

Why does Arakawa and Schubert's convective quasi-equilibrium closure not work? Mathematical analysis and implications

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Accepted Version

Yano, J.-I. and Plant, B. (2020) Why does Arakawa and Schubert's convective quasi-equilibrium closure not work? Mathematical analysis and implications. *Journal of the Atmospheric Sciences*, 77 (4). pp. 1371-1385. ISSN 1520-0469 doi: <https://doi.org/10.1175/jas-d-19-0165.1> Available at <https://centaur.reading.ac.uk/87845/>

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To link to this article DOI: <http://dx.doi.org/10.1175/jas-d-19-0165.1>

Publisher: American Meteorological Society

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1 **Why Does**
2 **Arakawa and Schubert's Convective Quasi-Equilibrium Closure**
3 **Not Work?**
4 **Mathematical Analysis and Implications**

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ABSTRACT

12 Arakawa and Schubert (1974) proposed convective quasi-equilibrium as a
13 guiding principle for the closure of convection parameterization. However,
14 empirical experiences from operational implementation efforts suggest that
15 its strict application does not work well. The purpose of the present paper
16 is to explain mathematically why this closure does not work in practice, and
17 to suggest that problems stem from physically unrealistic assumptions. For
18 this purpose, the closure hypothesis is examined in its original form, and
19 without imposing a condition of a positiveness to the convective mass fluxes.
20 The Jordan sounding with idealized large-scale forcing is used for diagno-
21 sis purposes. The question is addressed from several perspectives including
22 the completeness of the entraining plume spectrum, and a singular vector de-
23 composition of the interaction kernel matrix. The main problems with the
24 quasi-equilibrium closure are traced to: (i) the relatively slow response of
25 shallower convective modes to large-scale forcing; and, (ii) detrainment at
26 convection top producing strong cooling and moistening. A strict application
27 of the convective quasi-equilibrium principle leads to a singular response of
28 shallow convection. An explicit coupling of convection with stratiform clouds
29 would be crucial for preventing this unrealistic behavior, recognizing that the
30 re-evaporation of detrained cloudy-air is a relatively slow process.

32 **1. Introduction**

33 Closure is a key issue in the convection parameterization problem (*cf.*, Yano *et al.* 2013). Con-
 34 vective quasi-equilibrium, as originally proposed by Arakawa and Schubert (1974: see Yano and
 35 Plant 2012a as a review), remains an important guiding principle for the convective closure even
 36 today (*e.g.*, Zhang 2002, 2003, Donner and Phillips 2003, Bechtold *et al.* 2014), in spite of various
 37 criticisms (*e.g.*, Houze and Betts 1981, Mapes 1997).

38 The quasi-equilibrium closure may be formally stated for a spectral form of mass-flux convec-
 39 tion parameterization as:

$$40 \quad \mathbf{K}\vec{\mathbf{M}} + \vec{\mathbf{F}} = 0. \quad (1.1)$$

41 Here, \mathbf{K} is an interaction matrix (kernel in Arakawa and Schubert, 1974) that describes the feed-
 42 back from the mass-flux vector (spectrum), $\vec{\mathbf{M}}$, onto the large-scale tendency of an instability
 43 measure known as the cloud work function; $\vec{\mathbf{F}}$ is the spectrum of large-scale forcing for the cloud
 44 work function, which is also defined as a vector. The vector components correspond to convective
 45 plume types that represent a spectrum of convective towers. The cloud work function corresponds
 46 to the rate at which available potential energy is converted into convective kinetic energy, as nor-
 47 malized by the mass flux at the convection base (*cf.*, Yano *et al.* 2005a). Here, the equilibrium
 48 assumption states that the total tendency vanishes. Generalizations of the quasi-equilibrium ideas
 49 are discussed by Yano and Plant (2016).

50 Eq. (1.1) states that the convective response (1st term) is always in balance with the large-scale
 51 forcing (2nd term). This closure is, intuitively speaking, physically sound, because the convective
 52 process is much faster than the large-scale processes. However, in spite of a series of subsequent
 53 efforts, this original form of the closure has never become fully operational, but only in variant

54 forms (*e.g.*, Moorthi and Suarez 1992). This study will explain why the formulation given by
55 Eq. (1.1) is structurally difficult to implement as a closure from a mathematical point of view.

56 The original implementation (Lord and Arakawa 1980, Lord 1982, Lord *et al.* 1982) devoted
57 much attention to maintaining positiveness of the convective mass fluxes, because only convective
58 updrafts were considered. Unfortunately, in our opinion and as we will discuss below, a rather
59 elaborate iteration procedure introduced for this purpose may have obscured some more basic
60 issues with a strict convective quasi–equilibrium closure.

61 The present study focuses on the closure problem exactly as given by Eq. (1.1) without any
62 further restrictions. This strategy may be partially justified by considering negative mass–fluxes
63 as detraining downdrafts (*i.e.*, time–reversed updrafts). Importantly, regardless of whether this
64 reinterpretation stands or not, this simplification enables us to elucidate more clearly and cleanly
65 some basic problems with Arakawa and Schubert’s (1974) original convective quasi–equilibrium
66 closure.

67 For the same reason, the original assumption of a spectrum of purely entraining plumes is main-
68 tained in the present study, because we believe it is important to establish a baseline. In the lit-
69 erature, the problems with the oversimplified entraining–plume hypothesis have been extensively
70 discussed, and various alternative formulations have been proposed, as reviewed in *e.g.*, de Rooy
71 *et al.* (2013), Yano (2014a). Analysis with a more elaborate plume model would be considered a
72 future work.

73 A simple formulation for the terms in Eq. (1.1) is provided in the next section, and some basic
74 demonstrations of the problems are made in Sec. 3. The identified problems are investigated in
75 Sec. 4 by examining the completeness of the entraining-plume spectrum as well as the mathemat-
76 ical structure of the interaction (kernel) matrix.

77 The present paper focuses on a rather narrow question of mathematical difficulties with the orig-
78 inal closure formulation by Arakawa and Schubert (1974). Various physical issues associated with
79 this closure hypothesis as well as with the mass–flux formulation itself are extensively discussed
80 in the literature. Some of these may be found in a review of uasi-equilibrium by Yano and Plant
81 (2012), and more general issues associated with the mass–flux parameterization are covered by
82 Plant and Yano (2015). In concluding, in Sec. 5, the paper also turns to the physical implications
83 from the present findings, also referring to background issues.

84 2. Formulation

85 *a. Data*

86 A tropical climatology based on the Jordan sounding (Jordan 1958) is adopted for specifying
87 vertical profiles of temperature and moisture. The vertical resolution used for the profile data is
88 50 hPa from 1000 to 200 hPa, and with a surface value at 1015 hPa being separately given. Data
89 is also available further above at 175, 150, 125, 100, 80, 60, 50, 40, and 30 hPa levels.

90 We introduce idealized large–scale advective forcings defined by

$$91 \quad F_L[T] = -\bar{w} \left(\frac{\bar{T}}{\bar{\theta}} \right) \frac{d\bar{\theta}}{dz} \quad (2.1a)$$

$$92 \quad F_L[q_v] = -\bar{w} \frac{d\bar{q}_v}{dz} \quad (2.1b)$$

94 for temperature and moisture, respectively. Here, \bar{T} , $\bar{\theta}$, \bar{q}_v are the vertical profiles for the tem-
95 perature, the potential temperature, and the moisture as provided by the Jordan sounding. The
96 large–scale vertical velocity, \bar{w} , in Eq. (2.1) is prescribed by

$$97 \quad \bar{w}(z) = \begin{cases} w_0 \sin \pi \left[\frac{p(z) - p(z_T)}{p_0 - p(z_T)} \right] & \text{for } p_0 \geq p(z) \geq p(z_T) \\ 0 & \text{for } p(z_T) > p(z) \end{cases} \quad (2.2)$$

98 as a function of the pressure, $p(z)$, with $w_0 = 10^{-2} \text{ ms}^{-1}$, and $p_0 = 1015 \text{ hPa}$ the surface pressure.

99 Three types of large–scale forcing are considered: deep ($z_T = 15 \text{ km}$), shallow ($z_T = 5 \text{ km}$), and

100 very shallow ($z_T = 1.5$ km). The purpose of this idealization is to examine the convective response
101 to large-scale forcing strictly confined to a certain vertical range. These forcing profiles are shown
102 in Fig. 1(a). Here, as a drastic simplification, potential contributions to the forcing from boundary-
103 layer processes are neglected, despite their possible importance. Consistent with that assumption,
104 contributions from boundary-layer processes to the interaction matrix, \mathbf{K} , will also be neglected
105 in the analysis below.

106 The large-scale forcing on the cloud work function, $\vec{\mathbf{F}}$ from Eq. (1.1), is obtained by vertically
107 integrating a linear combination of two large-scale forcings, as explicitly given by Eq. (B33) in
108 Arakawa and Schubert (1974). The integration is defined with a weighting that is a function
109 of the fractional entrainment rate, ε (see next subsection), and the resulting integrated forcing
110 is presented in Fig. 1(b). We remark that the forcing has a relatively weak dependence on a
111 microphysical parameter, c_0 , which is defined by Eq. (2.5) below in Sec. 2.c. The vertical profile
112 of the large-scale forcing as defined by Eqs. (2.1a,b) and (2.2) has a well-defined vertical scale
113 but its projection onto the plume components in Fig. 1(b), presents a very broad distribution of
114 forcing as a function of the entrainment rate, despite the fact that the entrainment rate determines
115 the vertical scale of each plume mode. Moreover, the main difference from changing the vertical
116 scale of large-scale forcing is a change of the spectrum amplitude rather than a change of the
117 spectrum shape.

118 We diagnose the convective quasi-equilibrium closure of Eq. (1.1) by closely following the
119 mass-flux spectrum formulation introduced by Arakawa and Schubert (1974), and for formula-
120 tion details we refer to the original paper. In the following two subsections, we describe two major
121 assumptions for which some additional specifications are required: the entraining-plume spectrum
122 (Sec. 2.b) and the precipitation formulation (Sec. 2.c).

123 *b. Entraining-Plume Spectrum*

124 Arakawa and Schubert's (1974) entraining-plume spectrum is characterized by a set of constant
 125 fractional entrainment rates, ε_i , which are defined in this study by:

$$126 \quad \varepsilon_i = \frac{i - 1/2}{n} \varepsilon_{\max}, \quad (2.3)$$

127 where the vector index, i , spans for $i = 1, \dots, n$ with $n = 20$ plume types considered, and $\varepsilon_{\max} =$
 128 10^{-4} m^{-1} is the maximum fractional entrainment rate considered. The i -th entraining plume has
 129 a normalized mass-flux profile of

$$130 \quad \eta_i(z) = \begin{cases} \exp[-\varepsilon_i(z - z_B)] & \text{for } z_B \leq z \leq z_{Ti} \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

131 where z_B and z_{Ti} are the bottom and top levels of the plume. The base level, z_B , is taken to be
 132 950 hPa (583 m), approximately corresponding to the top of the convectively well-mixed boundary
 133 layer. The top, z_{Ti} , is defined by the level of neutral buoyancy, at which all of the plume air
 134 detrain into the environment. The top height, z_{Ti} , is diagnosed as a continuous function by taking
 135 a linear interpolation of values between the data height levels and we assume that the plume-top
 136 detrainment happens over a vertical layer spanning between these two levels.

137 For a larger fractional entrainment rate, ε , the in-plume air is more diluted by the environmental
 138 air, and so becomes less buoyant. As a result, the plume top height, z_{Ti} , decreases with increasing
 139 ε . In essence, the fractional entrainment rate, ε , becomes a reverse measure of the convection
 140 depth, z_T . Some examples of vertical profiles of entraining plumes for the Jordan sounding are
 141 shown in Fig. 2. A full mass-flux profile for the i -th plume is defined by $M_i \eta_i(z)$, where M_i is the
 142 mass flux at the plume base for the plume type and is the i -th component of the mass-flux vector
 143 \vec{M} in Eq. (1.1).

144 *c. Precipitation efficiency*

145 A very simple cloud microphysics is used, in which the precipitation rate, R_i , within the i -th
 146 plume at each vertical level is assumed to be proportional to the cloud–water vertical flux, $w_{ci}q_{ci}$,
 147 with a proportionality constant, c_0 , called the precipitation efficiency:

$$148 \quad R_i = c_0 w_{ci} q_{ci}.$$

149 Here, $w_{ci} = M_i \eta_i / \rho \sigma_i$ and q_{ci} are respectively the in-cloud vertical velocity and the cloud–water
 150 mixing ratio of the i -th plume type, and ρ is the air density. The precipitation rate, R_i , is defined
 151 in such a manner that the fractional area, σ_i , occupied by the i -th plume type does not appear in
 152 actual calculations of the total–water for a given plume type (*cf.*, Eq. 6.2b of Yano 2015).

153 The precipitation efficiency, c_0 , is chosen by following a curve shown in Hack *et al.* (1984: their
 154 Fig. 3). Specifically, we take

$$155 \quad c_0 = -\frac{2\Delta c}{\pi} \arctan\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_c}\right) + c_{00}, \quad (2.5)$$

156 where $c_{00} = (c_{\max} + c_{\min})/2$ and $\Delta c = (c_{\max} - c_{\min})/2$. Note that $c_0 \rightarrow c_{\max}$ and $c_0 \rightarrow c_{\min}$ as
 157 $\varepsilon \rightarrow 0$ and $\varepsilon \rightarrow \infty$, respectively; ε_0 marks a transition from a weakly–precipitating shallow (with
 158 large ε) to a heavily–precipitating deep (with small ε) regime. Here, the parameters are set as
 159 $\varepsilon_0 = 5 \times 10^{-5} \text{ m}^{-1}$, $c_{\max} = 4.5 \times 10^{-3} \text{ m}^{-1}$, $c_{\min} = 5 \times 10^{-5} \text{ m}^{-1}$, and $\varepsilon_c = 10^{-6} \text{ m}^{-1}$. Figure 3
 160 plots the precipitation efficiency, c_0 , as a function of the fractional entrainment rate, ε .

161 **3. Basic Analyses**

162 *a. Interaction matrix*

163 The interaction matrix (kernel), \mathbf{K} , is defined by Eq. (B32) and Eqs. (B35)–(B38) of Arakawa
 164 and Schubert (1974). Its evaluation using Eq. (2.5) for the precipitation efficiency is shown
 165 in Fig. 4(a). An element, K_{ij} , of the interaction matrix defines the rate at which a unit of the
 166 convection-base mass flux for the j -th plume type, M_j , changes the cloud-work function for the
 167 i -th plume type.

168 By referring to Eq. (144) and Fig. 11 of Arakawa and Schubert (1974), we find that the large–
 169 scale thermodynamic profiles are modified by convection in two major ways: (i) detrainment at
 170 the plume top, which cools and moistens the large–scale environment due to evaporation of the
 171 detrained cloudy air; and, (ii) a compensating descent in the large–scale environment, which leads
 172 to adiabatic heating and drying by downward transport of drier air from aloft. These two major
 173 processes modify the cloud–work function, and the interaction matrix can be separated into two
 174 dominant contributions:

$$175 \quad \mathbf{K} = \mathbf{K}_d + \mathbf{K}_v, \quad (3.1)$$

176 where \mathbf{K}_d and \mathbf{K}_v represent the effects of detrainment and environmental descent, respectively. A
 177 third part, \mathbf{K}_M , as defined by Eq. (B32) of Arakawa and Schubert (1974), is neglected because of
 178 our assumptions above about boundary layer processes.

179 The evaporative cooling associated with detrainment leads to a further destabilization of the
 180 atmosphere, and thus \mathbf{K}_d is positive definite (Fig. 4(b)). This tendency is stronger when a plume
 181 is less–strongly precipitating, and hence for the shallower plumes with larger ε' . Moreover, the
 182 detrainment effect is felt only by the plume types that extend higher than the detrainment level of
 183 the plume in question (*i.e.*, $\varepsilon < \varepsilon'$), and so \mathbf{K}_d is triangular. On the other hand, adiabatic heating
 184 by environmental descent leads to a stabilization, and thus \mathbf{K}_v is negative definite (Fig. 4(c)). The
 185 descent effect is stronger for deeper plumes with smaller ε' , and affects plume types of all depths.

186 *b. Response due to a single plume*

187 Once a value of M_i , as a component of the mass-flux vector, $\vec{\mathbf{M}}$, is specified [see also Eq. (3.2)
 188 below], the tendencies of temperature and moisture produced by each convective plume type,
 189 i , can be calculated respectively from Eqs. (3.6a) and (3.6b) of Yano (2015). Examples of the
 190 convective response from individual plume types are shown in Fig. 5. Here, we rather arbitrary
 191 assume $M_i = 10^{-2} \text{ kg m}^{-2} \text{ s}^{-1}$. For the cases of $\varepsilon = 6 \times 10^{-5}$ and $8 \times 10^{-5} \text{ m}^{-1}$, the resulting

192 plumes are relatively shallow, with relatively weak precipitation. This leads to strong cooling and
 193 moistening at the detrainment level associated with cloud evaporation. The effects are much less
 194 pronounced for the deep-plume example, because a high precipitation does not leave much cloud-
 195 water for detrainment at the plume top. The values obtained for the strong cooling and moistening
 196 associated with the detrained-air re-evaporation are shown in the Appendix to be consistent with
 197 a simple scale analysis.

198 The strongly-peaked character of the thermodynamic tendencies from individual plume types
 199 raises potential issues for construction of the total convective response, obtained by taking a linear
 200 sum of these individual tendencies weighted by the convection-base mass-flux values, M_i . The
 201 total response is considered next.

202 *c. Total Convective response*

203 The convective-base mass-flux vector, $\vec{M} = (M_i)$, is obtained from Eq. (1.1) by multiplying the
 204 inverted matrix, \mathbf{K}^{-1} , on the large-scale forcing, \vec{F} . The obtained \vec{M} , shown in Fig. 6(a) as a
 205 function of the fractional entrainment rate, ε , is marked by relatively large contributions from both
 206 small and large ε with modest contributions from intermediate values. This basic structure is not
 207 dependent on the depth of the large-scale forcing.

208 The resulting vertical profile of the total mass flux, $M(z)$, is given by

$$209 \quad M(z) = \sum_{i=1}^n \eta_i(z) M_i \quad (3.2)$$

210 and is shown in Fig. 6(b), where $\eta_i(z)$ is the vertical profile of the i -th plume type, as defined by
 211 Eq. (2.4).

212 The most noticeable feature is a strong downdraft below the 4-km level, which is the lowest
 213 height achieved by plumes with largest fractional entrainment rates, ε , under the given mean ther-
 214 mal profile. Above this level, a substantial updraft reaches the 14-km level under deep large-scale

215 forcing (solid curve), consistent with the depth of forcing in Fig. 1(a). It is replaced by an updraft
 216 that decreases linearly with height between 4 and 14-km under shallow large-scale forcing (long
 217 dash). This response is rather unintuitive considering the fact that shallow large-scale forcing only
 218 reaches the 5-km level (*cf.*, Fig. 1). Only when very shallow large-scale forcing is considered
 219 does the convective response above the 4-km level becomes negligible (short dash).

220 Figure 7 shows the corresponding convective tendency profiles for temperature (a) and moisture
 221 (b). Clearly these do not match well with the forcings in Fig. 1, even though the cloud work
 222 functions for each mode are in equilibrium by construction (*cf.*, Sec. 5.a). The sudden increase
 223 of mass flux at the 4-km level (Fig. 6(b)) is associated with unrealistically strong heating and
 224 drying, with magnitudes *c.a.*, 60 Kday⁻¹ for temperature and -120 Kday⁻¹ for moisture. The
 225 peaks are manifestations of those seen for individual plume types in Fig. 5, but with the signs
 226 reversed: entrainment (*i.e.*, negative cloud-top detrainment) at the top of detraining-downdraft
 227 plumes causes this tendency. On the other hand, tendencies with more reasonable magnitudes are
 228 found at the other vertical levels.

229 4. Further Analyses

230 a. Completeness of the spectrum of plumes

231 The basic idea of the spectrum model is to be able to represent every possible convective profile
 232 using a sum of profiles from the individual plumes. Thus, we now ask whether the ensemble of
 233 entraining plumes has such a capacity? The question may be more formally posed as the possibility
 234 of decomposing any given arbitrary mass-flux profile, $M(z)$, by a plume spectrum given by the set
 235 of functions $\{\eta_i\}$ ($i = 1, \dots, n$) as:

$$236 \quad M(z) = \sum_{i=1}^n \tilde{m}_i \eta_i(z), \quad (4.1)$$

237 where \tilde{m}_i are the expansion coefficients. Unfortunately, performing such a decomposition is not
 238 straightforward, because the exponential entraining plume profiles of Eq. (2.4) do not constitute

239 an orthogonal set. Nevertheless, it is instructive to consider the issues further by assessing decom-
 240 positions of both η_i and M using a complete orthonormal set. For this purpose, it is convenient to
 241 use the vertical normal modes, $W_i(z)$, for the vertical velocity defined for the hydrostatic primitive
 242 equation system (Kasahara and Puri 1981, Fulton and Schubert 1985). Thus, we set:

$$243 \quad \eta_j = \sum_{i=1}^n \hat{\eta}_{ij} W_i(z) \quad (4.2a)$$

$$244 \quad M = \sum_{i=1}^n \hat{m}_i W_i(z) \quad (4.2b)$$

246 with expansion coefficients $\hat{\eta}_{ij}$ and \hat{m}_i for η_j and M , respectively. By substituting Eq. (4.2a) into
 247 Eq. (4.1), and by comparing this result with Eq. (4.2b), we find

$$248 \quad \hat{m}_i = \sum_{j=1}^n \tilde{m}_j \hat{\eta}_{ij} \quad (4.3)$$

249 and so the expansion coefficients, \tilde{m}_j , are determined by inverting the matrix $\hat{\eta}_{ij}$. In order for the
 250 inverse to exist, the determinant of this matrix must be non-zero.

251 To investigate the structure of the matrix, we perform a singular vector decomposition:

$$252 \quad \hat{\eta}_{ij} = \sum_{k=1}^n \lambda_k w_{ik} \tilde{w}_{kj}, \quad (4.4)$$

253 with eigenvalues, λ_k , and eigenvectors, w_{ik} and \tilde{w}_{kj} , the subscript k designating the index for the
 254 eigenmode. These are defined by linear eigenvalue problems:

$$255 \quad \sum_{j=1}^n \hat{\eta}_{ij} w_{jk} = \lambda_k w_{ik} \quad (4.5a)$$

$$256 \quad \sum_{i=1}^n \tilde{w}_{ki} \hat{\eta}_{ij} = \lambda_k \tilde{w}_{kj}. \quad (4.5b)$$

258 These two vector sets are called the right- and the left-vectors, which satisfy the orthonormality

$$259 \quad \sum_{k=1}^n \tilde{w}_{ik} w_{kj} = \delta_{ij}. \quad (4.6)$$

260 As a result, the determinant and the inverse matrix are defined by

$$261 \quad \det(\hat{\eta}_{ij}) = \prod_{k=1}^n \lambda_k, \quad (4.7a)$$

$$262 \quad \hat{\eta}_{ij}^{-1} = \sum_{k=1}^n \lambda_k^{-1} w_{ik} \tilde{w}_{kj}, \quad (4.7b)$$

263
264 respectively. The eigenvalue spectrum, $\{\lambda_k\}$, characterizes a singularity of a given matrix. If any
265 of the eigenvalues, λ_k , are too small, the determinant becomes very small, and the inverse matrix
266 becomes singular.

267 Figure 8(a) and (b) shows the plume spectrum, $\{\eta_j(z)\}$, and the plume matrix, $(\hat{\eta}_{ij})$, respec-
268 tively. To ensure that we retain sufficient vertical modes for the decomposition, and henceforth for
269 the dimension of the matrix, $(\hat{\eta}_{ij})$, we re-set $n = 40$ in Eq. (2.3) only for the analysis of the present
270 subsection. The eigenvalues, λ_k , obtained by the singular vector decomposition of Eq. (4.4) are
271 plotted in Fig. 9: note that we have chosen to label the eigenvectors in order of decreasing mag-
272 nitude of the corresponding eigenvalues, $|\lambda_k|$. The eigenvalues fall to very small values above
273 $k \geq 20$, suggesting that the entraining plume decomposition is highly redundant, and as a result
274 the determinant of the matrix, $(\hat{\eta}_{ij})$, practically vanishes.

275 However, the singular vector decomposition can be used to regularize a matrix by removing all
276 the small eigenvalues, λ_k , with, say, $k > n_c$ (with $n_c < n$) from the summations in Eqs. (4.4) and
277 (4.7b). Thus, we obtain

$$278 \quad \hat{\eta}_{ij} \simeq \sum_{k=1}^{n_c} \lambda_k w_{ik} \tilde{w}_{kj}, \quad (4.8a)$$

$$279 \quad \hat{\eta}_{ij}^{-1} \simeq \sum_{k=1}^{n_c} \lambda_k^{-1} w_{ik} \tilde{w}_{kj}. \quad (4.8b)$$

281 Setting $n_c = 16$ yields a regularized matrix $(\hat{\eta}_{ij})$ shown in Fig. 10(a), and its transformation back
282 to real space leads to Fig. 10(b). The reconstruction is noisier than the original spectrum shown in
283 Fig. 8(a); nevertheless, the overall structure remains surprisingly similar.

284 In summary, the completeness analysis demonstrates the entraining–plume decomposition to be
 285 highly redundant, so that it does not directly permit a decomposition of any vertical mass flux
 286 profile under the formulae (4.1) and (4.3) due to a singularity of the matrix, $(\hat{\eta}_{ij})$. However, the
 287 singularity can easily be removed under a singular–vector decomposition, and the reconstructed
 288 nonsingular plume spectrum remains fairly close to the original entraining–plume spectrum. Thus,
 289 the redundancy of the entraining–plume decomposition is not a practical issue in applying the
 290 convective quasi–equilibrium closure.

291 *b. Eigenvalues and eigenvectors of the interaction matrix, \mathbf{K}*

292 The basic structure of the interaction matrix, \mathbf{K} , can also be elucidated by performing a singular–
 293 vector decomposition. Here, the right– and the left–eigenvectors, $\vec{\mathbf{M}}_l$ and $\tilde{\mathbf{M}}_l$, respectively, are
 294 defined by solving linear eigenvalue problems:

$$295 \quad \mathbf{K}\vec{\mathbf{M}}_l = \kappa_l\vec{\mathbf{M}}_l, \quad (4.9a)$$

$$296 \quad \tilde{\mathbf{M}}_l\mathbf{K} = \kappa_l\tilde{\mathbf{M}}_l \quad (4.9b)$$

297
 298 with the eigenvalues, κ_l ($l = 1, \dots, n$). Recall the orthonormality:

$$299 \quad \tilde{\mathbf{M}}_i \cdot \vec{\mathbf{M}}_j = \delta_{ij}. \quad (4.10)$$

300 The large–scale forcing vector, $\vec{\mathbf{F}}$, may then be represented in terms of the interaction matrix
 301 decomposition by

$$302 \quad \vec{\mathbf{F}} = \sum_l F_l \vec{\mathbf{M}}_l \quad (4.11)$$

303 with the expansion coefficients, F_l , being defined by

$$304 \quad F_l = \tilde{\mathbf{M}}_l \cdot \vec{\mathbf{F}}. \quad (4.12)$$

305 Similarly, the cloud–base mass–flux vector, $\vec{\mathbf{M}}$, may be represented as

$$306 \quad \vec{\mathbf{M}} = \sum_l \mu_l \vec{\mathbf{M}}_l. \quad (4.13)$$

307 Substitution of Eqs. (4.11) and (4.13) into Eq. (1.1) shows that the expansion coefficients are
 308 related by

$$309 \quad \mu_l = F_l / \kappa_l. \quad (4.14)$$

310 The interaction–matrix eigenvalues, κ_l , are plotted in Fig. 11 in decreasing order of their absolute
 311 values. From Eq. (4.14), if the large–scale forcing were to contribute with the same order to all
 312 of the eigenmodes [*cf.*, Fig. 1(b)], then the higher–order modes (say, $l \geq 14$) would dominate the
 313 convective response.

314 Considering the eigenmodes themselves, the spectra of the first eight right– and left–
 315 eigenvectors, $\vec{\mathbf{M}}_l$ and $\tilde{\mathbf{M}}_l$, are shown in Figs. 12 and 13, respectively. The most striking feature
 316 is that the right eigenvectors, $\vec{\mathbf{M}}_l$, are dominated by the high–entrainment (*i.e.*, high mode index)
 317 shallower modes, whereas the left–eigenvectors, $\tilde{\mathbf{M}}_l$, encompass relatively low–entrainment deep
 318 modes (from the 8th to the 14th mode index).

319 These features have significant consequences in defining the response of convection, $\vec{\mathbf{M}}$, against
 320 a given large–scale forcing, $\vec{\mathbf{F}}$. First, the expansion coefficients, F_l , for the large–scale forcing
 321 are defined by projecting the large–scale forcing, $\vec{\mathbf{F}}$, onto the left–eigenvectors, $\tilde{\mathbf{M}}_l$ by Eq. (4.12).
 322 Since $\tilde{\mathbf{M}}_l$ reflects the deeper modes, there is a tendency that the deeper the structure of the large–
 323 scale forcing, the stronger the projection onto the expansion coefficients, F_l , and hence onto μ_l ,
 324 through Eq. (4.14). However, the right–eigenvectors, $\vec{\mathbf{M}}_l$, are dominated by the shallow modes,
 325 and thus, the convective response, $\vec{\mathbf{M}}$, as defined by Eq. (4.13) is also dominated by shallow modes.
 326 Due to these different characteristics of the left– and the right–eigenvectors, we therefore find that
 327 convection responds most effectively to deeper modes of large–scale forcing, but that it manifests
 328 as a response primarily through the shallower modes. Note that this “twisted” relation stems from
 329 a strong asymmetry of the interaction matrix, as is demonstrated more explicitly using a simple
 330 idealized example matrix in the next subsection.

331 *c. Analysis of an idealized, highly-truncated interaction matrix*

332 It is also informative to take an analytical perspective on the singularities in strict convective
 333 quasi-equilibrium closure by examining an idealized interaction matrix, which captures its basic
 334 characteristics. Specifically, we consider a 3×3 interaction matrix, \mathbf{K} , of the form:

$$335 \quad \mathbf{K} = \begin{pmatrix} k & -k_d & -k_d \\ k_s & k & -k_d \\ k_s & k_s & -1 \end{pmatrix} \quad (4.15a)$$

336 The quasi-equilibrium closure of Eq. (1.1) reduces to:

$$337 \quad \mathbf{K} \begin{pmatrix} M_3 \\ M_2 \\ M_1 \end{pmatrix} + \begin{pmatrix} F_3 \\ F_2 \\ F_1 \end{pmatrix} = 0. \quad (4.15b)$$

338 Here, the order of the vector indices for $\vec{\mathbf{M}}$ and $\vec{\mathbf{K}}$ is reversed from a standard convention so that
 339 the matrix form defined by Eq. (4.15a) closely follows the matrix-element distributions shown
 340 in Fig. 4: the given distribution can directly be compared with the definition (4.15a) by flipping
 341 the horizontal direction in the figures. The idealized matrix is normalized by setting the right-
 342 lowest element to -1 ; k , k_s , and k_d are expected to be small values, where k and k_s represent
 343 destabilization tendencies of shallow convection modes acting on themselves and on the deeper
 344 modes, respectively, whereas $-k_d$ represents the stabilization from the deeper modes to shallower
 345 modes.

346 The solution to the matrix problem (4.15b) is:

$$347 \quad M_3 = [(2k - k_s + k_d - 1)k_s k_d - k^2]^{-1} [(k - k_s k_d)F_3 + (1 + k_s)k_d F_2 - (k + k_d)k_d F_1] \quad (4.16a)$$

$$348 \quad M_2 = [(2k - k_s + k_d - 1)k_s k_d - k^2]^{-1} [(k_d - 1)k_s F_3 + (k - k_s k_d)F_2 - (k - k_s)k_d F_1] \quad (4.16b)$$

$$349 \quad M_1 = [(2k - k_s + k_d - 1)k_s k_d - k^2]^{-2} [(k - k_s)k_s F_3 + (k + k_d)k_s F_2 - (k^2 + k_s k_d)F_1]. \quad (4.16c)$$

350

351 A further simplification is to set all the small parameters to the same value, $k_s = k_d = k$, so that
 352 the solution in Eq. (4.16) becomes:

$$353 \quad M_3 = \frac{1}{2(k-1)k} [(1-k)F_3 + (1+k)F_2 - 2kF_1] \quad (4.17a)$$

$$354 \quad M_2 = \frac{F_3 - F_2}{2k} \quad (4.17b)$$

$$355 \quad M_1 = \frac{-F_2 + F_1}{1-k}. \quad (4.17c)$$

356

357 When the limit of $k \rightarrow 0$ is taken, the above solution reduces to

$$358 \quad M_3 \rightarrow -\frac{F_3 + F_2}{2k}$$

$$359 \quad M_2 \rightarrow \frac{F_3 - F_2}{2k}$$

$$360 \quad M_1 \rightarrow -F_2 + F_1,$$

361

362 retaining only the leading terms with respect to k , and assuming all forcing components, F_j ($j =$
 363 $1, 2, 3$), to be of $O(1)$. Thus, the two shallowest convective modes, M_3 and M_2 , respectively, diverge
 364 in the limit of $k \rightarrow 0$. Also note that the signs of M_2 and M_1 sensitively depend on differences
 365 between F_3 and F_2 , and that between F_2 and F_1 , respectively.

366 In this manner, the idealized matrix (4.15a) provides a very simple demonstration for the origin
 367 of the singular behaviors of the quasi-equilibrium closure that were seen in previous sections.

368 *d. Perturbation Analysis*

369 The idealized matrix problem may be further developed by considering a perturbation expansion.

370 Noting that many of the matrix elements are small in \mathbf{K} , we can write:

$$371 \quad \mathbf{K} = \mathbf{K}^{(0)} + \delta\mathbf{K}^{(1)} \quad (4.18a)$$

$$372 \quad \vec{\mathbf{M}} = \vec{\mathbf{M}}^{(0)} + \delta\vec{\mathbf{M}}^{(1)} + \dots \quad (4.18b)$$

373

374 where δ is a small expansion parameter, and where the idealized matrix of Eq. (4.15a) can be
 375 decomposed as:

$$376 \quad \mathbf{K}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.19a)$$

$$377 \quad \delta \mathbf{K}^{(1)} = \begin{pmatrix} k & -k_d & -k_d \\ k_s & k & -k_d \\ k_s & k_s & 0 \end{pmatrix} \quad (4.19b)$$

378
 379 To $O(1)$, we obtain:

$$380 \quad \mathbf{K}^{(0)} \vec{\mathbf{M}}^{(0)} + \vec{\mathbf{F}} = 0$$

381 with $\det(\mathbf{K}^{(0)}) = 0$, because of the fact that large elements are localized, and hence there is no
 382 solution available for $\vec{\mathbf{M}}^{(0)}$. To avoid this problem, we need to re-formulate the expansion of
 383 Eq. (4.18b) as:

$$384 \quad \vec{\mathbf{M}} = \frac{1}{\delta} \vec{\mathbf{M}}^{(-1)} + \vec{\mathbf{M}}^{(0)} + \dots \quad (4.20)$$

385 so that we obtain to $O(1/\delta)$:

$$386 \quad \mathbf{K}^{(0)} \vec{\mathbf{M}}^{(-1)} = 0, \quad (4.21)$$

387 which, with the matrix (4.15a), leads to

$$388 \quad M_1^{(-1)} = 0$$

389 and leaves the other two components, $M_2^{(-1)}$ and $M_3^{(-1)}$, as undetermined. At $O(1)$ we have,

$$390 \quad \mathbf{K}^{(0)} \vec{\mathbf{M}}^{(0)} + \mathbf{K}^{(1)} \vec{\mathbf{M}}^{(-1)} + \vec{\mathbf{F}} = 0, \quad (4.22)$$

391 which makes the problem solvable. Specifically for the case with Eq. (4.15a), this $O(1)$ relation
 392 defines $M_2^{(-1)}$, $M_3^{(-1)}$, and $M_1^{(0)}$.

393 Thus, the perturbation analysis here more explicitly demonstrates how a strict application of the
 394 convective quasi-equilibrium condition tends to lead to an abnormally strong response of shallow
 395 convection to large-scale forcing.

396 5. Physical Implications

397 The present paper has focused on a rather narrow question of mathematical difficulties with
 398 the original closure formulation of Arakawa and Schubert (1974). In concluding, we turn to the
 399 physical implications from the present findings, also referring to background issues.

400 a. Free-ride principle

401 The convective quasi-equilibrium closure of Eq. (1.1) is based on stationarity of the cloud work
 402 function, which is a vertically-integrated quantity (*cf.*, Eq. 133 of Arakawa and Schubert 1974).
 403 Thus, the closure is also formulated in terms of vertically-integrated quantities. However, we
 404 might intuitively expect that a certain quasi-equilibrium state (*i.e.*, a balance condition) is achieved
 405 at each vertical level, at least to a good approximation, if a large enough number of convective
 406 modes is considered. The different modes provide different weighting functions and upper limits
 407 for the integrals in question.

408 It is observationally known that the large-scale tropical atmosphere satisfies a free-ride state
 409 (Fraedrich and McBride 1989: later Sobel *et al.* 2001 term it alternatively as “weak temperature
 410 gradient”), with a close balance between the large-scale tendency and the convective response in
 411 both the heat and moisture equations:

$$412 \quad w \frac{d\bar{\theta}}{dz} \simeq Q_1 + Q_R, \quad (5.1a)$$

$$413 \quad w \frac{d\bar{q}_v}{dz} \simeq -\frac{C_p}{L} Q_2. \quad (5.1b)$$

415 Here, Q_1 and Q_2 are tendencies due to non-advective processes, apart from radiative heating,
 416 Q_R , in the context of large-scale modeling (*i.e.*, convective-scale advections are not explicitly

417 considered). See Fig. 1 of Yano (2001: also reproduced as Fig. 4.2 in Ch. 4 of Plant and Yano
 418 2015) for a graphical demonstration. Although the literature tends to refer only to the balance
 419 (5.1a), here, it is seen that the second balance (5.1b) is equally valid. In the large-scale tropical
 420 atmosphere, Q_1 and Q_2 are mostly due to convection (*i.e.*, Q_c). On the other hand, the vertical
 421 advection and the radiation terms may be combined to define the total large-scale forcing, F_L .
 422 Thus, the free-ride state may be equivalently expressed in the form

$$423 \quad Q_c + F_L \simeq 0 \quad (5.2)$$

424 for both variables. Eq. (5.2) may be considered as a statement of convective quasi-equilibrium,
 425 but defined separately on each vertical level, rather than as an integral constraint.

426 Hence, we are led to ask whether, given enough plume modes in Eq. (1.1), we obtain a free-
 427 ride state corresponding to Eq. (5.2): will this be actually accomplished in practice by the quasi-
 428 equilibrium closure?

429 *b. Completeness of the plume spectrum*

430 Equivalence between Eqs. (1.1) and (5.2) could be established if the mass–flux spectrum were
 431 able to represent any possible convective response that may be required to satisfy the free–ride
 432 state. Thus, a first consideration is whether the mass–flux spectrum is flexible enough to represent
 433 any possible vertical profile. This has been examined using normal–mode and singular–vector de-
 434 compositions in Sec. 4.a. The entraining–plume decomposition is shown to be highly redundant,
 435 as expected from the individual plume profiles (*cf.*, Fig. 2), and so a decomposition of the entrain-
 436 ing plumes into normal modes does not provide well–defined expansion coefficients. However,
 437 this ill-posedness of the decomposition can be resolved by removing all the singular vectors with
 438 almost–vanishing eigenvalues from the expansion. A reconstructed plume spectrum still remains
 439 fairly close to the original entraining–plume spectrum, but practically removing the redundancy.

440 Here, the mathematical question of the completeness of a plume spectrum addresses its capacity
441 and flexibility to represent any physically-feasible vertical structure of convection. As we have
442 seen, the conclusion obtained is rather mixed, and further investigations from a more practical
443 perspective could be warranted.

444 *c. Convective response under the spectrum mass flux*

445 The next consideration is how an individual plume mode modifies the large-scale thermody-
446 namic state (*i.e.*, convective response: Sec. 3.b). The effect of an individual entraining plume
447 is comprised of two main parts: (i) detrainment that cools and moistens the large-scale by re-
448 evaporation of the detrained cloudy air; and, (ii) compensating environmental descent, in response
449 to the convective updraft, that induces adiabatic heating and drying. A major difference between
450 these two effects is that the detrainment effect is found only at a single level at the plume top,
451 whereas the environmental descent is felt at all of the vertical levels spanned by the plume. As a
452 result, the detrainment effect focused on a single vertical level tends to be abnormally strong, with
453 cooling and moistening rates far exceeding 10 Kday^{-1} and so strongly dominant at that level over
454 the environmental-descent effect.

455 The consequence is that a straightforward inversion of the interaction matrix in the closure con-
456 dition of Eq. (1.1) produces a full convective response against a given large-scale forcing that be-
457 comes very singular (Sec. 3.c). For idealized large-scale forcing profiles with a half-sine shaped
458 large-scale uplifting, we find that the convective response is dominated by singularly strong warm-
459 ing and cooling induced at the top of the detraining-plume downdrafts (*i.e.*, entraining-plume
460 updraft modes with a negative amplitude). Due to the tendency of entraining-plume modes to pro-
461 duce a singular response, the convective quasi-equilibrium closure condition does not achieve a
462 thermodynamic state close to the free ride balances. Thus, the mathematical analysis herein points

463 out in an explicit manner how and why a physically unrealistic feature of the entraining–plume
464 model causes a problem.

465 A very simple way of removing these singular cooling–moistening effects would be to neglect
466 all of the detrainment effects from the interaction matrix, \mathbf{K} , by setting $\mathbf{K}_d = 0$ in Eq. (3.1) so
467 that the interaction matrix, \mathbf{K} , is replaced by \mathbf{K}_v . However, totally removing this effect from the
468 convective equilibrium problem would not be very realistic for reasons discussed in Sec. 4.e.

469 *d. Interaction matrix analysis*

470 Another important aspect of the convective response under the convective quasi–equilibrium
471 closure is the dominance of shallow plumes regardless of the vertical extent of large–scale forcing.
472 This is rather unintuitive. However, one must remember that as a matter of principle, large–scale
473 forcing is projected to all the plume modes by design, as explicitly shown by Fig. 1(a). The
474 resulting spectrum of the convective response is rather nontrivial, mathematically taking the form
475 of a matrix inversion. This character of the problem means that we need to pay attention to the
476 mathematical behaviour of the inversion calculation in order to better understand the structural
477 issues involved.

478 First, a singular–vector decomposition is performed on the interaction matrix in Sec. 4.b. The
479 left–eigenvector spectra are dominated by middle–height plume modes, with maximum heights
480 of 8–10 km, thus relatively deep components of large–scale forcing lead to a strong response by
481 convection. On the other hand, the right–eigenvector spectra are dominated by shallow plume
482 modes, and thus relatively–deep large–scale forcing modes are strongly projected onto shallow
483 convective modes.

484 This rather strong asymmetry between the left and the right eigenvectors stems from a strong
485 asymmetry in the interaction matrix itself. In turn, the asymmetry of the interaction matrix stems
486 from the nature of the detrainment effect of a plume mode onto other plume modes: only the

487 deeper plume modes are affected by detrainment from a given plume mode, and this gives rise to
488 the triangular structure apparent within the interaction matrix (*cf.*, Fig. 4(b)).

489 With increasing precipitating efficiency, the detrainment effect becomes weaker as less cloudy
490 air is available to detrain at plume top. In a fully-precipitating limit for all of the plume modes,
491 then the asymmetry of the interaction matrix would disappear, and the singular response to the
492 large-scale forcing would be removed. However, additional calculations (not shown) indicate that
493 even a weak asymmetry of the interaction matrix can lead to a singular response. A relatively
494 strong sensitivity of the convective response to the transition scale, ϵ_0 , in precipitation-efficiency
495 [Eq. (2.5)] has also been found because this parameter controls the relative contribution of detrain-
496 ment effects to the interaction matrix.

497 An idealized 3×3 interaction matrix (Sec. 4.c-d) is able to reproduce the character of these
498 results. A singular perturbation expansion is required for describing the convective quasi-
499 equilibrium closure due the fact that the matrix elements related to shallow convection tend to
500 be substantially smaller than those for the interactions between deep convection. As a result,
501 shallow convection tends to respond to large-scale forcing in a singular manner.

502 *e. Further Physical Implications*

503 An important feature throughout the present analysis is the strong cooling and moistening in-
504 duced by re-evaporation of the detrained cloudy air. When this contribution is suppressed, the con-
505 vective response under the quasi-equilibrium closure becomes much more reasonable. It is worth
506 noting that some alternative formulations of mixing, beyond the simple entrainment formulation
507 of pure Arakawa and Schubert (1974), may help to alleviate the problem (de Rooy *et al.* 2013,
508 Yano 2015). Another legitimate way of suppressing this effect is to couple the convection param-
509 eterization with a stratiform cloud representation, and to transfer the detrained cloudy convective
510 air to form part of a stratiform cloud rather than immediately re-evaporating it into the environ-

511 ment. The importance of this procedure would probably be needless to emphasize, because such a
512 coupling of convection with stratiform clouds is already accomplished in most of the operational
513 global models already. However, its significance, to the extent revealed here, appears to be not
514 widely appreciated.

515 At the same time, completely suppressing the evaporative cooling of the detrained cloudy air
516 would likely not be wise. Yano and Plant (2012b) suggest that the resulting destabilization ten-
517 dency of shallow convection can be a key mechanism driving transformations from shallow to
518 deep convection. Two solutions may be considered for this remedy. The first is to retain the ten-
519 dency explicitly for shallow convection, rather than imposing a strict equilibrium constraint. In
520 this case, a singular response of shallow convection to large-scale forcing associated with evap-
521 orative cooling must be tamed in a different manner. The second is to transfer the role of this
522 destabilization tendency to the stratiform cloud scheme: the mechanism may be represented by
523 the cloud-top entrainment process (*cf.*, Deardorff 1980, Randall 1980) under this reformulation,
524 which is also expected to lead to an equivalent destabilization.

525 Another important implication from the present study is a much slower response time scale
526 for the shallower convective modes than for the deep convection, as indicated by the relatively
527 small elements in the interaction matrix. This implication can be seen directly from the prognostic
528 equation for the cloud work function spectrum, $\vec{\mathbf{A}}$, from which the quasi-equilibrium closure (1.1)
529 is derived:

$$530 \quad \frac{\partial \vec{\mathbf{A}}}{\partial t} = \mathbf{KM} + \vec{\mathbf{F}}. \quad (5.3)$$

531 The quasi-equilibrium closure has been justified based on an argument that an *overall* time scale
532 for the response of convection to large-scale forcing is so short that we can drop the time tendency
533 of the cloud work function on the left hand side, which is expected to evolve by following a slow
534 large-scale time scale.

535 However, more precisely, the response time scale is short only for deep convection, but *not*
536 for shallow convection. As a result, Eq. (5.3) may be approximated by Eq. (1.1) for the deep
537 convection part only. In other words, the full convective ensemble does not immediately respond
538 to any slow large-scale forcing, as originally envisioned by Arakawa and Schubert (1974). Rather,
539 a finite time-scale for the convective response to large-scale forcing should explicitly be taken into
540 account by retaining the temporal tendency of the cloud work function on the left hand side of
541 Eq. (5.3), so that the closure becomes fully prognostic. Suitable formulations are already in place
542 (*e.g.*, Pan and Randall 1998, Yano and Plant 2012c). Here, we point out a solid reason for moving
543 towards this direction.

544 The issues appear to be further involved, because observational analyses by Zhang (2002, 2003),
545 Donner and Phillips (2003) suggest that the boundary-layer processes controlling the evolution of
546 the convective available potential energy (CAPE), and thus also likely of the cloud work functions,
547 are of a much shorter time scale than those found in the free atmosphere. Thus, boundary-layer
548 processes, neglected in the analysis herein, may further contribute to break down a strict applica-
549 tion of convective quasi-equilibrium closure. Those implications warrant further investigations.

550 The present study further suggests needs for re-considering the mass-flux convection param-
551 eterization formulation from more general perspectives. Such investigations are already under
552 way (*e.g.*, Yano *et al.* 2005b, Yano 2014b, 2016). These developments should more seriously be
553 considered in operational contexts.

554 **Appendix: Scale Analysis**

555 The purpose of this Appendix is to estimate the order of magnitude of cooling and moistening
556 associated with re-evaporation of the detrained cloudy air.

We begin with the simple point that latent heating due to the condensation of a unit of water vapor, $q = 1 \text{ g kg}^{-1}$, leads to an increase of temperature by

$$qL/C_p = 2.5 \text{ K}$$

using the latent heating, $L = 2.5 \times 10^6 \text{ J kg}^{-1}$, and the specific heat capacity, $C_p = 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$ for air at constant pressure. A typical mass flux value under convective quasi-equilibrium is $M \sim 10^{-2} \text{ kg m}^{-2} \text{ s}^{-1}$, or $M/\rho \sim w_0 = 10^{-2} \text{ ms}^{-1}$ in units of vertical velocity. At the convective cloud top, all of the mass flux detrains under the entraining-plume hypothesis. The associated heat flux is thus

$$(M/\rho)(q_c L/C_p) \sim 2.5 \times 10^{-2} \times q_c \text{ K m s}^{-1} \sim 2.5 \times 10^3 \times q_c \text{ K m day}^{-1} \quad (\text{A.1})$$

557 where the detrained cloud-water mixing ratio, q_c , is expressed in units of [g/kg]. If the detrainment
 558 occurs over a layer of, say, 1 km in depth, it will amount to a cooling rate of 2.5 K day^{-1} for
 559 $q_c = 1 \text{ g kg}^{-1}$.

The last piece of estimate is the amount of cloud water, q_c , expected at the convective cloud top at the height of, say, $H \sim 10 \text{ km}$. To obtain this, we note that within a convective updraft, condensative heating is well balanced by adiabatic cooling (a local realization of free-ride state: *cf.*, Eq. 5.1). Thus,

$$\frac{L}{C_p} \frac{dq_v^*}{dz} + \frac{d\theta}{dz} \simeq 0.$$

This relation leads to an estimate for the rate of decrease of saturated water-vapor with height,

$$\frac{dq_v^*}{dz} \sim -\frac{C_p}{L} \frac{d\theta}{dz} \sim -10^{-6} \text{ m}^{-1}.$$

Thus, neglecting fall out due to precipitation, the accumulation of condensed water in lifting through a height $H \sim 10 \text{ km}$ is estimated as:

$$q^* = -H \frac{dq_v^*}{dz} \sim 10^{-6} \text{ m}^{-1} \times 10^4 \text{ m} \sim 10^{-2} \text{ kg kg}^{-1} \sim 10 \text{ g kg}^{-1}. \quad (\text{A.2})$$

560 Substitution of Eq. (A.2) into Eq. (A.1) leads to an estimate of the cooling rate of 25 K day^{-1} for
561 a 1 km–deep detrainment layer.

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639 **LIST OF FIGURES**

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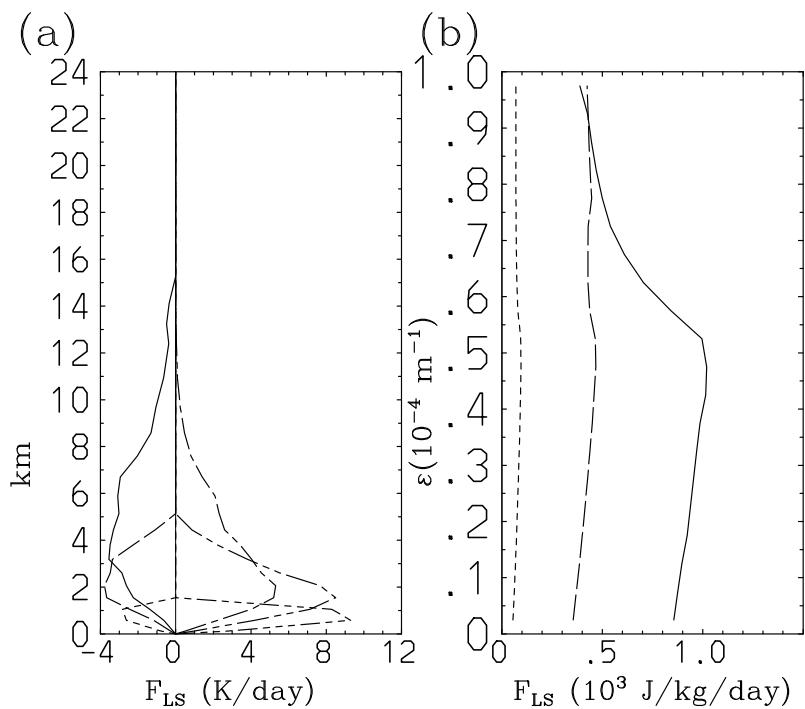
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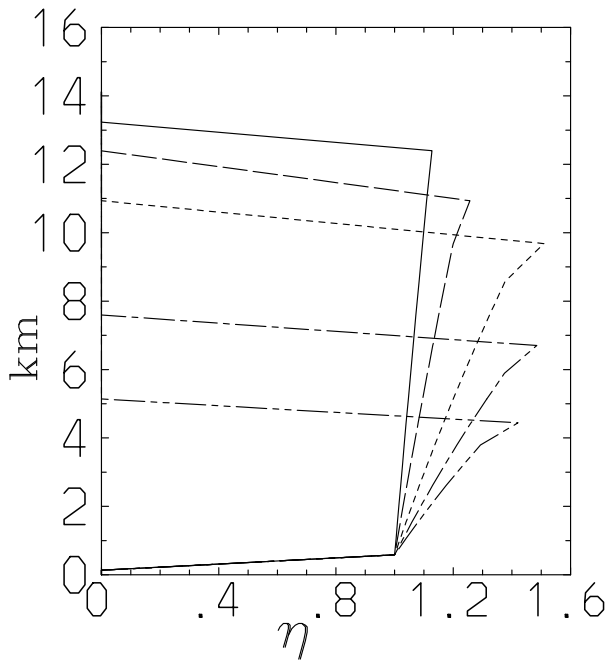
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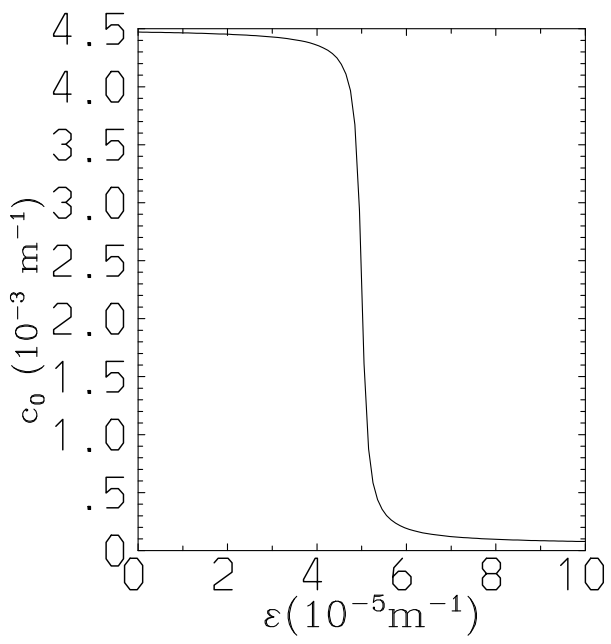
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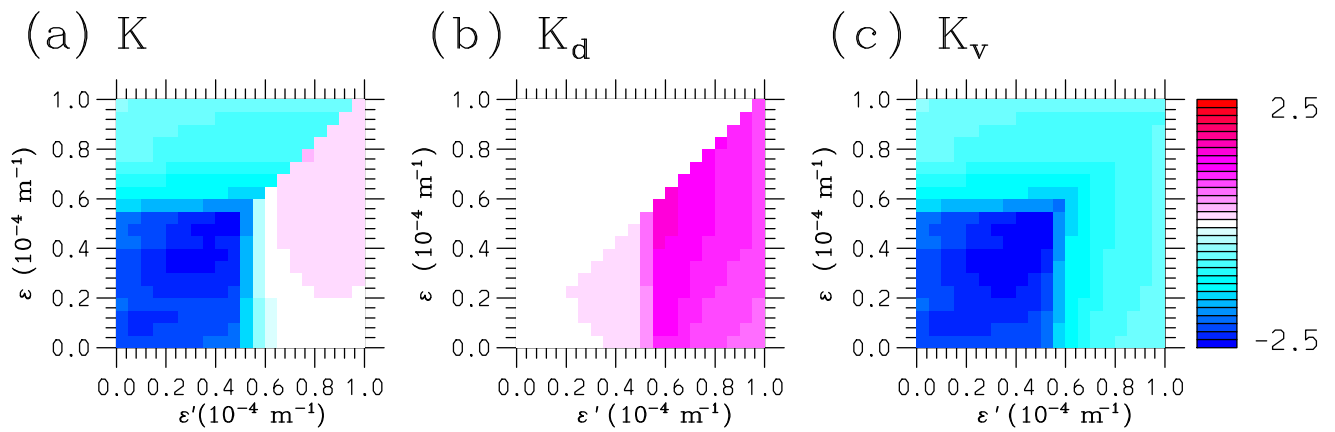
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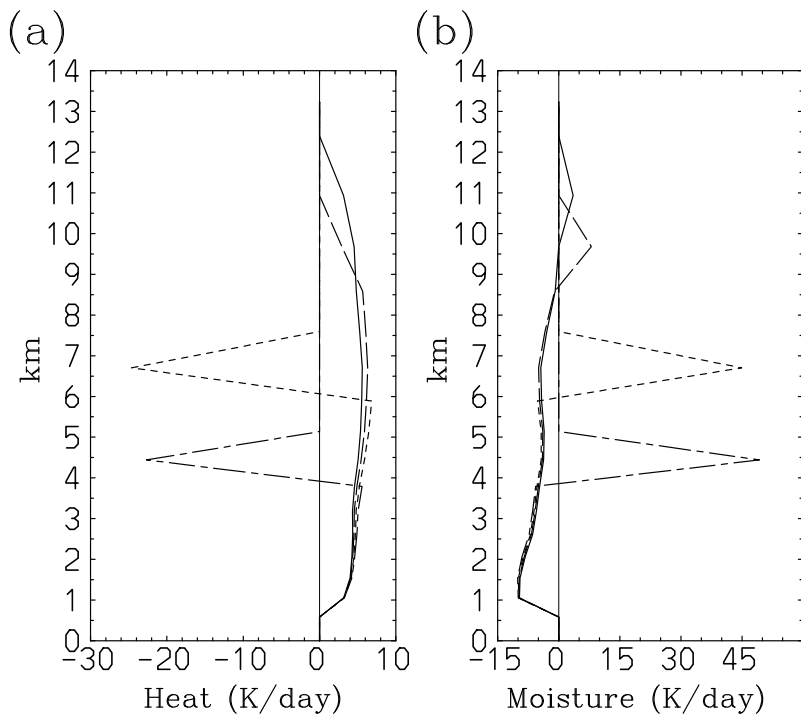
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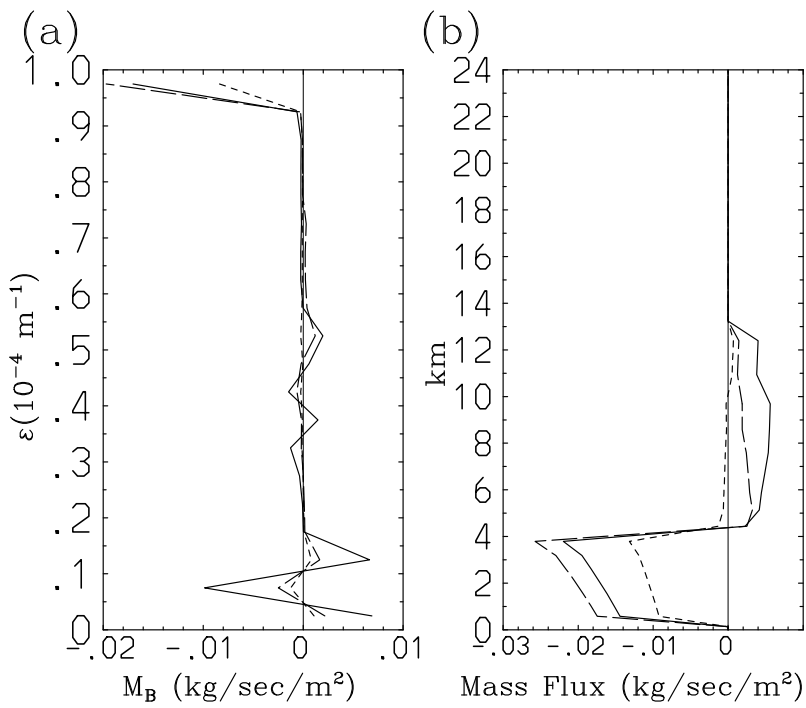
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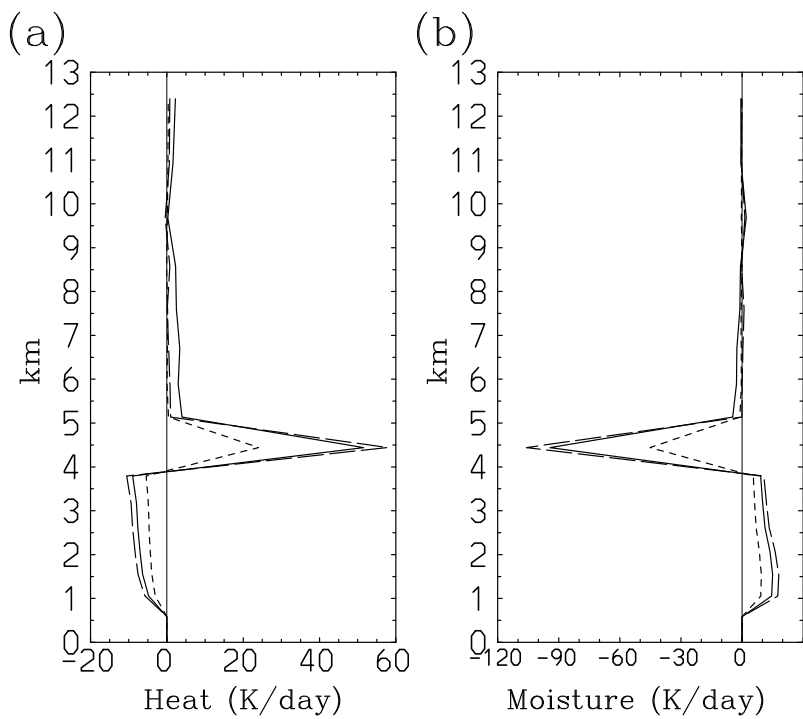
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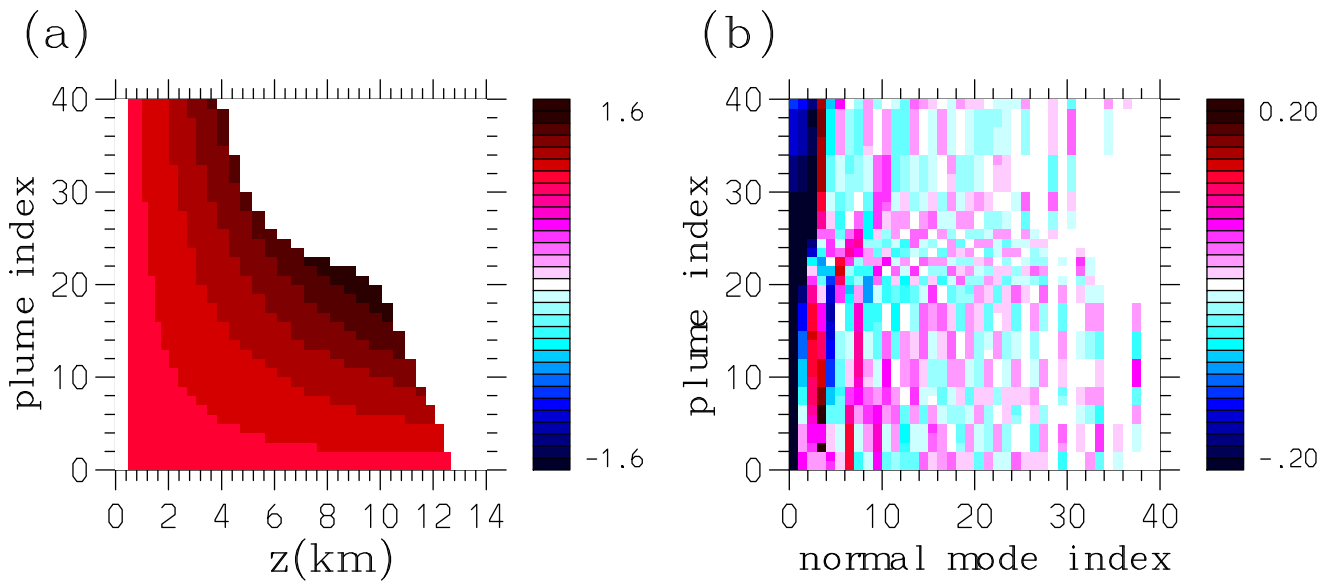
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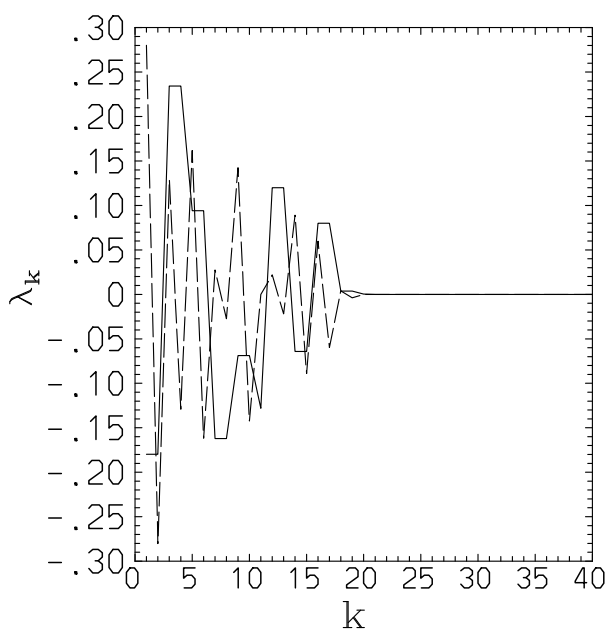
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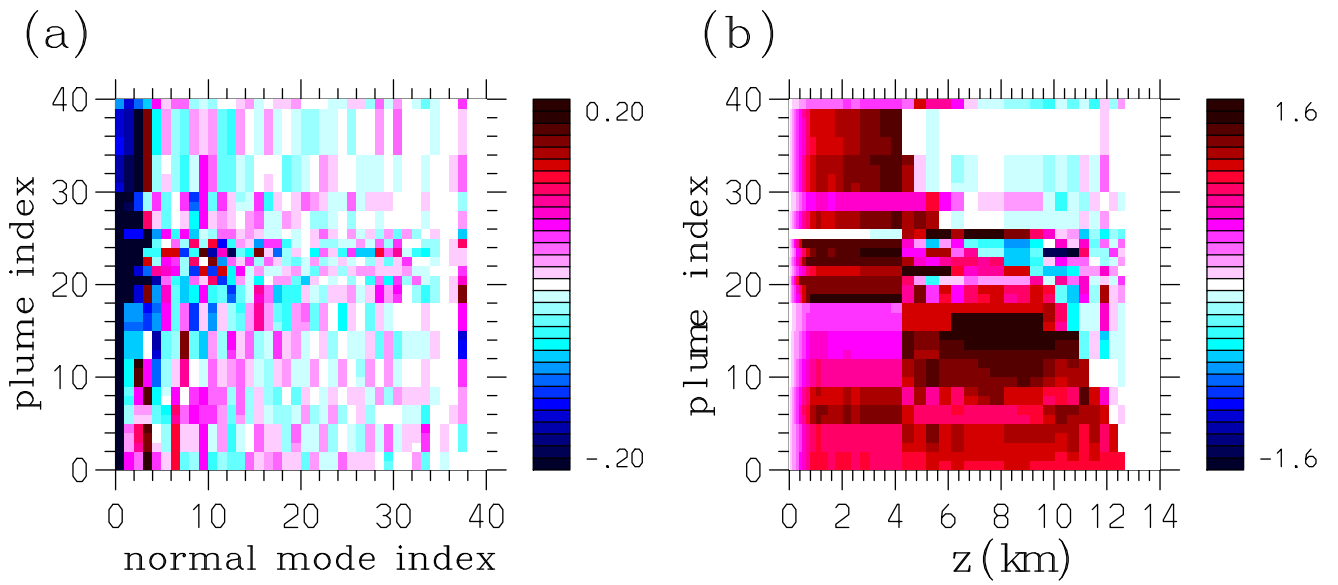
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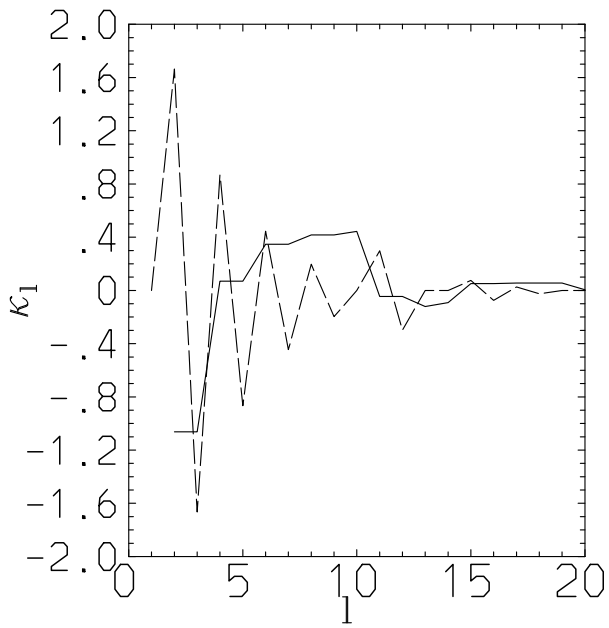
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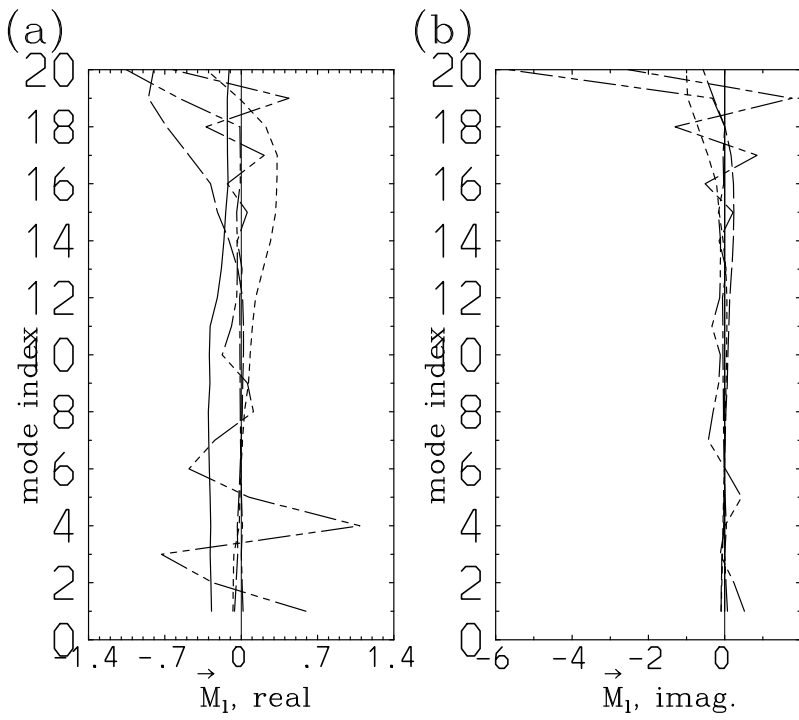
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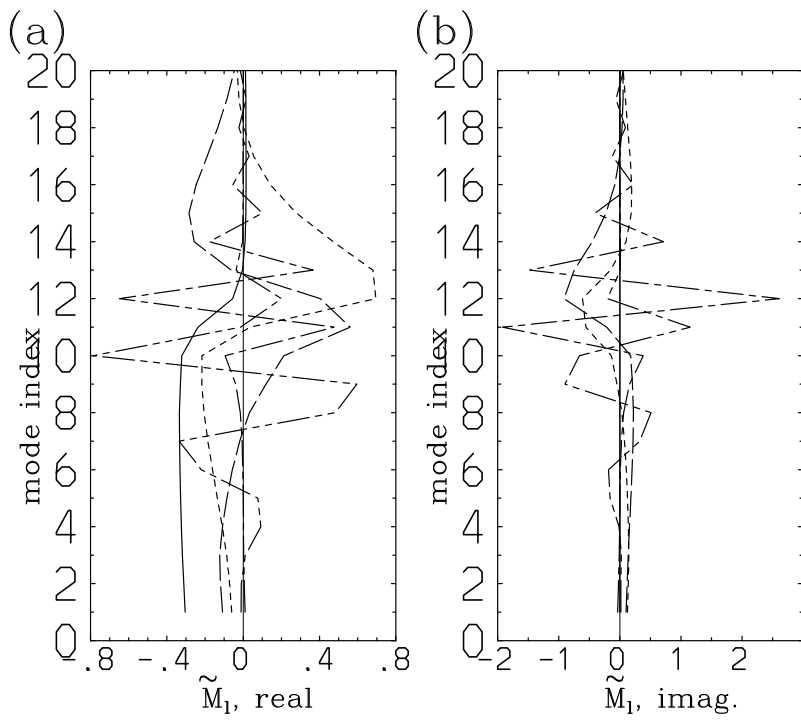
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