

*Rank deficiency of Kalman error
covariance matrices in linear time-varying
system with deterministic evolution*

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3 **RANK DEFICIENCY OF KALMAN ERROR COVARIANCE**
4 **MATRICES IN LINEAR TIME-VARYING SYSTEM WITH**
5 **DETERMINISTIC EVOLUTION***

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8 **Abstract.** We prove that for-linear, discrete, time-varying, deterministic system (perfect-model)
9 with noisy outputs, the Riccati transformation in the Kalman filter asymptotically bounds the rank
10 of the forecast and the analysis error covariance matrices to be less than or equal to the number
11 of nonnegative Lyapunov exponents of the system. Further, the support of these error covariance
12 matrices is shown to be confined to the space spanned by the unstable-neutral backward Lyapunov
13 vectors, providing the theoretical justification for the methodology of the algorithms that perform
14 assimilation only in the unstable-neutral subspace. The equivalent property of the autonomous
15 system is investigated as a special case.

16 **Key words.** Kalman filter, data assimilation, linear dynamics, control theory, covariance matrix,
17 rank

18 **AMS subject classifications.** 93E11, 93C05, 93B05, 60G35, 15A03

19 **DOI.** 10.1137/15M1025839

20 **1. Introduction.** The problem of estimating the state of an evolving system
21 from an incomplete set of noisy observations is the central theme of the state es-
22 timation and optimal control theory [7], also referred to as data assimilation (DA)
23 in geosciences [6, 20]. In the filtering procedure, based on the concept of recursive
24 processing, measurements are utilized sequentially, as they become available [7]. For
25 linear dynamics, and when a linear relation exists between measurements and the
26 state variables, and when the errors associated to all sources of information are Gaus-
27 sian, the solution can be expressed via the Kalman filter (KF) equations [8]. The KF
28 provides a closed set of equations for the first two moments of the posterior probabil-
29 ity density function of the system state, conditioned on the observations. In the case
30 of nonlinear dynamics, the first order extension of the KF is known as the extended
31 Kalman filter (EKF) [7], whereas a Monte Carlo approximation is the basis of a set
32 of methods known as the ensemble Kalman filter, both of which have been studied
33 extensively in geophysical contexts [13, 5].

34 Atmosphere and ocean are example of dissipative chaotic systems. This implies
35 sensitivity to the initial condition [11] and the fact that the estimation error strongly

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36 projects on the unstable manifold of the dynamics [18], which has inspired the devel-
 37 opment of a class of algorithms known as assimilation in the unstable subspace (AUS)
 38 [23]. In AUS, the span of the leading Lyapunov vectors (to be defined precisely in
 39 later sections), or a suitable approximation of this span, is used explicitly in the anal-
 40 ysis step: the analysis update is confined to the unstable subspace [16]. AUS has
 41 been formalized in the framework of the EKF, the EKF-AUS [22], and the variational
 42 (smoothing) procedure, 4DVar-AUS [21]. Applications with atmospheric, oceanic, and
 43 traffic models [24, 3, 17] showed that even in high dimensional systems, an efficient
 44 error control is achieved by monitoring only a limited number of unstable directions,
 45 making AUS a computationally efficient alternative to standard procedures. The AUS
 46 methodology is based on and at the same time hints at a fundamental observation:
 47 the span of the estimation error covariance matrices asymptotically (in time) tends
 48 to the subspace spanned by the unstable-neutral Lyapunov vectors.

49 The search for a formal proof of this aforesaid property is the basic motivation of
 50 the present work, which is focused on linear nonautonomous and linear autonomous
 51 perfect-model dynamical systems. The main results of the paper are as follows. In
 52 Theorem 3.5 we show that the error covariance matrices, independent of the initial
 53 condition, asymptotically become rank deficient in time, and then in Theorem 3.7 we
 54 characterize their null spaces by proving that the restriction of these matrices onto
 55 the stable backward Lyapunov vectors converges to zero in time. When restricted to
 56 the linear, autonomous system with the time invariant propagator A , we establish that
 57 the stable space of the time independent backward Lyapunov vectors equals the stable
 58 space of A^T —span of generalized eigen-vectors of A^T corresponding to eigen-values
 59 less than one in absolute magnitude—in Theorem B.3. Consequently, in Corollary 4.2
 60 we show that the null space of the error covariance matrices contain the stable space
 61 of A^T asymptotically.

62 The paper is organized as follows. After describing the general notation in sec-
 63 tion 2, the nonautonomous case is considered in section 3. The assumptions used
 64 in proving our main result, other useful results such as the Oseledets theorem, and
 65 the concepts of observability and controllability for noiseless systems are described
 66 in sections 3.1, 3.2, and 3.3. Theorem 3.5 discussing the rank deficiency of error co-
 67 variance matrices is presented in section 3.4 and the proof of Theorem 3.7 using the
 68 geometric viewpoint of Kalman filtering [2, 25, 1] is detailed in section 3.5. Section 3.6
 69 presents some numerical results buttressing the theorem. Section 4 includes the proof
 70 of Corollary 4.2 along with a numerical illustration supporting the analytical findings
 71 for autonomous systems. We conclude in section 5.

72 Although the extension of these results to the general nonlinear case is the object
 73 of active research [19], the current findings already provide a formal justification to
 74 the AUS foundation and further motivate its use as a DA strategy in nonlinear chaotic
 75 dynamics.

76 **2. Notation.** The dimension of the state space is represented by d . For any
 77 square matrix $Z \in \mathbb{C}^{d \times d}$ let the set $\{\lambda_1(Z), \dots, \lambda_d(Z)\}$ represent the eigen-values of
 78 Z , where $|\lambda_1(Z)| \geq \dots \geq |\lambda_d(Z)|$. Similarly, let the set $\{\sigma_1(Z), \dots, \sigma_d(Z)\}$ stand for
 79 the singular values of Z with $\sigma_1(Z) \geq \dots \geq \sigma_d(Z)$. We define the column vectors
 80 of the matrix $V_Z = [\mathbf{v}_1(Z), \dots, \mathbf{v}_d(Z)]$ to be the generalized eigen-vectors of Z for
 81 satisfying the relation $ZV_Z = V_ZJ(Z)$, where $J(Z)$ is the Jordan canonical form of
 82 Z . In the event that Z is diagonalizable ($J(Z)$ is diagonal), let the entries of the
 83 diagonal matrix $\Lambda_Z = J(Z)$ symbolize the eigen-values of Z and the columns of V_Z —
 84 the eigen-vectors—be of unit magnitude. Z^* denotes the adjoint of Z for the scalar

85 product under consideration in \mathbb{C}^d and Z^\dagger represents the conjugate transpose of Z .
 86 For the canonical scalar product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v}$ in \mathbb{C}^d , $Z^* = Z^\dagger$, and when confined to
 87 the real space \mathbb{R}^d where $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$, $Z^* = Z^T$. Unless explicitly stated we assume a
 88 real vector space endowed with a canonical scalar product. The matrix norm $\|Z\|$ we
 89 consider is the largest singular value $\sigma_1(Z)$ of Z . The notation $Z > 0$ ($Z \geq 0$) is used
 90 when Z is symmetric positive-definite (positive-semidefinite). For any two symmetric
 91 matrices Z_1, Z_2 , the notation $Z_1 \geq Z_2$ means $Z_1 - Z_2 \geq 0$. The following definitions
 92 are useful.

93 **DEFINITION 2.1** (real span). *The real span of a complex vector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ where*
 94 *$\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ is the vector space $\mathcal{T}_{\mathbf{w}} \subset \mathbb{R}^d$ defined as*

$$95 \quad \mathcal{T}_{\mathbf{w}} \equiv \{\alpha \mathbf{u} + \beta \mathbf{v} : \alpha, \beta \in \mathbb{R}\}.$$

96 **DEFINITION 2.2** (α -eigenspace). *Given $\alpha > 0$, the α -eigenspace of a square matrix*
 97 *Z denoted by $\mathcal{E}^\alpha(Z)$ is the real span of the generalized eigen-vectors of Z corresponding*
 98 *to eigen-values λ with $|\lambda| < \alpha$.*

99 3. Nonautonomous systems.

100 **3.1. Setup and assumptions.** We define the general linear nonautonomous
 101 dynamical system at time $n \geq 0$ by

$$102 \quad (3.1) \quad \begin{aligned} \mathbf{x}_{n+1} &= A_{n+1} \mathbf{x}_n + F_{n+1} \mathbf{p}_{n+1}, \\ 103 \quad \mathbf{y}_{n+1} &= H_{n+1} \mathbf{x}_{n+1} + \mathbf{q}_{n+1}, \end{aligned}$$

105 where $\mathbf{x}_n \in \mathbb{R}^d$, $\mathbf{q}_n \in \mathbb{R}^q$, $\mathbf{p}_n \in \mathbb{R}^p$. The \mathbf{x}_n are the state variables, \mathbf{p}_n represents
 106 model noise, \mathbf{y}_n represents observational variables, and \mathbf{q}_n is the observational noise
 107 term. The basic random variables $\{\mathbf{x}_0, \mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{p}_1, \mathbf{p}_2, \dots\}$ are all assumed to be
 108 independent and Gaussian with

$$109 \quad \mathbf{x}_0 \sim \mathcal{N}(\mathbf{x}_{0|0}, \Delta_0), \quad \mathbf{q}_n \sim \mathcal{N}(0, Q_n), \quad \mathbf{p}_n \sim \mathcal{N}(0, I)$$

110 such that $\Delta_0 \in \mathbb{R}^{d \times d}$ is the initial error covariance matrix of the state variable \mathbf{x}_0 ,
 111 $Q_n \in \mathbb{R}^{q \times q}$ is the observation error covariance matrix at time n , and $F_n \in \mathbb{R}^{d \times p}$.
 112 The matrices $\Delta_0, Q_n, F_n, A_n, H_n$ are known for all time n . Further, A_n and Q_n are
 113 considered to be nonsingular, $\|A_n\| \leq c_A$, $\|Q_n\| \leq c_Q$, and $\|H_n\| \leq c_H \forall n \geq 1$, where
 114 c_A, c_Q , and c_H are positive constants. The model noise error covariance is given by
 115 $P_n \equiv F_n F_n^T$. Unless explicitly stated $\Delta_0 > 0$, i.e., its eigen-values are strictly positive.

116 Filtering theory deals with the properties of the conditional distribution, called
 117 the *analysis* in the context of DA, of the state \mathbf{x}_n at time n conditioned on observations
 118 $Y_{0:n} = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n]$ up to time n where the first observation \mathbf{y}_1 is assumed to occur
 119 at time $n = 1$. This conditional distribution provides an optimal state estimate in
 120 the least squares sense [7]. Under the assumptions of linearity and Gaussianity stated
 121 above, this conditional distribution is Gaussian, with mean and covariance denoted
 122 by $\mathbf{x}_{n|n}$, and Δ_n respectively:

$$123 \quad \mathbf{x}_{n|n} = \mathbb{E}[\mathbf{x}_n | Y_{0:n}] \quad \text{and} \quad \Delta_n = \mathbb{E}[(\mathbf{x}_n - \mathbf{x}_{n|n})(\mathbf{x}_n - \mathbf{x}_{n|n})^T | Y_{0:n}].$$

124 We also note that the conditional distribution, called the *forecast* in DA literature,
 125 of the state \mathbf{x}_{n+1} conditioned on observations $Y_{0:n}$ up to time n is Gaussian with its
 126 mean and covariance denoted by $\mathbf{x}_{n+1|n}$ and Σ_{n+1} , respectively:

$$127 \quad \mathbf{x}_{n+1|n} = \mathbb{E}[\mathbf{x}_{n+1} | Y_{0:n}] \quad \text{and} \quad \Sigma_{n+1} = \mathbb{E}[(\mathbf{x}_{n+1} - \mathbf{x}_{n+1|n})(\mathbf{x}_{n+1} - \mathbf{x}_{n+1|n})^T | Y_{0:n}].$$

128 In this work we concern ourselves with systems that have no model error, i.e.,
 129 $F_n \equiv 0 \forall n \geq 1$, and investigate the dynamics

$$130 \quad (3.2) \quad \mathbf{x}_{n+1} = A_{n+1}\mathbf{x}_n \quad \text{and} \quad \mathbf{y}_{n+1} = H_{n+1}\mathbf{x}_{n+1} + \mathbf{q}_{n+1}.$$

132 We will be interested in asymptotic properties of the conditional error covariances
 133 Σ_n and Δ_n . The KF provides a closed form, iterative formula for obtaining these
 134 quantities [7]. Under the assumption of no model noise, the update equation for the
 135 forecast error covariance is

$$136 \quad (3.3) \quad \Sigma_n = A_n \Delta_{n-1} A_n^T.$$

137 By defining the Kalman gain matrix K_n as

$$138 \quad (3.4) \quad K_n \equiv \Sigma_n H_n^T [H_n \Sigma_n H_n^T + Q_n]^{-1},$$

139 the analysis error covariance equals

$$140 \quad (3.5) \quad \Delta_n = (I - K_n H_n) \Sigma_n.$$

141 The update equations for the means are given by

$$142 \quad (3.6) \quad \mathbf{x}_{n+1|n} = A_{n+1} \mathbf{x}_{n|n},$$

$$143 \quad (3.7) \quad \mathbf{x}_{n+1|n+1} = \mathbf{x}_{n+1|n} + K_{n+1} (y_{n+1} - H_{n+1} \mathbf{x}_{n+1|n}).$$

145 Defining the sequence of matrices M_n as

$$146 \quad (3.8) \quad M_1 \equiv (I - K_1 H_1) A_1, \quad M_n \equiv (I - K_n H_n) A_n M_{n-1}$$

147 and writing the propagator $B_{m:m+n}$ from time m to time $m+n$ by

$$148 \quad (3.9) \quad B_{m:m+n} \equiv A_{m+n} A_{m+n-1} \cdots A_{m+1},$$

150 the analysis covariance at time n can be expressed as

$$151 \quad (3.10) \quad \Delta_n = (I - K_n H_n) A_n \cdots (I - K_1 H_1) A_1 \Delta_0 A_1^T \cdots A_n^T = M_n \Delta_0 B_{0:n}^T.$$

153 This equation clearly shows that the asymptotic properties of Δ_n are closely related to
 154 those of $B_{0:n}$ and M_n . The notation in (3.10) is suggestive of the line of argument we
 155 will take in the following sections. To outline, we may consider the singular-value de-
 156 composition of the propagator $B_{0:n}^T = V_n S_n U_n^T$ and decompose the error covariances
 157 into a basis of the left singular vectors. In particular, we know that this decom-
 158 position may be written as a function of the singular values, provided we have an
 159 appropriate bound on M_n in (3.10). Moreover, the left singular vectors of the prop-
 160 agator $B_{0:n}$ will become arbitrarily close to the backward Lyapunov vectors of the
 161 system.

162 The properties of $B_{0:n}$ are basically determined by the dynamical system and
 163 are discussed in the next section, while those of M_n are commonly discussed in the
 164 context of control theory and are discussed in section 3.3, where we prove a useful
 165 bound on its matrix norm in Lemma 3.3.

166 **3.2. Oseledets theorem.** Note that the boundedness condition on A_n implies
 167 the bound $\|B_{0:n}\| \leq (c_A)^n \forall n$. Then the Oseledets multiplicative ergodic theorem in
 168 [15] states that for each nonzero vector $\mathbf{u} \in \mathbb{R}^d$ the limit

$$169 \quad \mu = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\|B_{0:n} \mathbf{u}\|}{\|\mathbf{u}\|}$$

170 exists and assumes up to d distinct values $\mu_1 \geq \dots \geq \mu_d$ which are called the Lyapunov
 171 exponents. We will assume

$$172 \quad (3.11) \quad 0 > \mu_{d_0+1}$$

174 so that exactly $d_0 < d$ of the Lyapunov exponents are nonnegative. Further, defining
 175 the matrices

$$176 \quad (3.12) \quad E_n^b(m) \equiv [B_{m-n:m}(B_{m-n:m})^*]^{\frac{1}{2n}}, \quad E_n^f(m) \equiv [(B_{m:m+n})^* B_{m:m+n}]^{\frac{1}{2n}},$$

177 the Oseledets theorem guarantees that the following limits exist, namely,

$$178 \quad (3.13) \quad E^b(m) \equiv \lim_{n \rightarrow \infty} E_n^b(m),$$

$$179 \quad (3.14) \quad E^f(m) \equiv \lim_{n \rightarrow \infty} E_n^f(m).$$

181 The eigen-vectors of $E^b(m)$ and $E^f(m)$ represented as the column vectors of $L^b(m) =$
 182 $[\mathbf{l}_1^b(m), \dots, \mathbf{l}_{d_0}^b(m)]$ and $L^f(m) = [\mathbf{l}_1^f(m), \dots, \mathbf{l}_d^f(m)]$, respectively, are defined as the
 183 *backward* and the *forward* Lyapunov vectors at time m [10]. We note that the
 184 asymptotic results in later sections will essentially use the backward Lyapunov vectors
 185 $L^b(m)$.

186 The convergence of the individual matrix entries in (3.13) and (3.14) guarantee the
 187 convergence of their characteristic polynomials—whose coefficients are well-defined
 188 functions of the matrix entries—the roots of which are the eigen-values. Therefore,

$$189 \quad \lim_{n \rightarrow \infty} \Lambda_{E_n^b(m)} = \Lambda_{E^b(m)}, \quad \lim_{n \rightarrow \infty} \Lambda_{E_n^f(m)} = \Lambda_{E^f(m)},$$

190 where we recall that Λ_Z is a diagonal matrix comprising eigen-values of Z . Using the
 191 notation from section 2 we additionally find

$$192 \quad \|\lambda_j(E^b(m)) \mathbf{v}_j(E_n^b(m)) - E^b(m) \mathbf{v}_j(E_n^b(m))\| \leq |\lambda_j(E^b(m)) - \lambda_j(E_n^b(m))| \\ 193 \quad \quad \quad + \|E_n^b(m) - E^b(m)\|$$

195 from which we can infer that

$$196 \quad \lim_{n \rightarrow \infty} \|\lambda_j(E^b(m)) \mathbf{v}_j(E_n^b(m)) - E^b(m) \mathbf{v}_j(E_n^b(m))\| = 0$$

197 leading to $\lim_{n \rightarrow \infty} V_{E_n^b(m)} = V_{E^b(m)} = L^b(m)$. Similarly, $\lim_{n \rightarrow \infty} V_{E_n^f(m)} = V_{E^f(m)} =$
 198 $L^f(m)$.

199 The Oseledets theorem also asserts the eigen-values of $E^b(m)$ or $E^f(m)$ do not
 200 depend on the initial time m , are the same for the forward and backward matrices,
 201 and relate to the Lyapunov exponents as

$$202 \quad (3.15) \quad \mu_j = \log(\lambda_j(E)), \quad j \in \{1, \dots, d\},$$

203 where we deliberately drop the index m and the superscript b or f on E . However,
 204 the forward and backward Lyapunov vectors are different from each other and they
 205 also depend on the time m , i.e., $L^b(k) \neq L^b(m) \neq L^f(m) \neq L^f(k)$ for $k \neq m$.

206 Consider the singular-value decomposition $B_{0:n} \equiv U_n S_n (V_n)^T$ so that under the
 207 canonical inner product

$$208 \quad E_n^f(0) = [(B_{0:n})^T B_{0:n}]^{\frac{1}{2n}} = [V_n (S_n)^2 (V_n)^T]^{\frac{1}{2n}} = V_n (S_n)^{\frac{1}{n}} (V_n)^T,$$

209 implying $V_{E_n^f(0)} = V_n$ and

$$211 \quad (3.16) \quad \lim_{n \rightarrow \infty} \|\mathbf{v}_{j,n} - \mathbf{l}_j^f(0)\| = 0,$$

212 where $\mathbf{v}_{j,n}$ (and similarly $\mathbf{u}_{j,n}$ below) is the j th column vector of V_n (respectively,
 213 U_n). Likewise, we obtain

$$214 \quad E_n^b(n) = [B_{0:n} (B_{0:n})^T]^{\frac{1}{2n}} = [U_n (S_n)^2 (U_n)^T]^{\frac{1}{2n}} = U_n (S_n)^{\frac{1}{n}} (U_n)^T,$$

216 from which we can deduce that $V_{E_n^b(n)} = U_n$ and

$$217 \quad (3.17) \quad \lim_{n \rightarrow \infty} \|\mathbf{u}_{j,n} - \mathbf{l}_j^b(n)\| = 0.$$

218 We also infer that

$$219 \quad (3.18) \quad (\sigma_j(B_{0:n}))^{\frac{1}{n}} = \lambda_j(E_n^b(n)) = \lambda_j(E_n^f(0)).$$

220 **3.3. Controllability and observability for linear dynamics.** The notions of
 221 observability and controllability are dual notions within filtering problems. Roughly,
 222 observability is the condition that given sufficiently many observations, the initial
 223 state of the system can be reconstructed by using a finite number of observations.
 224 Similarly, controllability can be described as the ability to move the system from any
 225 initial state to a desired state over a finite time interval. This is formally stated as
 226 follows.

227 **DEFINITION 3.1.** *The system (3.1) is defined to be completely observable if*
 228 $\forall n \geq 1,$

$$229 \quad (3.19) \quad \det \left(\sum_{m=0}^{d-1} (B_{n:n+m})^T H_{n+m}^T Q_{n+m}^{-1} H_{n+m} B_{n:n+m} \right) \neq 0,$$

230 and it is defined to be completely controllable if $\forall n \geq 0,$

$$231 \quad (3.20) \quad \det \left(\sum_{m=1}^d B_{n+m:n+d} F_{n+m} F_{n+m}^T (B_{n+m:n+d})^T \right) \neq 0.$$

232 *In addition we describe the system as uniformly completely observable (respectively,*
 233 *uniformly completely controllable) if (3.19) (respectively, (3.20)) is bounded from zero*
 234 *uniformly in n .*

235 We will assume that the system in (3.2) is uniformly completely observable, i.e.,
 236 the inequality (3.19) is uniformly bounded away from zero. Note, however, that
 237 this system *cannot* be controllable since the determinant in (3.20) is identically zero
 238 for a deterministic, perfect-model system as $F_n = 0 \forall n$. The hypothesis of uniform
 239 complete observability ensures that the error covariance matrices remain bounded
 240 over time, as seen below.

241 LEMMA 3.2. *Suppose that the linear, nonautonomous system (3.2) where the ini-*
 242 *tial state \mathbf{x}_0 has a Gaussian law with mean $\mathbf{x}_{0|0}$ and covariance Δ_0 is uniformly*
 243 *completely observable (Definition 3.1). Then the error covariance matrices remain*
 244 *bounded for all time, i.e., there exist constants c_Σ and c_Δ such that $\forall n$, $\|\Delta_n\| \leq c_\Delta$*
 245 *and $\|\Sigma_n\| \leq c_\Sigma$.*

246 *Proof.* The result is proven for autonomous systems in Kumar and Varaiya [9,
 247 Chapter 7, equations (2.36) and (2.37)]. Extension to the nonautonomous case is
 248 straightforward by rehashing the steps and changing the constants of the autonomous
 249 system to their time-varying counterparts. \square

250 One should note the recent work of Ni and Zhang [14] has demonstrated a stronger
 251 result: in continuous, perfect-model systems the assumption of uniform complete
 252 observability is sufficient to demonstrate the stability of the KF. In particular this
 253 shows that all solutions to the continuous Riccati equation for any choice of initial
 254 error covariance are bounded and converge to the same solution asymptotically. This
 255 strongly suggests the same can be shown for the discrete time system, and we will
 256 return to this point in our discussion of results in section 5.

257 Utilizing only the boundedness of the error covariance matrices, we demonstrate
 258 that the matrix M_n stays bounded in the following lemma.

259 LEMMA 3.3. *Consider the uniformly completely observable, perfect-model, linear,*
 260 *nonautonomous system (3.2) where the initial state \mathbf{x}_0 has a Gaussian law with co-*
 261 *variance $\Delta_0 > 0$. Then the matrix M_n defined in (3.8) is uniformly bounded, i.e.,*
 262 *there exists a constant c_M such that $\|M_n\| \leq c_M \forall n$.*

263 *Proof.* We first show that the analysis error covariance matrix satisfies the recur-
 264 sive equation

$$265 \quad (3.21) \quad \Delta_n = (I - K_n H_n) A_n \Delta_{n-1} A_n^T (I - K_n H_n)^T + K_n Q_n K_n^T.$$

266 Plugging in the Kalman update equations (3.3) and (3.3), the right-hand side of (3.21)
 267 equals $\Delta_n - (\Delta_n H_n^T - K_n Q_n) K_n^T$. Equation (4.29) in [4] establishes the equality
 268 $K_n = \Delta_n H_n^T Q_n^{-1}$ from which the recursion (3.21) follows, further implying that

$$269 \quad \Delta_n \geq (I - K_n H_n) A_n \Delta_{n-1} A_n^T (I - K_n H_n)^T.$$

270 Recursively applying the above inequality gives $\Delta_n \geq M_n \Delta_0 M_n^T$. Decomposing $\Delta_0 =$
 271 $V_{\Delta_0} \Lambda_{\Delta_0} V_{\Delta_0}^T$ and employing Lemma 3.2 we find

$$272 \quad \left\| M_n V_{\Delta_0} \Lambda_{\Delta_0}^{\frac{1}{2}} \right\|^2 \leq \|\Delta_n\| \leq c_\Delta.$$

273 As $\|M_n\| \leq \|M_n V_{\Delta_0} \Lambda_{\Delta_0}^{\frac{1}{2}}\| \|\Lambda_{\Delta_0}^{-\frac{1}{2}} V_{\Delta_0}^T\|$ the result follows. Note that as $\Delta_0 > 0$ the
 274 matrix $\Lambda_{\Delta_0}^{-\frac{1}{2}}$ is well-defined. \square

275 Bearing this bound in mind we shall proceed to discuss the asymptotic properties
 276 of the error covariance matrices.

277 **3.4. The asymptotic rank deficiency of the error covariance.** We begin
 278 by introducing a lemma which allows us to formally describe the collapse of the
 279 eigen-values of the error covariance matrix.

280 LEMMA 3.4. *For a given $\epsilon > 0$, let $Z \in \mathbb{R}^{d \times d}$ be a symmetric matrix such that*
 281 *there is a $k \leq d$ dimensional subspace $\mathcal{W} \subset \mathbb{R}^d$ for which*

$$282 \quad \sup\{\|Z\mathbf{u}\| : \|\mathbf{u}\| = 1, \mathbf{u} \in \mathcal{W}\} < \epsilon.$$

283 Then $\dim(\mathcal{E}^\epsilon(Z)) \geq k$, where the subspace \mathcal{E}^ϵ is in accordance with Definition 2.2.

284 *Proof.* Let $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ be an orthonormal eigen-vector basis for Z corresponding
285 to $|\lambda_1(Z)| \geq \dots \geq |\lambda_d(Z)|$, and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a basis for \mathcal{W} of unit magnitude,
286 such that we write

$$287 \quad \mathbf{u}_l = \sum_{j=1}^d \beta_{l,j} \mathbf{v}_j; \quad l \in \{1, 2, \dots, k\},$$

288 and the matrix of coefficients

$$289 \quad \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,d-k+1} & 0 & \cdots & 0 \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,d-k+1} & \beta_{2,d-k+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{k-1,1} & \beta_{k-1,2} & \cdots & \cdots & \cdots & \beta_{k-1,d-1} & 0 \\ \beta_{k,1} & \beta_{k,2} & \cdots & \cdots & \cdots & \beta_{k,d-1} & \beta_{k,d} \end{pmatrix}$$

290 is in column echelon form where for every column index $j > d - k + 1$, the entries

$$291 \quad \beta_{1,j} = \cdots = \beta_{k+j-d-1,j} = 0$$

292 and for every row index $l \leq k$, $\sum_{j=1}^{d-k+l} \beta_{l,j}^2 = 1$ corresponding $\|u_l\| = 1$. Furthermore,
293 as Z is symmetric its eigen-vectors form an orthonormal basis and hence $\|Z\mathbf{u}_l\|^2 =$
294 $\sum_{j=1}^{d-k+l} \beta_{l,j}^2 \lambda_j^2(Z)$. For every $1 \leq l \leq k$, setting $s = k - l + 1$ we find

$$295 \quad \epsilon^2 > \|Z\mathbf{u}_s\|^2 = \sum_{j=1}^{d-k+s} \beta_{s,j}^2 \lambda_j^2(Z) \geq \lambda_{d-k+s}^2(Z) = \lambda_{d-l+1}^2(Z).$$

296 Hence the k smallest eigen-values in absolute magnitude satisfy

$$297 \quad |\lambda_d(Z)| \leq \cdots \leq |\lambda_{d-k+1}(Z)| < \epsilon$$

298 and the result follows. \square

299 **THEOREM 3.5.** *Consider the uniformly completely observable, perfect-model, lin-*
300 *ear, nonautonomous system (3.2) where the initial state \mathbf{x}_0 has a Gaussian law with*
301 *covariance Δ_0 . Then $\forall \epsilon > 0, \exists n_1 > 0$ such that if $n \geq n_1$, Σ_n and Δ_n will each*
302 *have at least $d - d_0$ eigen-values which are less than ϵ where $d - d_0$ is the number of*
303 *negative Lyapunov exponents of the system (3.2), i.e.,*

$$304 \quad (3.22) \quad \dim(\mathcal{E}^\epsilon(\Sigma_n)) \geq d - d_0, \quad \text{and} \quad \dim(\mathcal{E}^\epsilon(\Delta_n)) \geq d - d_0,$$

305 where the subspace \mathcal{E}^ϵ is in accordance with Definition 2.2.

306 *Proof.* As denoted earlier, let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_d$ be the Lyapunov exponents of
307 the system (3.2) where $d_0 < d$ of them are nonnegative. The forward stable Lyapunov
308 vectors based at time zero are the set $\{\mathbf{I}_j^f(0)\}_{j=d_0+1}^d$ which by definitions (3.13) and
309 (3.15) satisfy

$$310 \quad (3.23) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\left\| B_{0:n} \mathbf{I}_j^f(0) \right\| \right) = \mu_j.$$

312 Rewriting the analysis error covariance update equation in terms of the transpose

$$313 \quad \Delta_n = M_n \Delta_0 B_{0:n}^T = B_{0:n} \Delta_0 M_n^T$$

314 we get $\Delta_n M_n^{-T} \Delta_0^{-1} = B_{0:n}$ and in particular

$$315 \quad \Delta_n M_n^{-T} \Delta_0^{-1} \mathbf{1}_j^f(0) = B_{0:n} \mathbf{1}_j^f(0).$$

316 Let us therefore define the sequence of vectors

$$317 \quad (3.24) \quad \mathbf{w}_{j,n} \equiv M_n^{-T} \Delta_0^{-1} \mathbf{1}_j^f(0).$$

319 By Lemma 3.3 we know that M_n is bounded above, so that the sequence of vectors
 320 $\mathbf{w}_{j,n} = M_n^{-T} \Delta_0^{-1} \mathbf{1}_j^f(0)$ must be bounded below. As such, there is a constant $c_{\mathbf{w}}$ such
 321 that $c_{\mathbf{w}} \leq \|\mathbf{w}_{j,n}\| \forall n$ and $j \in \{d_0 + 1, \dots, d\}$. Choose a $\rho > 0$ such that for each
 322 $j \in \{d_0 + 1, \dots, d\}$, $\rho + \mu_j < 0$. Define $\bar{\mathbf{w}}_{j,n} \equiv \frac{\mathbf{w}_{j,n}}{\|\mathbf{w}_{j,n}\|}$. Then for a given $\epsilon > 0$, $\exists n_1$
 323 such that for $n \geq n_1$

$$324 \quad (3.25) \quad \|\Delta_n \bar{\mathbf{w}}_{j,n}\| = \frac{1}{\|\mathbf{w}_{j,n}\|} \|B_{0:n} \mathbf{1}_j^f(0)\| \leq \frac{1}{c_{\mathbf{w}}} e^{(\mu_j + \rho)n} < \epsilon.$$

326 The theorem is therefore an immediate consequence of Lemma 3.4. The proof for Σ_n
 327 follows along similar lines. \square

328 **3.5. Null space characterization and assimilation in the unstable sub-**
 329 **space.** The sequence of subspaces defined by the span of $\{\mathbf{w}_{j,n}\}_{j=d_0+1}^d$ will be the
 330 object of study for the remainder of this section. In particular, we wish to estab-
 331 lish the connection between this sequence of subspaces and AUS which utilizes the
 332 *backward Lyapunov vectors*.

333 **DEFINITION 3.6.** Define $\Lambda_{E_n^f(0)}^s$ to be the $d - d_0 \times d - d_0$ diagonal matrix with
 334 diagonal entries given by $\{\lambda_j(E_n^f(0))\}_{j=d_0+1}^d$. Also, let us define the following $d \times$
 335 $d - d_0$ operators:

$$336 \quad (3.26) \quad U_n^s = [\mathbf{u}_{d_0+1,n}, \dots, \mathbf{u}_{d,n}],$$

$$337 \quad (3.27) \quad V_n^s = [\mathbf{v}_{d_0+1,n}, \dots, \mathbf{v}_{d,n}],$$

$$338 \quad (3.28) \quad L_n^{bs} = [\mathbf{l}_{d_0+1}^b(n), \dots, \mathbf{l}_d^b(n)].$$

340 Note that (3.17) implies that

$$341 \quad (3.29) \quad \lim_{n \rightarrow \infty} \|U_n^s - L_n^{bs}\| = 0.$$

343 Consider (3.10), namely, $\Delta_n = M_n \Delta_0 V_n S_n U_n^T$, for the analysis error covariance
 344 Δ_n at time n in terms of the matrix M_n and the singular-value decomposition of the
 345 propagator $B_{0:n}$. Noting that $B_{0:n}^T \mathbf{u}_{j,n} = \sigma_j(B_{0:n}) \mathbf{v}_{j,n}$ and utilizing the relation (3.18)
 346 we get

$$347 \quad (3.30) \quad \Delta_n U_n^s (U_n^s)^T = M_n \Delta_0 V_n^s \left(\Lambda_{E_n^f(0)}^s \right)^n (U_n^s)^T.$$

349 Likewise, recalling that $\Sigma_n = A_n \Delta_{n-1} A_n^T$, we can express the restriction of the fore-
 350 cast error covariances as

$$351 \quad (3.31) \quad \Sigma_n U_n^s (U_n^s)^T = A_n M_{n-1} \Delta_0 V_n^s \left(\Lambda_{E_n^f(0)}^s \right)^n (U_n^s)^T.$$

353 Making use of the above relations we now prove one of our main results, which
 354 states that the norm of the restriction of the analysis and forecast error covariances
 355 onto the backward stable Lyapunov subspaces must tend to zero.

356 THEOREM 3.7. Consider the uniformly completely observable, perfect-model, lin-
 357 ear, nonautonomous system (3.2) where the initial state \mathbf{x}_0 has a Gaussian law with
 358 covariance Δ_0 . The restriction of Δ_n and Σ_n into the span of the backward stable
 359 Lyapunov vectors, $\{\mathbf{l}_j^b(n)\}_{j=d_0+1}^d$, tends to zero as $n \rightarrow \infty$. That is,

$$360 \quad (3.32) \quad \lim_{n \rightarrow \infty} \|\Delta_n L_n^{bs} (L_n^{bs})^T\| = 0,$$

$$361 \quad (3.33) \quad \lim_{n \rightarrow \infty} \|\Sigma_n L_n^{bs} (L_n^{bs})^T\| = 0.$$

363 *Proof.* By definition $\log(\lambda_j(E^f(0))) = \mu_j$, so that the eigen-values $\lambda_j(E^f(0)) < 1$
 364 correspond to the stable Lyapunov exponents. Recalling that $\lambda_{d_0+1}(E_n^f(0)) \geq \dots \geq$
 365 $\lambda_d(E_n^f(0))$ we find $\|\Lambda_{E_n^f(0)}^s\| = \lambda_{d_0+1}(E_n^f(0))$ and

$$366 \quad (3.34) \quad \lim_{n \rightarrow \infty} \left\| \Lambda_{E_n^f(0)}^s \right\| = \lambda_{d_0+1}(E^f(0)) < 1.$$

367 Consequent to (3.34) we can choose a small $0 < \rho < 1$ and sufficiently large n_1 such
 368 that when $n \geq n_1$, $\|\Lambda_{E_n^f(0)}^s\| \leq 1 - \rho$.

369 The restriction of Δ_n into the span of the columns of U_n^s is given by (3.30). Note
 370 the column vectors of V_n^s and U_n^s are orthogonal and of unit norm, hence $\|V_n^s\| =$
 371 $\|U_n^s\| = 1$. We then find for $n \geq n_1$

$$372 \quad (3.35) \quad \|\Delta_n U_n^s (U_n^s)^T\| \leq \left\| \Lambda_{E_n^f(0)}^s \right\|^n \|M_n\| \|\Delta_0\| \leq (1 - \rho)^n c_M \|\Delta_0\|.$$

373 Consider

$$374 \quad (3.36) \quad \|\Delta_n L_n^{bs} (L_n^{bs})^T\| \leq \|\Delta_n\| \|L_n^{bs} (L_n^{bs})^T - U_n^s (U_n^s)^T\| + \|\Delta_n U_n^s (U_n^s)^T\|,$$

376 and Lemma 3.2 states $\|\Delta_n\|$ is bounded. Therefore,

$$377 \quad (3.37) \quad \lim_{n \rightarrow \infty} \|\Delta_n L_n^{bs} (L_n^{bs})^T\| = 0$$

379 by (3.17) and (3.35). This may be similarly stated for the forecast error covariance. \square

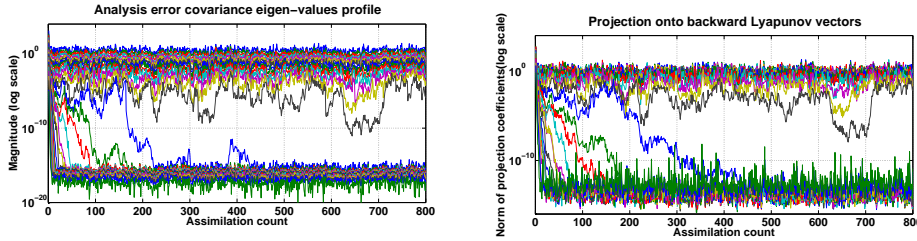
380 The forecast and analysis error covariance matrices for a generic nonautonomous
 381 system in general do not converge, but the above results entail that asymptotically the
 382 only relevant directions for the error covariance matrices are the backward unstable-
 383 neutral Lyapunov directions validating the central hypothesis made by Trevisan and
 384 Palatella [22] in their proposed reduced rank Kalman filtering algorithms.

385 An intriguing consequence from (3.25) in Theorem 3.5 is the following corollary.

386 COROLLARY 3.8. Suppose that for some $\epsilon_0 > 0$, $N_0 > 0$, and for every $0 < \epsilon < \epsilon_0$,
 387 $n > N_0$,

$$388 \quad (3.38) \quad \dim(\mathcal{E}^\epsilon(\Delta_n)) = d - d_0,$$

390 *i.e.*, asymptotically the rank deficiency of the analysis error covariance Δ_n is exactly of
 391 dimension $d - d_0$. Then the transformation $M_n^{-T} \Delta_0^{-1}$ asymptotically maps the forward
 392 stable vectors $\{\mathbf{l}_j^f(0)\}_{j=d_0+1}^d$ into the span of the backward stable vectors $\{\mathbf{l}_j^b(n)\}_{j=d_0+1}^d$
 393 as $n \rightarrow \infty$.



416 FIG. 1. Profile of the eigen-values of Δ_n .
 417 Counting establishes that the bottom 16 eigen-
 418 values converge to zero.

416 FIG. 2. Norm of the projection coefficients
 417 $\|\Delta_n \mathbf{u}_{j,n}\|$ for varying observation time n .

394 **3.6. Numerical results for a 30-dimensional system.** Below we provide
 395 an illustration for this *asymptotic rank deficiency* property of the error covariance
 396 matrices. The state space vector \mathbf{x}_n and the observation vector \mathbf{y}_n have dimension
 397 $d = 30$ and $q = 10$, respectively. This choice is arbitrary and our simulations with
 398 different d and q have shown qualitatively equivalent results.

399 The time-varying, invertible propagators $A_n \in \mathbb{R}^{30 \times 30}$, the observation error co-
 400 variance matrices $Q_n \in \mathbb{R}^{10 \times 10}$, and the observation matrices $H_n \in \mathbb{R}^{10 \times 30}$ were all
 401 randomly generated for sufficiently large n . We employed the *QR* method [10] to
 402 numerically compute the Lyapunov vectors and the Lyapunov exponents and it was
 403 found that the number of nonnegative Lyapunov exponents was $d_0 = 14$. Starting
 404 from a random positive-definite Δ_0 , the sequence (Σ_n, Δ_n) was generated based on
 405 the Kalman update equations (3.3)–(3.5). For every n we computed the eigen-values
 406 of Δ_n sorted in descending order.

407 Figure 1 shows the eigen-values of Δ_n as a function of n . Barring the dominant 14
 408 eigen-values, the rest converge to zero, serving as a visual testament to Theorem 3.5.
 409 Furthermore, we also calculated the norm $\|\Delta_n \mathbf{u}_{j,n}\|, j \in \{1, 2, \dots, d\}, \forall n$ and plot
 410 them in Figure 2. These norm values are unsorted, meaning that the topmost line in
 411 Figure 2 represents the values $\|\Delta_n \mathbf{u}_{1,n}\|$ and the bottommost line denotes $\|\Delta_n \mathbf{u}_{d,n}\|$
 412 for different values of n . For $j > d_0 = 14$, $\|\Delta_n \mathbf{u}_{j,n}\|$ approaches zero, suggesting that
 413 as $n \rightarrow \infty$, the row space of Δ_n (and also Σ_n) coincides the space spanned by the
 414 unstable-neutral, backward Lyapunov vectors, i.e., the bounds in inequalities (3.22)
 415 are saturated.

4. Autonomous linear dynamical systems.

422 **4.1. Null space characterization for autonomous systems.** The noiseless,
 423 linear autonomous system can be defined from (3.2), with the additional assumptions
 424 that $A_n \equiv A, H_n \equiv H, Q_n \equiv Q$ are fixed matrices $\forall n$ —therefore the results about
 425 the asymptotic rank deficiency property of the error covariance matrices in section 3
 426 also apply to autonomous systems. However, a stronger statement can be made for
 427 time invariant systems because the backward Lyapunov vectors will not vary in time.
 428 In fact, the result in this section is valid even for the case when only the dynamical
 429 system is autonomous ($A_n \equiv A$) but the observation process is time dependent (H_n
 430 and Q_n depend on n).

431 Akin to the nonautonomous case we define

$$432 \quad (4.1) \quad E_n^b \equiv [A^n (A^n)^*]^{\frac{1}{2n}}, \quad E_n^f \equiv [(A^n)^* A^n]^{\frac{1}{2n}}$$

and the similarity with (3.12) can readily be seen by setting $B_{m:m+n} = A^n \forall m$ in (3.9) (hence the omission of the time index m). As before, the existence of the limits

$$(4.2) \quad E^b \equiv \lim_{n \rightarrow \infty} E_n^b, \quad E^f \equiv \lim_{n \rightarrow \infty} E_n^f$$

is guaranteed by the Oseledets theorem [10]. The eigen-vectors of E^b and E^f are called the backward and forward Lyapunov vectors, represented here as the column vectors of L^b and L^f ordered left to right from the most unstable direction—corresponding to the largest Lyapunov exponent—to the most stable direction—corresponding to the smallest Lyapunov exponent. Specifically, the Lyapunov vectors are defined globally and have *no* dependence on the time in the linear, autonomous case. Without the time dependence on the backward stable Lyapunov vectors, we obtain a stronger statement about the asymptotic null space of the covariance matrices.

DEFINITION 4.1. Let $L^{bs} \equiv L_n^{bs} = [\mathbf{l}_{d_0+1}^b, \dots, \mathbf{l}_d^b]$. Note that Theorem B.3 proved in Appendix B states that the span of the columns of L^{bs} is equal to $\mathcal{E}^1(A^T)$.

COROLLARY 4.2. Consider the uniformly completely observable, perfect-model, linear, autonomous system defined from (3.2) where $A_n \equiv A$, but H_n and Q_n may depend on n and the initial state \mathbf{x}_0 has a Gaussian law with covariance Δ_0 . Then the restriction of the analysis and forecast error covariances onto $\mathcal{E}^1(A^T)$ tend to zero as $n \rightarrow \infty$. That is,

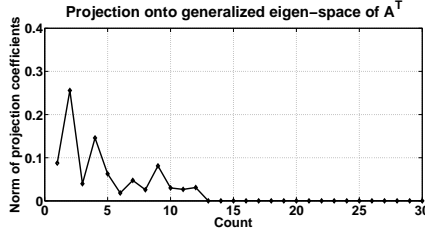
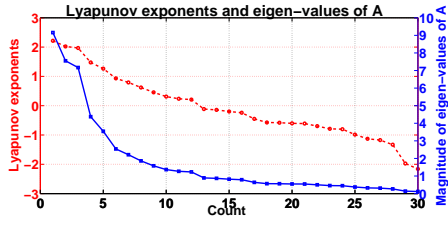
$$(4.3) \quad \lim_{n \rightarrow \infty} \|\Delta_n L^{bs} (L^{bs})^T\| = 0,$$

$$(4.4) \quad \lim_{n \rightarrow \infty} \|\Sigma_n L^{bs} (L^{bs})^T\| = 0.$$

Proof. Combining Theorem 3.7 with Theorem B.3 this is a straightforward consequence. \square

In our numerical simulations with arbitrary (and completely observable) choices of A , H , and Q we have additionally observed *convergence* of Δ_n and Σ_n to a fixed Δ and Σ , respectively, and seen their null spaces contain $\mathcal{E}^1(A^T)$ as stated by Corollary 4.2 (refer to section 4.2). Considering the recent work of Ni and Zhang [14], this strongly suggests that the classical result of the stable Riccati equation for completely observable and controllable, discrete autonomous systems [9] has an analogue in the case of completely observable, perfect-model systems.

4.2. Numerical results for linear autonomous system. We choose a non-singular matrix $A \in \mathbb{R}^{30 \times 30}$ ($d = 30$) consisting of random entries and set $d_0 = 12$ of its eigen-values to be greater than or equal to one in absolute magnitude. We ran the Kalman filtering system long enough and observed that the analysis error covariances do converge to a fixed Δ and then projected Δ onto the generalized eigen-space of A^T . Figure 3 plots the absolute magnitude of eigen-values of A sorted in descending order ($|\lambda_1(A)| \geq \dots \geq |\lambda_d(A)|$) in blue and shows the Lyapunov exponents for this system in red, where we note that the number of nonnegative Lyapunov exponents is exactly 12 tantamount to the number of eigen-values of A greater than or equal to one in magnitude. Additionally, it can be verified that the Lyapunov exponents are just the logarithm (to the base e) of the absolute magnitude eigen-values of A . Recalling the definition of the Lyapunov exponents from (3.15), this equality also lends credence to our Theorem A.3. The plot in Figure 4 displays $\|\Delta(\mathbf{v}_j(A^T))\|; j \in \{1, 2, \dots, d\}$, where $\mathbf{v}_j(A^T)$ is the generalized eigen-vector of $\lambda_j(A)$. Observe that when $j > 12$, the norm of the projected coefficients is zero, rendering a visual confirmation to Corollary 4.2.



479 FIG. 3. *Lyapunov exponents in blue and the*
 480 *magnitude of the eigen-values of A in red.* 482

FIG. 4. *Norm of the projection coefficients*
onto the generalized eigen-space of A^T .

483 **5. Discussion.** We have shown that under sequential Kalman filtering, the error
 484 covariance for a linear, perfect-model, conditionally Gaussian system asymptotically
 485 collapses to the subspaces spanned by the backward unstable Lyapunov vectors. This
 486 has been known to practitioners in the forecasting community [1] but had yet to be
 487 stated in precise mathematical terms. In particular, this foundational work validates
 488 the underlying assumptions and methodology of AUS.

489 At the same time, these results open many new questions for ongoing research
 490 related to AUS algorithms. For instance, the present results do not formally show
 491 the equivalence of a fully reduced-rank algorithm such as EKF-AUS applied in such a
 492 setting. The conditions that imply the convergence of the covariance matrices, given
 493 arbitrary low rank symmetric matrices chosen as initial conditions have yet to be
 494 established. Recent work strongly suggests that filter stability for discrete, perfect-
 495 model systems can be demonstrated under sufficient observability hypotheses alone
 496 [14]. Determining the necessary hypotheses for stability of the discrete with low rank
 497 initializations of the prior covariance matrix in perfect-model systems will be the
 498 subject of the sequel to our work.

499 Additionally there are conceptual issues to be resolved in bridging the results
 500 for linear systems to nonlinear settings, the former having the advantage of Lyapunov
 501 vectors being defined globally in space, whereas the formulation must change in a non-
 502 linear setting, respecting the dependence on the underlying path. Both of these direc-
 503 tions of inquiry open rich areas for mathematical research and future algorithm design.

504 While the ultimate goal of DA is a precise estimate of state for chaotic dynam-
 505 ics, it is critical to understand the uncertainty of the prediction. An exact calcula-
 506 tion of the posterior distribution of states for a high dimensional, complex system is
 507 computationally intractable; as computational resources increase, so will model com-
 508 plexity and thus computational efficiency alone will not resolve this issue. This work
 509 provides an idealized but general framework for future investigations into low dimen-
 510 sional approximations for uncertainty calculation. We hope that a precise mathemat-
 511 ical framework for understanding the nature of uncertainty for linear systems will lead
 512 to innovative research to surmount these challenges.

513 **Appendix A. Eigen-values, singular values, and Lyapunov exponents** 514 **of linear autonomous systems.**

515 The results established in this appendix and
 516 Appendix B should be treated as an independent body of work elucidating the rela-
 517 tionship between various concepts in linear, autonomous systems and not restricted
 518 to the domain of DA and filtering theory. While these relationships are known and
 519 can be retrieved from multiple sources in the literature, we have explicitly proved
 520 them here for completeness. Readers familiar with these mathematical connections
 may choose to skip through these sections without any loss of continuity.

521 Based on the definition of the matrix E_n^f in (4.1) we find $\lambda_j(E_n^f) = [\sigma_j(A^n)]^{\frac{1}{n}}$.
 522 As $E_n^f \rightarrow E^f$ we also have

$$523 \quad (\text{A.1}) \quad \lim_{n \rightarrow \infty} \lambda_j(E_n^f) = \lim_{n \rightarrow \infty} [\sigma_j(A^n)]^{\frac{1}{n}} = \lambda_j(E^f) \quad j \in \{1, 2, \dots, d\},$$

524 where the eigen-values λ_j and singular values σ_j are ordered descending in norm.
 525 Dropping the label for brevity let $J = V_A^{-1} A V_A$ (instead of $J(A)$) be the Jordan
 526 canonical form of A . It is straightforward to see that $A^n = V_A J^n V_A^{-1}$ for any integer
 527 n . The following inequality stated in Theorem 9 of [12] is quite useful. For any two
 528 square matrices Z_1 and Z_2 we have

$$529 \quad (\text{A.2}) \quad \sigma_j(Z_1) \sigma_d(Z_2) \leq \sigma_j(Z_1 Z_2) \leq \sigma_j(Z_1) \sigma_1(Z_2).$$

530 Since the singular values of both the matrix and its transpose are the same, it follows
 531 that

$$532 \quad (\text{A.3}) \quad \sigma_d(Z_1) \sigma_j(Z_2) \leq \sigma_j(Z_1 Z_2) \leq \sigma_1(Z_1) \sigma_j(Z_2).$$

533

534 LEMMA A.1. For any square matrix $Z = V_Z J(Z) V_Z^{-1}$

$$535 \quad \lim_{n \rightarrow \infty} [\sigma_j(Z^n)]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} [\sigma_j(J(Z)^n)]^{\frac{1}{n}}.$$

536 *Proof.* Inequalities (A.2) and (A.3) lead to

$$537 \quad \sigma_d(V_Z) \sigma_d(V_Z^{-1}) \sigma_j(J(Z)^n) \leq \sigma_j(Z^n) \leq \sigma_1(V_Z) \sigma_1(V_Z^{-1}) \sigma_j(J(Z)^n).$$

538 Raising each term to the power $1/n$ and letting $n \rightarrow \infty$ proves the result. \square

539 COROLLARY A.2. For any matrix A let E^f be defined as in (4.2) and J be the
 540 Jordan canonical form of A . Then $\lambda_j(E^f) = \lim_{n \rightarrow \infty} [\sigma_j(J^n)]^{\frac{1}{n}}, j \in \{1, 2, \dots, d\}$.

541 *Proof.* The results follow immediately when we employ Lemma A.1 setting $Z = A$
 542 in conjunction with (A.1). \square

543 The theorem below establishes the relation between the eigen-values of the time
 544 invariant propagator A and the limit matrix E^f .

545 THEOREM A.3. For any matrix A let the matrix E^f be defined as in (4.2). Then
 546 the eigen-values of E^f equal the absolute magnitude eigen-values of A , i.e., $\lambda_j(E^f) =$
 547 $|\lambda_j(A)|, j \in \{1, 2, \dots, d\}$.

548 *Proof.* We consider two different cases.

549 *Case 1: A is diagonalizable.* When J is diagonal then $\sigma_j(J) = |\lambda_j(J)| = |\lambda_j(A)|$.
 550 Recalling that $\lambda_j(J^n) = [\lambda_j(J)]^n \forall n$, we get $[\sigma_j(J^n)]^{\frac{1}{n}} = |\lambda_j(A)|$ and the result
 551 follows from Corollary A.2.

552 *Case 2: A is not diagonalizable.* Let $J_\lambda(A)$ denote the Jordan block of size $k \times k$
 553 corresponding to an eigen-value λ of A of the form

$$554 \quad (\text{A.4}) \quad J_\lambda(A) \equiv \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

555 The following lemma is useful in proving Theorem A.3.

556 LEMMA A.4. For any matrix A let $J_\lambda(A)$ be a Jordan block corresponding to
 557 eigen-value λ of A as defined in (A.4). Then the singular values of $J_\lambda(A)$ respect the
 558 following equality, namely,

$$559 \quad (A.5) \quad \lim_{n \rightarrow \infty} [\sigma_j(J_\lambda^n)]^{\frac{1}{n}} = |\lambda|, \quad j \in \{1, 2, \dots, k\},$$

560 i.e., the limiting singular-values are the absolute magnitude of their respective eigen-
 561 values.

562 *Proof.* Following the standard proof technique for equality results we individually
 563 show that

$$564 \quad (A.6) \quad \lim_{n \rightarrow \infty} [\sigma_j(J_\lambda^n)]^{\frac{1}{n}} \leq |\lambda|, \quad j \in \{1, 2, \dots, k\},$$

565 and

$$566 \quad (A.7) \quad \lim_{n \rightarrow \infty} [\sigma_j(J_\lambda^n)]^{\frac{1}{n}} \geq |\lambda|, \quad j \in \{1, 2, \dots, k\}.$$

567 Let the Nilpotent matrix $N \equiv J_\lambda - \lambda I$ with $N^k = \mathbf{0}$. When $n \geq k - 1$ we get

$$568 \quad J_\lambda^n = (\lambda I + N)^n = \sum_{r=0}^{k-1} \binom{n}{r} \lambda^{n-r} N^r.$$

569 Further, the highest singular-value $\sigma_1(N^r) = 1$ for $r \in \{0, 1, \dots, k-1\}$. If $\lambda = 0$, then
 570 $J_\lambda^n = \mathbf{0}$ when $n \geq k - 1$ and the result is trivially true. Suppose $\lambda \neq 0$ define $\delta \equiv \frac{1}{\lambda}$.
 571 Using the identity that for any two matrices Z_1 and Z_2 , $\sigma_1(Z_1 + Z_2) \leq \sigma_1(Z_1) + \sigma_1(Z_2)$
 572 as stated in Theorem 6 of [12], we have

$$573 \quad (A.8) \quad \sigma_1(J_\lambda^n) \leq |\lambda|^n \left[\sum_{r=0}^{k-1} \binom{n}{r} |\delta|^r \right].$$

574 Let $|\delta| = \epsilon \xi$ for any $0 < \epsilon \leq |\delta|$. Then

$$575 \quad \sigma_1(J_\lambda^n) \leq |\lambda|^n \xi^k \left[\sum_{r=0}^{k-1} \binom{n}{r} \epsilon^r \right]$$

$$576 \quad \leq |\lambda|^n \xi^k \left[\sum_{r=0}^n \binom{n}{r} \epsilon^r \right] = |\lambda|^n \xi^k (1 + \epsilon)^n.$$

578 Raising to the power $1/n$ and taking the limit we get

$$579 \quad \lim_{n \rightarrow \infty} [\sigma_1(J_\lambda^n)]^{\frac{1}{n}} \leq |\lambda|(1 + \epsilon).$$

580 The above inequality is also true for the rest of the singular values as $\sigma_1(\cdot)$ is the
 581 largest. Since ϵ is *arbitrary* the first inequality (A.6) follows. If $\lambda = 0$ we get the
 582 desired, stronger equality result in (A.5) as the singular values by definition are non-
 583 negative. It suffices to focus on the case $\lambda \neq 0$, where J_λ is invertible.

584 To establish the reverse inequality (A.7), let T_λ be the Jordan canonical form of
 585 J_λ^{-1} given by

$$586 \quad T_\lambda \equiv \begin{pmatrix} \frac{1}{\lambda} & 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{1}{\lambda} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix}.$$

587 Lemma A.1 entails that

$$588 \quad \lim_{n \rightarrow \infty} \left[\sigma_j \left((J_\lambda^{-1})^n \right) \right]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} [\sigma_j(T_\lambda^n)]^{\frac{1}{n}}.$$

589 Applying the inequality (A.6) on T_λ gives us

$$590 \quad \lim_{n \rightarrow \infty} [\sigma_j(T_\lambda^n)]^{\frac{1}{n}} \leq \frac{1}{|\lambda|}, \quad j \in \{1, 2, \dots, k\}.$$

591 In particular,

$$592 \quad \lim_{n \rightarrow \infty} \left[\sigma_1 \left((J_\lambda^{-1})^n \right) \right]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{[\sigma_k(J_\lambda^n)]^{\frac{1}{n}}} \leq \frac{1}{|\lambda|},$$

593 where the equality stems from the fact that for any invertible matrix Z of size $k \times k$

$$594 \quad \sigma_j(Z^{-1}) = \frac{1}{\sigma_{k-j+1}(Z)}.$$

595 We then get

$$596 \quad (\text{A.9}) \quad \lim_{n \rightarrow \infty} [\sigma_k(J_\lambda^n)]^{\frac{1}{n}} \geq |\lambda|.$$

597 Since $\sigma_k(\cdot)$ is the smallest singular value the inequality (A.9) is also valid for the
598 rest. \square

599 Now to prove Theorem A.3 note that for any n

$$600 \quad J^n = \begin{pmatrix} J_{\lambda_1}^n & 0 & \cdots & 0 \\ 0 & J_{\lambda_2}^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\lambda_l}^n \end{pmatrix}$$

601 is a block diagonal matrix and the eigen-(singular) values of J^n equal the *disjoint*
602 *union* of eigen-(singular) values of individual Jordan blocks $J_{\lambda_1}^n, \dots, J_{\lambda_l}^n$. In accor-
603 dance with Corollary A.2 and Lemma A.4 we find $\forall j \in \{1, 2, \dots, d\}$,

$$604 \quad \lambda_j(E^f) = \lim_{n \rightarrow \infty} [\sigma_j(J^n)]^{\frac{1}{n}} = |\lambda_j(J)| = |\lambda_j(A)|. \quad \square$$

606 **Appendix B. Eigen-spaces and Lyapunov vectors of linear autonomous**
607 **systems.** By a suitable coordinate transformation, namely, $\mathbf{z}_n = V_A^{-1} \mathbf{x}_n$, studying
608 the dynamics $\mathbf{x}_{n+1} = A \mathbf{x}_n$ is tantamount to investigating $\mathbf{z}_{n+1} = J \mathbf{z}_n$, where $J =$
609 $V_A^{-1} A V_A$ is the Jordan canonical form of A . Indeed,

$$610 \quad \mathbf{z}_{n+1} = J \mathbf{z}_n = V_A^{-1} A V_A V_A^{-1} \mathbf{x}_n = V_A^{-1} \mathbf{x}_{n+1}.$$

612 Corresponding to the definitions of the matrices E_n^f and E^f in (4.1)–(4.2), let $G_n \equiv$
613 $[(J^n)^* J^n]^{\frac{1}{2n}}$ and let $G \equiv \lim_{n \rightarrow \infty} G_n$.

614 We consider the two systems in the different d dimensional spaces \mathbb{R}_A^d and \mathbb{C}_J^d ,
615 where the underlying propagators are A and J , respectively. Note that as the matrix
616 V_A might be complex (though A is real) the dynamics for the propagator J is examined
617 in a complex state space.

LEMMA B.1. *If the scalar product in \mathbb{C}_J^d is the canonical one, namely, $\langle \mathbf{u}, \mathbf{v} \rangle_J = \mathbf{u}^\dagger \mathbf{v}$, then $V_G = I_d$, where I_d is the $d \times d$ identity matrix.*

Proof. We find it convenient to handle the following scenarios separately.

Case 1: A is diagonalizable. J is diagonal and so is J^n . In the canonical inner product setting the entries of the diagonal G_n are the absolute magnitude entries of J . It follows that G is diagonal and $V_G = V_J = I_d$.

Case 2: A is not diagonalizable. As before, consider the Jordan block J_λ given in (A.4) of size $k \times k$ corresponding to the eigen-value λ . Define $G_\lambda \equiv \lim_{n \rightarrow \infty} [(J_\lambda^n)^* J_\lambda^n]^{\frac{1}{2n}}$. Since G_λ is symmetric it is diagonalizable and by Theorem A.3 we have $\lambda_j(G_\lambda) = |\lambda| \forall j \in \{1, 2, \dots, k\}$. As all the eigen-values of G_λ are equal, it is a scalar matrix and therefore we can choose $V_{G_\lambda} = I_k$. Since

$$G = \begin{pmatrix} G_{\lambda_1} & 0 & \cdots & 0 \\ 0 & G_{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{\lambda_l} \end{pmatrix}$$

the result follows. \square

LEMMA B.2. *Under the definition of the scalar products $\langle \mathbf{u}, \mathbf{v} \rangle_J = \mathbf{u}^\dagger V_A^\dagger V_A \mathbf{v}$ in \mathbb{C}_J^d and $\langle \mathbf{u}, \mathbf{v} \rangle_A = \mathbf{u}^T \mathbf{v}$ in \mathbb{R}_A^d , $V_G = V_A^{-1} V_{E^f}$.*

Proof. For the aforesaid considerations of the scalar products in \mathbb{C}_J^d and \mathbb{R}_A^d , $J^* = (V_A^\dagger V_A)^{-1} J^\dagger V_A^\dagger V_A$ and $A^* = A^T$, respectively. Recalling that $J = V_A^{-1} A V_A$ we have

$$\begin{aligned} (J^n)^* &= \left(V_A^\dagger V_A \right)^{-1} V_A^\dagger (A^n)^T (V_A^{-1})^\dagger V_A^\dagger V_A = V_A^{-1} (A^n)^T V_A \\ \Rightarrow G_n &= \left[V_A^{-1} (A^n)^T V_A V_A^{-1} A^n V_A \right]^{\frac{1}{2n}} = \left[V_A^{-1} (A^n)^T A^n V_A \right]^{\frac{1}{2n}}. \end{aligned}$$

As $(E_n^f)^{2n} = (A^n)^T A^n$ is symmetric, it is diagonalizable by an orthonormal matrix $V_{E_n^f}$ and carries a representation $(E_n^f)^{2n} = V_{E_n^f} \Lambda_{E_n^f}^{2n} V_{E_n^f}^T$. We find $\Lambda_{G_n} = \Lambda_{E_n}$ and $V_{G_n} = V_A^{-1} V_{E_n} \forall n$ and the result follows by letting $n \rightarrow \infty$. \square

Recall the real span $\mathcal{T}_{\mathbf{w}}$ from Definition 2.1 bearing in mind the complex generalized eigen-vectors of any matrix Z always occur in conjugate pairs $\{\mathbf{w}, \overline{\mathbf{w}}\}$ with $\mathcal{T}_{\mathbf{w}} = \mathcal{T}_{\overline{\mathbf{w}}}$. We have the following theorem.

THEOREM B.3 (eigenspace equality). *For any matrix A let the matrix E^f be defined as in (4.2). Then for any $\alpha \geq 0$ the corresponding α -eigenspaces of E^f and A are the same, i.e., $\mathcal{E}^\alpha(E^f) = \mathcal{E}^\alpha(A)$. Equivalently, $\mathcal{E}^\alpha(E^b) = \mathcal{E}^\alpha(A^T)$.*

Proof. By Theorem A.3 we have $\lambda_j(G) = |\lambda_j(J)| = |\lambda_j(A)| = \lambda_j(E^f)$. Recall that the eigen-values are ordered with $\lambda_1(G)$ and $\lambda_d(G)$ being the largest and the smallest, respectively. The Oseledets theorem states that there exists a sequence of embedded subspaces

$$0 \subset \mathcal{F}_d \subset \mathcal{F}_{d-1} \subset \cdots \subset \mathcal{F}_1 = \mathbb{C}_J^d$$

such that on the complement $\mathcal{F}_j \setminus \mathcal{F}_{j+1}$ of \mathcal{F}_{j+1} in \mathcal{F}_j the growth rate is at most $\lambda_j(G)$ [15]. The subspaces \mathcal{F}_j can be obtained as the direct sum of the eigen-vectors $\mathbf{v}_j(G)$

as

$$\mathcal{F}_j = \mathbf{v}_d(G) \oplus \mathbf{v}_{d-1}(G) \oplus \cdots \oplus \mathbf{v}_j(G),$$

657 where $\mathbf{v}_j(G)$ is the eigen-vector of G corresponding to $\lambda_j(G)$. Further, though the
 658 eigen-vectors of G depend on the underlying scalar product in \mathbb{C}_J^d , the embedded
 659 subspaces \mathcal{F}_j and the eigen-values $\lambda_j(G)$ are *independent* of it [10].

660 Corresponding to the two inner product definitions in \mathbb{C}_J^d , specifically $\langle \mathbf{u}, \mathbf{v} \rangle_J =$
 661 $\mathbf{u}^\dagger \mathbf{v}$ and $\langle \mathbf{u}, \mathbf{v} \rangle_J = \mathbf{u}^\dagger V_A^\dagger V_A \mathbf{v}$, we denote the respective eigen-vectors with the super-
 662 script symbols 1 and 2. By Lemma B.1 we have $V_G^1 = I_d = V_A^{-1} V_A$ and Lemma B.2
 663 declares that $V_G^2 = V_A^{-1} V_{E^f}$, where V_{E^f} is computed using the canonical inner product
 664 in \mathbb{R}_A^d . For the given α let $q = \operatorname{argmin}_j \lambda_j(G) \leq \alpha$. The invariance of the embedded
 665 subspace \mathcal{F}_q to the underlying scalar product signifies that the real span of the vectors
 666 $\{V_A \mathbf{v}_d^1(G), \dots, V_A \mathbf{v}_q^1(G)\}$ equals the real span of the vectors $\{V_A \mathbf{v}_d^2(G), \dots, V_A \mathbf{v}_q^2(G)\}$.
 667 As $\forall j \in \{1, 2, \dots, d\}$, $V_A \mathbf{v}_j^1(G) = \mathbf{v}_j(A)$ and $V_A \mathbf{v}_j^2(G) = \mathbf{v}_j(E^f)$, the result follows. \square

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