projection

$$\llbracket V(x) \rrbracket^\perp := \text{Proj}_{\tilde{N}(V(x)^\top)}, \quad \llbracket V(\cdot) \rrbracket^\perp : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N \times N},$$

where $\tilde{N}(V(x)^\top)$ is the reduced nullspace, given by

$$\begin{aligned} \tilde{N}(V(x)^\top) := \left\{ \xi \in N(V(x)^\top) \mid \exists \varepsilon > 0 \ \& \ \exists \bar{\xi} \in C^1(\mathbb{R}^n; \mathbb{R}^N) : \right. \\ \left. \bar{\xi}(x) = \xi \ \& \ \bar{\xi}(y) \in N(V(y)^\top), \ \forall y \in \mathbb{B}_\varepsilon(x) \right\}. \end{aligned}$$

It is a trivality to check that $\tilde{N}(V(x)^\top)$ is indeed a vector space and that

$$\llbracket V(x) \rrbracket^\perp [V(x)]^\perp = \llbracket V(x) \rrbracket^\perp,$$

where $[V(x)]^\perp = \text{Proj}_{N(V(x)^\top)}$. Note that the definition could be written in a more concise manner by using the algebraic language of *sheaves and germs*, but we refrained from doing so as there is no real benefit in this simple case.

The first main result in this paper is the next variational characterisation of the Aronsson system (1.4).

Theorem 4 (Variational Structure of Aronsson's system). *Let $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ be a map in $C^2(\Omega; \mathbb{R}^N)$. Then:*

(I) *If u is a rank-one absolute minimiser for (1.1) on Ω (Definition 2(i)), then it solves*

$$(1.8) \quad \mathbb{H}_P(\cdot, u, Du) D(\mathbb{H}(\cdot, u, Du)) = 0 \quad \text{on } \Omega.$$

The opposite is true if in addition \mathbb{H} does not depend on $\eta \in \mathbb{R}^N$ and $\mathbb{H}_P(\cdot, Du)$ has full rank on Ω .

(II) *If u has ∞ -minimal area for (1.1) on Ω (Definition 2(ii)), then it solves*

$$(1.9) \quad \mathbb{H}(\cdot, u, Du) \llbracket \mathbb{H}_P(\cdot, u, Du) \rrbracket^\perp \left(\text{Div}(\mathbb{H}_P(\cdot, u, Du)) - \mathbb{H}_\eta(\cdot, u, Du) \right) = 0 \quad \text{on } \Omega.$$

The opposite is true if in addition for any $x \in \Omega$, $\mathbb{H}(x, \cdot, \cdot)$ is convex on $\mathbb{R}^n \times \mathbb{R}^{N \times n}$.

(III) *If u is ∞ -minimal map for (1.1) on Ω , then it solves the (reduced) Aronsson system*

$$\begin{aligned} A_\infty u &:= \mathbb{H}_P(\cdot, u, Du) D(\mathbb{H}(\cdot, u, Du)) \\ &+ \mathbb{H}(\cdot, u, Du) \llbracket \mathbb{H}_P(\cdot, u, Du) \rrbracket^\perp \left(\text{Div}(\mathbb{H}_P(\cdot, u, Du)) - \mathbb{H}_\eta(\cdot, u, Du) \right) = 0. \end{aligned}$$

The opposite is true if in addition \mathbb{H} does not depend on $\eta \in \mathbb{R}^N$, $\mathbb{H}_P(\cdot, Du)$ has full rank on Ω and for any $x \in \Omega$ $\mathbb{H}(x, \cdot)$ is convex in $\mathbb{R}^{N \times n}$.

The emergence of two distinct sets of variations and a pair of separate PDE systems comprising (1.4) might seem at first glance mysterious. However, it is a manifestation of the fact that the (reduced) Aronsson system in fact consists of two linearly independent differential operators because of the perpendicularity between $\llbracket \mathbb{H}_P \rrbracket^\perp$ and \mathbb{H}_P ; in fact, one may split $A_\infty u = 0$ to

$$\begin{cases} \mathbb{H}_P(\cdot, u, Du) D(\mathbb{H}(\cdot, u, Du)) = 0, \\ \mathbb{H}(\cdot, u, Du) \llbracket \mathbb{H}_P(\cdot, u, Du) \rrbracket^\perp \left(\text{Div}(\mathbb{H}_P(\cdot, u, Du)) - \mathbb{H}_\eta(\cdot, u, Du) \right) = 0. \end{cases}$$

Theorem 4 makes clear that Aronsson's absolute minimisers do **not** characterise the Aronsson system when $N \geq 2$, at least when the additional natural assumptions hold true. This owes to the fact that, unlike the scalar case, *the Aronsson system admits arbitrarily smooth non-minimising solutions*, even in the model case of the ∞ -Laplacian. For details we refer to [37].

Since Aronsson's absolute minimisers do not characterise the Aronsson system, the natural question arises as to what is their PDE counterpart. The next theorem which is our second main result answers this question:

Theorem 5 (Divergence PDE characterisation of Absolute minimisers). *Let $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ be a map in $C^1(\Omega; \mathbb{R}^N)$. Consider the following statements:*

(I) *u is a vectorial absolute minimiser for (1.1) on Ω .*

(II) *For any $\mathcal{O} \Subset \Omega$, there exists a non-empty compact set*

$$(1.10) \quad \mathbb{K} \subseteq \text{Argmax}\{\mathbb{H}(\cdot, u, Du) : \overline{\mathcal{O}}\}$$

such that, for any Radon probability measure $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ satisfying $\text{supp}(\sigma) \subseteq \mathbb{K}$, we have

$$(1.11) \quad -\text{div}(\mathbb{H}_P(\cdot, u, Du)\sigma) + \mathbb{H}_\eta(\cdot, u, Du)\sigma = 0,$$

in the dual space $(C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N))^*$.

(III) For any $\mathcal{O} \Subset \Omega$, there exists a non-empty compact set K satisfying (1.10) and such that

$$(1.12) \quad \mathbb{H}_P(\cdot, u, Du) : D\psi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \psi = 0 \quad \text{on } K,$$

for any $\psi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$.

Then, (I) \implies (II) \implies (III). If additionally $\mathbb{H}(x, \cdot, \cdot)$ is convex for any $x \in \Omega$, then (III) \implies (I) and all three statements are equivalent.

The result above provides an interesting characterisation of Aronsson's concept of Absolute minimisers in terms of divergence PDE systems with measures as parameters. The exact distributional meaning of (1.11) is

$$\int_{\overline{\mathcal{O}}} \left(\mathbb{H}_P(\cdot, u, Du) : D\psi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \psi \right) d\sigma = 0$$

for all $\psi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$, where the “:” notation in the PDE symbolises the Euclidean (Frobenius) inner product in $\mathbb{R}^{N \times n}$.

The idea of Theorem 5 is inspired by the paper [25] of Evans and Yu, wherein a particular case of the divergence system is derived (in the special scalar case $N = 1$ for the ∞ -Laplacian and only for $\Omega = \mathcal{O}$), as well as by new developments on higher order Calculus of variations in L^∞ in [36, 38, 40].

Note that, it does not suffice to consider only $\Omega = \mathcal{O}$ as in [25] in order to describe absolute minimisers. For a subdomain $\mathcal{O} \subseteq \Omega$, it may well happen that the only measure σ “charging” the points of $\overline{\mathcal{O}}$ where the energy density $\mathbb{H}(\cdot, u, Du)$ is maximised is the Dirac measure at a single point $x \in \partial\mathcal{O}$. This is for instance the case for the standard “Aronsson solution” of the ∞ -Laplacian on \mathbb{R}^2 , given by $u(x, y) = |x|^{4/3} - |y|^{4/3}$, as well as for any other ∞ -Harmonic function which is nowhere Eikonal (i.e. $|Du|$ is non-constant on all open subsets).

By specialising Theorems 4-5 to the scalar case $N = 1$, we obtain the next set of equivalences (for simplicity stated in the C^2 case only):

Corollary 6 (The scalar case $N = 1$). *Let $u \in C^2(\Omega)$. Suppose that $\mathbb{H}(x, \cdot, \cdot)$ is convex on $\mathbb{R} \times \mathbb{R}^n$ and that $\{\mathbb{H}_P = 0\} \subseteq \{\mathbb{H} = 0\}$. Then, all the statements below are equivalent:*

- (I) u is an absolute minimiser for (1.1) on Ω .
- (II) For any $\mathcal{O} \Subset \Omega$, there exists a non-empty compact set K satisfying (1.10) such that for any Radon probability measure $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ with $\text{supp}(\sigma) \subseteq K$, (u, σ) solves (1.11) in the dual space $(C_0^1(\overline{\mathcal{O}}))^*$.
- (III) For any $\mathcal{O} \Subset \Omega$, there exists a non-empty compact set K satisfying (1.10) such that u satisfies (1.12) on K , for any $\psi \in C_0^1(\overline{\mathcal{O}})$.
- (IV) u solves the Aronsson equation (1.3) on Ω .

We conclude this introduction by noting that the two vectorial variational concepts we are considering herein (Definitions 1-2) do not exhaust the plethora variational concepts in L^∞ . In particular, in the paper [43] the concept of *tight maps* was introduced in the case of $\mathbb{H}(x, \eta, P) = \|P\|$ where $\|\cdot\|$ is the operator norm on $\mathbb{R}^{N \times n}$. Additionally, in the papers [1, 35] a concept of special affine variations was considered which also characterises the Aronsson system, in fact in the generality of merely locally Lipschitz \mathcal{D} -solutions. Finally, in the paper [8] new concepts

of absolute minimisers for constrained minimisation problems have been proposed, whilst results relevant to variational principles in L^∞ and applications appear in [15, 16, 18, 26, 41, 42].

2. PROOFS AND A MAXIMUM-MINIMUM PRINCIPLE FOR $H(\cdot, u, Du)$

In this section we prove our main results Theorems 4-5. Before delving into that, we establish a result of independent interest, which generalises a corresponding result from [30].

Proposition 7 (Maximum-Minimum Principles). *Suppose Let $u \in C^2(\Omega; \mathbb{R}^N)$ be a solution to (1.8), such that H satisfies*

- (a) $H_P(\cdot, u, Du)$ has full rank on Ω ,
- (b) there exists $c > 0$ such that

$$(\xi^\top H_P(x, \eta, P)) \cdot (\xi^\top P) \geq c |\xi^\top H_P(x, \eta, P)|^2,$$

for all $\xi \in \mathbb{R}^N$ and all $(x, \eta, P) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

Then, for any $\mathcal{O} \Subset \Omega$ we have:

$$(2.1) \quad \sup_{\mathcal{O}} H(\cdot, u, Du) = \max_{\partial \mathcal{O}} H(\cdot, u, Du),$$

$$(2.2) \quad \inf_{\mathcal{O}} H(\cdot, u, Du) = \min_{\partial \mathcal{O}} H(\cdot, u, Du).$$

The proof is based on the usage of the following flow with parameters:

Lemma 8. *Let $u \in C^2(\Omega; \mathbb{R}^N)$. Consider the parametric ODE system*

$$(2.3) \quad \begin{cases} \dot{\gamma}(t) = \xi^\top H_P(\cdot, u, Du)|_{\gamma(t)}, & t \neq 0, \\ \gamma(0) = x, \end{cases}$$

for given $x \in \Omega$ and $\xi \in \mathbb{R}^N$. Then, we have

$$(2.4) \quad \frac{d}{dt} \left(H(\cdot, u, Du)|_{\gamma(t)} \right) = \xi^\top H_P(\cdot, u, Du) D(H(\cdot, u, Du))|_{\gamma(t)},$$

$$(2.5) \quad \frac{d}{dt} \xi^\top u(\gamma(t)) \geq c \left| \xi^\top H_P(\cdot, u, Du)|_{\gamma(t)} \right|^2.$$

Proof of Lemma 8. The identity (2.4) follows by a direct computation and (2.3). For the inequality (2.5), we have

$$\begin{aligned} \frac{d}{dt} \xi^\top u(\gamma(t)) &= \left(\xi^\top Du(\gamma(t)) \right) \cdot \dot{\gamma}(t) \\ &= \left(\xi^\top Du(\gamma(t)) \right) \cdot \left(\xi^\top H_P(\cdot, u, Du)|_{\gamma(t)} \right) \\ &\geq c \left| \xi^\top H_P(\cdot, u, Du)|_{\gamma(t)} \right|^2. \end{aligned}$$

The lemma ensues. □

Proof of Proposition 7. Fix $\mathcal{O} \Subset \Omega$. Without loss of generality, we may suppose \mathcal{O} is connected. Consider first the case where $\text{rk}(H_P(\cdot, u, Du)) \equiv n \leq N$. Then, the matrix-valued map $H_P(\cdot, u, Du)$ is pointwise left invertible. Therefore, by (1.8),

$$(H_P(\cdot, u, Du))^{-1} H_P(\cdot, u, Du) D(H(\cdot, u, Du)) = 0$$

which, by the connectivity of \mathcal{O} , gives $H(\cdot, u, Du) \equiv \text{const}$ on \mathcal{O} . The latter equality readily implies the desired conclusion. Consider now the case where $\text{rk}(H_P(\cdot, u, Du)) \equiv N \leq n$. Fix $x \in \mathcal{O}$ and a unit vector $\xi \in \mathbb{R}^n$ and consider the parametric ODE system (2.3) of Lemma 8. By the fullness of the rank of $H_P(\cdot, u, Du)$, we have that

$$|\xi^\top H_P(\cdot, u, Du)| \geq c_1 > 0 \quad \text{on } \mathcal{O}.$$

We will now show that the trajectory $\gamma(t)$ reaches $\partial\mathcal{O}$ in finite time. To this end, we estimate

$$\|Du\|_{L^\infty(\mathcal{O})} \text{diam}(\mathcal{O}) \geq \|Du\|_{L^\infty(\mathcal{O})} |\gamma(t) - \gamma(0)| \geq \left| \frac{d}{dt} \Big|_{\hat{t}} \xi^\top u(\gamma(t)) \right| t,$$

for some $\hat{t} \in (0, t)$, by the mean value theorem. Hence,

$$\begin{aligned} \|Du\|_{L^\infty(\mathcal{O})} \text{diam}(\mathcal{O}) &\geq \left| \frac{d}{dt} \Big|_{\hat{t}} \xi^\top u(\gamma(t)) \right| t \\ &= \left| \xi^\top Du(\gamma(\hat{t})) \cdot \dot{\gamma}(\hat{t}) \right| t \\ &= \left| \xi^\top Du(\gamma(\hat{t})) \cdot \left(\xi^\top H_P(\cdot, u, Du) \Big|_{\gamma(\hat{t})} \right) \right| t \\ &\geq c_0 \left| \xi^\top H_P(\cdot, u, Du) \Big|_{\gamma(\hat{t})} \right|^2 t \\ &\geq (c_0 c_1^2) t. \end{aligned}$$

This proves the desired claim. Further, since u solves (1.8), by (2.4) of Lemma 8 it follows that $H(\cdot, u, Du)$ is constant along the trajectory. Thus, if $x \in \mathcal{O}$ is chosen as a point realising either the maximum or the minimum in $\overline{\mathcal{O}}$, then by moving along the trajectory, we reach a point $y \in \partial\mathcal{O}$ such that $H(\cdot, u, Du) \Big|_x = H(\cdot, u, Du) \Big|_y$. This establishes both the maximum and minimum principle. The proposition ensues. \square

Remark 9 (Danskin's theorem and some of its consequences). The central ingredient in the proofs of Theorems 4-5 is the next consequence of Danskin's theorem: for any $\mathcal{O} \Subset \Omega$ and any $u, \phi \in C^1(\Omega; \mathbb{R}^N)$, we have the identities

$$(2.6) \quad \begin{cases} \frac{d}{dt} \Big|_{t=0^+} E_\infty(u + t\phi, \mathcal{O}) = \max_{\mathcal{O}(u)} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right), \\ \frac{d}{dt} \Big|_{t=0^-} E_\infty(u + t\phi, \mathcal{O}) = \min_{\mathcal{O}(u)} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right), \end{cases}$$

where

$$\mathcal{O}(u) := \text{Argmax}\{H(\cdot, u, Du) : \overline{\mathcal{O}}\}.$$

Indeed, by [24, Theorem 1, page 643] and the chain rule we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0^+} E_\infty(u + t\phi, \mathcal{O}) &= \frac{d}{dt} \Big|_{t=0^+} \left(\max_{\overline{\mathcal{O}}} H(\cdot, u + t\phi, Du + tD\phi) \right) \\ &= \max_{\mathcal{O}(u)} \left(\frac{d}{dt} \Big|_{t=0^+} H(\cdot, u + t\phi, Du + tD\phi) \right) \\ &= \max_{\mathcal{O}(u)} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right). \end{aligned}$$

This establishes the first identity of (2.6). The second one follows through the substitutions $\phi \rightsquigarrow -\phi$, $t \rightsquigarrow -t$. As a consequence, if

$$(2.7) \quad E_\infty(u + t\phi, \mathcal{O}) \geq E_\infty(u, \mathcal{O}), \quad \text{for all } t \in \mathbb{R},$$

then directly by (2.7) and the definition of one-sided derivatives, we have

$$(2.8) \quad \frac{d}{dt} \Big|_{t=0^-} E_\infty(u + t\phi, \mathcal{O}) \leq 0 \leq \frac{d}{dt} \Big|_{t=0^+} E_\infty(u + t\phi, \mathcal{O}).$$

Therefore, by continuity, (2.8) and (2.6) yield the existence of a non-empty compact set $K = K_\phi \subseteq \mathcal{O}(u)$ such that

$$(2.9) \quad (H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi) \Big|_K = 0.$$

Conversely, if (2.9) holds true for some non-empty compact set $K \subseteq \mathcal{O}(u)$, then by (2.6) we have that (2.8) is true. If further $H(x, \cdot, \cdot)$ is convex for all $x \in \Omega$, then by Lemma 10 below, $t \mapsto E_\infty(u + t\phi, \mathcal{O})$ is minimised at $t = 0$ and (2.7) holds true.

Lemma 10. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. If the one-sided derivatives $f'(0^\pm)$ exist and $f'(0^-) \leq 0 \leq f'(0^+)$, then $t = 0$ is the global minimum of f on \mathbb{R} .*

Proof of Lemma 10. By the convexity of f on \mathbb{R} , for any fixed $s \in \mathbb{R}$ there exists a sub-differential $p_s \in \mathbb{R}$ such that

$$(2.10) \quad f(t) - f(s) \geq p_s(t - s), \quad \text{for all } t \in \mathbb{R}.$$

For the choice $t = 0$ and $s > 0$, we have

$$\frac{f(s) - f(0)}{s} \leq p_s$$

and note also that since convex functions are locally Lipschitz, the set $(p_s)_{0 < s < 1}$ is bounded. Thus, since $f'(0^+)$ exists and is non-negative, we obtain

$$0 \leq f'(0^+) \leq \liminf_{s \rightarrow 0^+} p_s < \infty.$$

Hence, by passing to the limit as $s \rightarrow 0^+$ in the inequality (2.10) for $t > 0$ fixed, we obtain $f(t) - f(0) \geq 0$. The case of $t < 0$ follows by arguing similarly. \square

Now we may establish Theorem 4.

Proof of Theorem 4. (I) Suppose first that u is a rank-one absolute minimiser on Ω . The aim is to show that (1.8) is satisfied on Ω . This conclusion in fact follows by the results in [28], but below we provide a new shorter proof. To this end, fix $x \in \Omega$ and $\rho \in (0, \text{dist}(x, \partial\Omega))$ and let $\mathcal{O} := \mathbb{B}_\rho(x)$. We fix also $\xi \in \mathbb{R}^N$ and choose

$$\phi(y) := \xi(|y - x|^2 - \rho^2).$$

Then, $\phi \in C_0^1(\bar{\mathbb{B}}_\rho(x); \text{span}[\xi])$ and by invoking Remark 9, we obtain

$$(2.11) \quad \xi^\top \left(2H_P(\cdot, u, Du) \Big|_{x_\rho} (x_\rho - x) + H_\eta(\cdot, u, Du) \Big|_{x_\rho} (|x_\rho - x|^2 - \rho^2) \right) = 0,$$

for some point x_ρ with $|x_\rho - x| \leq \rho$ which lies in the argmax set

$$(\mathbb{B}_\rho(x))(u) = \text{Argmax}\{H(\cdot, u, Du) : \bar{\mathbb{B}}_\rho(x)\}.$$

If x_ρ lies in the interior of $\mathbb{B}_\rho(x)$, then it is an interior maximum and therefore

$$D(H(\cdot, u, Du)) \Big|_{x_\rho} = 0.$$

This means that (1.8) is satisfied at x_ρ . If x_ρ lies on the boundary of $\mathbb{B}_\rho(x)$, then this means that

$$\forall y \in \bar{\mathbb{B}}_\rho(x), \text{ we have } H(\cdot, u, Du)|_y \leq H(\cdot, u, Du)|_{x_\rho}.$$

The above can be rewritten as

$$\bar{\mathbb{B}}_\rho(x) \subseteq \mathcal{H}(x_\rho) := \left\{ H(\cdot, u, Du) \leq H(\cdot, u, Du)|_{x_\rho} \right\},$$

and note also that $x_\rho \in \partial\mathbb{B}_\rho(x) \cap \partial\mathcal{H}(x_\rho)$. Hence, the sublevel set $\mathcal{H}(x_\rho)$ satisfied an interior sphere condition at x_ρ . If $D(H(\cdot, u, Du))|_{x_\rho} = 0$ then (1.8) is again satisfied at x_ρ . If on the other hand

$$D(H(\cdot, u, Du))|_{x_\rho} \neq 0$$

then $\partial\mathcal{H}(x_\rho)$ is a C^1 manifold near x_ρ and the gradient above is the normal vector at the point x_ρ . Due to the interior sphere condition, this implies that this is also the normal vector to the sphere $\partial\mathbb{B}_\rho(x)$ at x_ρ . Thus, there exists $\lambda \neq 0$ such that

$$(2.12) \quad x_\rho - x = \lambda D(H(\cdot, u, Du))|_{x_\rho}.$$

By inserting (2.12) into (2.11) and noting that $|x_\rho - x| = \rho$, we infer that

$$2\lambda \xi^\top \left(H_P(\cdot, u, Du) D(H(\cdot, u, Du)) \right)|_{x_\rho} = 0.$$

By dividing by 2λ and letting $\rho \rightarrow 0$, we deduce that (1.8) is satisfied at the arbitrary $x \in \Omega$.

Conversely, suppose that u satisfies (1.8) on Ω , together with the additional assumptions of the statement. Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C_0^1(\bar{\mathcal{O}}; \text{span}[\xi])$. Without loss of generality, we may suppose \mathcal{O} is connected. Since $\phi = (\xi^\top \phi)\xi$, for convenience we set $g := \xi^\top \phi$ and then we may write $\phi = g\xi$ with $g \in C_0^1(\bar{\mathcal{O}})$. Then, the matrix-valued map $H_P(\cdot, Du)$ is pointwise left invertible. Therefore, by (1.8)

$$(H_P(\cdot, Du))^{-1} H_P(\cdot, Du) D(H(\cdot, Du)) = 0 \text{ on } \mathcal{O},$$

which, by the connectivity of \mathcal{O} , gives

$$H(\cdot, Du) \equiv \text{const on } \mathcal{O}.$$

Since $g \in C^1(\mathbb{R}^n)$ with $g = 0$ on $\partial\mathcal{O}$, there exists at least one interior critical point $\bar{x} \in \mathcal{O}$ such that $Dg(\bar{x}) = 0$. By the previous, we have

$$\begin{aligned} E_\infty(u, \mathcal{O}) &= H(\bar{x}, Du(\bar{x})) \\ &= H\left(\bar{x}, Du(\bar{x}) + \xi \otimes Dg(\bar{x})\right) \\ &= H\left(\bar{x}, Du(\bar{x}) + D\phi(\bar{x})\right) \\ &\leq \sup_{x \in \mathcal{O}} H\left(x, Du(x) + D\phi(x)\right) \\ &= E_\infty(u + \phi, \mathcal{O}). \end{aligned}$$

The conclusion ensues.

(II) Suppose that u has ∞ -minimal area. Fix $x \in \Omega$ and $\rho \in (0, \text{dist}(x, \partial\Omega))$. Fix

$$\xi \in \tilde{N}\left(H_P(\cdot, u, Du)^\top|_x\right),$$

noting also that by Definition 3 the above set is the reduced nullspace of $H_P(\cdot, u, Du)^\top$ at x . This implies that there exists a C^1 extension $\bar{\xi} \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\bar{\xi}(x) = \xi$ and $(\bar{\xi})^\top H_P(\cdot, u, Du) = 0$ on the closed ball $\bar{\mathbb{B}}_\varepsilon(x)$ for some $\varepsilon \in (0, \rho)$. By differentiating the relation $(\bar{\xi})^\top H_P(\cdot, u, Du) = 0$ and taking its trace, we obtain

$$(2.13) \quad \bar{\xi} \cdot \operatorname{div}(H_P(\cdot, u, Du)) + D\bar{\xi} : H_P(\cdot, u, Du) = 0,$$

on $\bar{\mathbb{B}}_\varepsilon(x)$. Since u has ∞ -minimal area and $\bar{\xi}$ is an admissible normal variation, by Remark 9 it follows that

$$(2.14) \quad \left(\bar{\xi} \cdot H_\eta(\cdot, u, Du) + D\bar{\xi} : H_P(\cdot, u, Du) \right) \Big|_{x_\varepsilon} = 0$$

for some $x_\varepsilon \in (\mathbb{B}_\varepsilon(x))(u)$, where

$$(\mathbb{B}_\varepsilon(x))(u) = \operatorname{Argmax}\{H(\cdot, u, Du) : \bar{\mathbb{B}}_\varepsilon(x)\}.$$

By (2.13)-(2.14), we infer that

$$\bar{\xi}(x_\varepsilon) \cdot \left(\operatorname{div}(H_P(\cdot, u, Du)) - H_\eta(\cdot, u, Du) \right) \Big|_{x_\varepsilon} = 0$$

and by letting $\varepsilon \rightarrow 0$, we deduce that

$$\xi \cdot \left(\operatorname{div}(H_P(\cdot, u, Du)) - H_\eta(\cdot, u, Du) \right) \Big|_x = 0,$$

for any $\xi \in \tilde{N}(H_P(\cdot, u, Du)^\top \Big|_x)$. Hence, u satisfies (1.9) at the arbitrary $x \in \Omega$.

Conversely, suppose that u solves (1.9) on Ω . Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\phi^\top H_P(\cdot, u, Du) = 0$ on \mathcal{O} . Note further that by the continuity up to the boundary of all functions involved, the latter identity in fact holds on $\bar{\mathcal{O}}$. By the satisfaction of (1.9) and Definition 3, it follows that

$$\phi \cdot \left(\operatorname{div}(H_P(\cdot, u, Du)) - H_\eta(\cdot, u, Du) \right) = 0,$$

on $\bar{\mathcal{O}} \subseteq \Omega$. By differentiating $\phi^\top H_P(\cdot, u, Du) = 0$, we obtain

$$\phi \cdot \operatorname{div}(H_P(\cdot, u, Du)) + D\phi : H_P(\cdot, u, Du) = 0,$$

on $\bar{\mathcal{O}}$. By the above two identities, we deduce

$$\phi \cdot H_\eta(\cdot, u, Du) + D\phi : H_P(\cdot, u, Du) = 0$$

on $\bar{\mathcal{O}}$. Since $\mathcal{O}(u) \subseteq \bar{\mathcal{O}}$, Remark 9 yields that u is a critical point since the left and right derivative of $E_\infty(u + t\phi, \mathcal{O})$ at $t = 0$ coincide and vanish. Since by assumption $H(x, \cdot, \cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N \times n}$, it follows that $E_\infty(\cdot, \mathcal{O})$ is convex on $W^{1,\infty}(\mathcal{O}; \mathbb{R}^N)$. Hence, the critical point u is in fact a minimum point for this class of variations. This establishes our claim.

(III) This is an immediate corollary of items (I) and (II). \square

Now we conclude by establishing Theorem 5.

Proof of Theorem 5. (I) \implies (II): If u is an absolute minimiser, then by Remark 9, there exists a compact set $K \subseteq \mathcal{O}(u) \subseteq \bar{\mathcal{O}}$ (where possibly K is a singleton set) such that

$$\left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) \Big|_K = 0.$$

Then, for any Radon probability measure $\sigma \in \mathcal{P}(\bar{\mathcal{O}})$ with $\operatorname{supp}(\sigma) \subseteq K$, we have

$$\int_{\bar{\mathcal{O}}} \left(H_P(\cdot, u, Du) : D\phi + H_\eta(\cdot, u, Du) \cdot \phi \right) d\sigma = 0$$

for all $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. Hence, we have shown that

$$-\operatorname{div}(\mathbb{H}_P(\cdot, u, Du)\sigma) + \mathbb{H}_\eta(\cdot, u, Du)\sigma = 0,$$

in the dual space $(C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N))^*$.

(II) \implies (III): By assumption there exists a non-empty compact set $K \subseteq \mathcal{O}(u)$ such that

$$\int_{\overline{\mathcal{O}}} \left(\mathbb{H}_P(\cdot, u, Du) : D\phi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \phi \right) d\sigma = 0$$

for all $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$ and all Radon probability measures $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ with $\operatorname{supp}(\sigma) \subseteq K$. Fix any $\bar{x} \in K$. By choosing the Dirac measure $\bar{\sigma} \in \mathcal{P}(\overline{\mathcal{O}})$ given by

$$\bar{\sigma} := \delta_{\bar{x}}$$

which evidently satisfies $\operatorname{supp}(\bar{\sigma}) = \{\bar{x}\} \subseteq K$, we obtain

$$\begin{aligned} & \left(\mathbb{H}_P(\cdot, u, Du) : D\phi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \phi \right) \Big|_{\bar{x}} \\ &= \int_{\overline{\mathcal{O}}} \left(\mathbb{H}_P(\cdot, u, Du) : D\phi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \phi \right) d\bar{\sigma} \\ &= 0, \end{aligned}$$

for any $\bar{x} \in K$. The conclusion ensues.

(III) \implies (I): By assumption we have

$$\mathbb{H}_P(\cdot, u, Du) : D\phi + \mathbb{H}_\eta(\cdot, u, Du) \cdot \phi = 0,$$

on the non-empty compact set $K \subseteq \mathcal{O}(u)$. By Remark 9 and Lemma 10, this implies u is in fact a minimum point and this completes the proof. \square

Conflict of Interest: The authors declare that they have no conflict of interest.

Acknowledgement. N.K. would like to thank Roger Moser, Giles Shaw and Tristan Pryer for their inspiring scientific discussion about L^∞ variational problems.

REFERENCES

1. B. Ayanbayev, N. Katzourakis, *A Pointwise Characterisation of the PDE system of vectorial Calculus of variations in L^∞* , Proc. Royal Soc. Edinburgh A, in press.
2. G. Aronsson, *Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$* , Arkiv für Mat. 6 (1965), 33 - 53.
3. G. Aronsson, *Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$ II*, Arkiv für Mat. 6 (1966), 409 - 431.
4. G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Arkiv für Mat. 6 (1967), 551 - 561.
5. G. Aronsson, *On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$* , Arkiv für Mat. 7 (1968), 395 - 425.
6. G. Aronsson, *Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$ III*, Arkiv für Mat. (1969), 509 - 512.
7. G. Aronsson, *On Certain Minimax Problems and Pontryagin's Maximum Principle*, Calculus of Variations and PDE 37, 99 - 109 (2010).
8. G. Aronsson, E.N. Barron, *L^∞ Variational Problems with Running Costs and Constraints*, Appl. Math. Optim. 65, 53 - 90 (2012).
9. S.N. Armstrong, M.G. Crandall, V. Julin, C.K. Smart, *Convexity Criteria and Uniqueness of Absolutely Minimising Functions*, Arch. Rational Mech. Anal. 200, 405-443 (2011).
10. G. Aronsson, M. Crandall, P. Juutinen *A tour of the theory of absolutely minimizing functions*, Bulletin of the AMS, New Series 41, 439-505 (2004).

11. E.N. Barron, *Viscosity solutions and analysis in L^∞* . In: Clarke F.H., Stern R.J., Sabidussi G. (eds) *Nonlinear Analysis, Differential Equations and Control*. NATO Science Series (Series C: Mathematical and Physical Sciences), vol 528. Springer, Dordrecht, 1999.
12. E. N. Barron, L. C. Evans, R. Jensen, *The infinity Laplacian, Aronsson's equation and their generalizations*, Trans. Amer. Math. Soc. 360 (2008), 77 - 101.
13. E. N. Barron, R. Jensen and C. Wang, *The Euler equation and absolute minimisers of L^∞ functionals*, Arch. Rational Mech. Analysis 157, 255–283 (2001).
14. N. Barron, R. Jensen, C. Wang, *Lower Semicontinuity of L^∞ Functionals*, Ann. I. H. Poincaré 18, 495–517 (2001).
15. M. Bocea, V. Nesi, Γ -convergence of power-law functionals, variational principles in L^∞ and applications, SIAM J. Math. Anal. 39, 15501576 (2008).
16. M. Bocea, C. Popovici, *Variational principles in L^∞ with applications to antiplane shear and plane stress plasticity*, Journal of Convex Analysis Vol. 18 No. 2, (2011) 403-416.
17. T. Champion, L. De Pascale, *Principles of comparison with distance functions for absolute minimizers*, J. Convex Anal. 14, 515541 (2007).
18. T. Champion, L. De Pascale, F. Prinari, Γ -convergence and absolute minimizers for supremal functionals, COCV ESAIM: Control, Optimisation and Calculus of Variations (2004), Vol. 10, 1427.
19. M. G. Crandall, *A visit with the ∞ -Laplacian*, in *Calculus of Variations and Non-Linear Partial Differential Equations*, Springer Lecture notes in Mathematics 1927, CIME, Cetraro Italy 2005.
20. M. Crandall, *An Efficient Derivation of the Aronsson Equation*, Archive for Rational Mechanics and Analysis 167 (4), 271279 (2003).
21. M.G. Crandall, L.C. Evans, R. Gariepy, *Optimal Lipschitz extensions and the infinity Laplacian*, Calc. Var. PDE 13, 123139 (2001).
22. G. Croce N. Katzourakis, G. Pisante, *\mathcal{D} -solutions to the system of vectorial Calculus of Variations in L^∞ via the singular value problem*, Discrete and Continuous Dynamical Systems 37:12, 6165-6181 (2017).
23. B. Dacorogna, *Direct Methods in the Calculus of Variations*, 2nd Edition, Volume 78, Applied Mathematical Sciences, Springer, 2008.
24. J.M. Danskin, *The theory of min-max with application*, SIAM Journal on Applied Mathematics 14, 641 - 664 (1966).
25. L.C. Evans, Y. Yu, *Various properties of solutions to the Infinity-Laplacian equation*, Communications in PDE 30:9, 1401 - 1428 (2005).
26. A. Garroni, V. Nesi, M. Ponsiglione, *Dielectric breakdown: optimal bounds*, Proceedings of the Royal Society A 457, issue 2014 (2001).
27. R. Jensen, *Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient*, Arch. Rational Mech. Analysis 123 (1993), 51-74.
28. N. Katzourakis, *L^∞ Variational Problems for Maps and the Aronsson PDE System*, J. Differential Equations 253 (7), 2123-2139 (2012).
29. N. Katzourakis, *Explicit 2D ∞ -Harmonic Maps whose Interfaces have Junctions and Corners*, Comptes Rendus Acad. Sci. Paris, Ser.I, 351, 677 - 680 (2013).
30. N. Katzourakis, *∞ -Minimal Submanifolds*, Proceedings of the AMS 142, 2797-2811 (2014).
31. N. Katzourakis, *On the Structure of ∞ -Harmonic Maps*, Communications in PDE 39:11, 2091 - 2124 (2014).
32. N. Katzourakis, *An Introduction to Viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in L^∞* , Springer Briefs in Mathematics, 2015, DOI 10.1007/978-3-319-12829-0.
33. N. Katzourakis, *Generalised solutions for fully nonlinear PDE systems and existence-uniqueness theorems*, Journal of Differential Equations 23, 641 - 686 (2017).
34. N. Katzourakis, *Absolutely minimising generalised solutions to the equations of vectorial Calculus of Variations in L^∞* , Calculus of Variations and PDE 56 (1), 1 - 25 (2017) (DOI: 10.1007/s00526-016-1099-z).
35. N. Katzourakis, *A New Characterisation of ∞ -Harmonic and p -Harmonic Mappings via Affine Variations in L^∞* , Electronic Journal of Differential Equations 2017:29, 1 - 19 (2017).
36. N. Katzourakis, R. Moser, *Existence, Uniqueness and Structure of Second Order Absolute Minimisers*, Archives for Rational Mechanics and Analysis, published online 06/09/2018, DOI: 10.1007/s00205-018-1305-6.

37. N. Katzourakis, G. Shaw, *Counterexamples in Calculus of Variations in L^∞ through the vectorial Eikonal equation*, Comptes Rendus Mathematique Ser. I 356:5 (2018), 498-502, <https://doi.org/10.1016/j.crma.2018.04.010>.
38. N. Katzourakis, T. Pryer, *2nd order L^∞ variational problems and the ∞ -Polylaplacian*, Advances in Calculus of Variations, Published Online: 27-01-2018, DOI: <https://doi.org/10.1515/acv-2016-0052>.
39. Q. Miao, C. Wang, Y. Zhou, *Uniqueness of Absolute Minimizers for L^∞ -Functionals Involving Hamiltonians $H(x, p)$* , Archive for Rational Mechanics and Analysis 223 (1), 141-198 (2017).
40. G. Papamikos, T. Pryer, *A Lie symmetry analysis and explicit solutions of the two-dimensional ∞ -Polylaplacian*, Studies in Applied Mathematics, Online 17 September 2018, <https://doi.org/10.1111/sapm.12232>.
41. F. Prinari, *On the lower semicontinuity and approximation of L^∞ -functionals*, NoDEA 22, 1591 - 1605 (2015).
42. A.N. Ribeiro, E. Zappale, *Existence of minimisers for nonlevel convex functionals*, SIAM J. Control Opt., Vol. 52, No. 5, (2014) 3341 - 3370.
43. S. Sheffield, C.K. Smart, *Vector Valued Optimal Lipschitz Extensions*, Comm. Pure Appl. Math. 65(1), 128 - 154 (2012).
44. Y. Yu, *Viscosity solutions of Aronssons equations*, Arch. Ration. Mech. Anal. 182, 153180 (2006).

B.A.: DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF READING, WHITEKNIGHTS,
PO BOX 220, READING RG6 6AX, UNITED KINGDOM
E-mail address: b.ayanbayev@pgr.reading.ac.uk

N.K.: (CORRESPONDING AUTHOR) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY
OF READING, WHITEKNIGHTS, PO BOX 220, READING RG6 6AX, UNITED KINGDOM
E-mail address: n.katzourakis@reading.ac.uk