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**On the Momentum Flux of Vertically-Propagating Orographic Gravity Waves
Excited in Nonhydrostatic Flow over Three-dimensional Orography**

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Abstract

32
33 This work studies nonhydrostatic effects (NHE) on the momentum flux of orographic
34 gravity waves (OGWs) forced by isolated three-dimensional orography. Based on linear wave
35 theory, an asymptotic expression for low horizontal Froude number ($Fr = \frac{\sqrt{U^2 + (\gamma V)^2}}{Na}$ where (U, V)
36 is the mean horizontal wind, γ and a are the orography anisotropy and half-width and N is the
37 buoyancy frequency) is derived for the gravity wave momentum flux (GWMF) of vertically-
38 propagating waves. According to this asymptotic solution, which is quite accurate for any value of
39 Fr , NHE can be divided into two terms (NHE1 and NHE2). The first term contains the high-
40 frequency parts of the wave spectrum that are often mistaken as hydrostatic waves, and only
41 depends on Fr . The second term arises from the difference between the dispersion relationships of
42 hydrostatic and nonhydrostatic OGWs. Having an additional dependency on the horizontal wind
43 direction and orography anisotropy, this term can change the GWMF direction. Examination of
44 NHE for OGWs forced by both circular and elliptical orography reveals that the GWMF is reduced
45 as Fr increases, at a faster rate than for two-dimensional OGWs forced by a ridge. At low Fr , the
46 GWMF reduction is mostly attributed to the NHE2 term, whereas the NHE1 term starts to
47 dominate above about $Fr = 0.4$. The behavior of NHE is mainly determined by Fr , while horizontal
48 wind direction and orography anisotropy play a minor role. Implications of the asymptotic GWMF
49 expression for the parameterization of nonhydrostatic OGWs in high-resolution and/or variable-
50 resolution models are discussed.

51 **1 Introduction**

52 Orographic gravity waves (OGWs) triggered by stably stratified airflow over topography
53 have been the subject of many studies over the last century. These waves can propagate upward
54 and thus have great importance for the large-scale circulation in the middle atmosphere (Fritts and
55 Alexander 2003). They are also closely related to various severe weather phenomena, like clear
56 air turbulence (CAT) and downslope windstorms occurring in the troposphere (Smith 1985). Given
57 that their horizontal spatial scales vary from a few to hundreds of kilometers, OGWs cannot be
58 fully resolved by numerical weather prediction (NWP) and general circulation models (GCMs).
59 As a result, the impacts of unresolved OGWs need to be parameterized (Kim et al. 2003).

60 Many parameterization schemes have been developed for subgrid-scale OGWs since the
61 1980s (e.g., Palmer et al. 1986; McFarlane 1987; Kim and Arakawa 1995; Lott and Miller 1997;
62 Scinocca and MacFarlane 2000; Kim and Doyle 2005), which are now routinely implemented in
63 various operational models for both weather forecasts and climate simulations. In general, these
64 schemes share many common assumptions, such as the columnar propagation of OGWs
65 (Plougonven et al. 2020). They also assume that OGWs are generated in a non-rotating and
66 hydrostatic framework. A state-of-the-art NWP model, the Integrated Forecasting System (IFS)
67 model of the European Centre for Medium-range Weather Forecasts (ECMWF), has horizontal
68 resolutions typically on the order of 10 km. In these circumstances, the non-rotating assumption is
69 justified because the subgrid-scale OGWs are too short to be affected by the earth's rotation.
70 However, this is not the case with the assumption of hydrostatic OGWs.

71 For small-scale OGWs with horizontal wavenumber comparable to the Scorer parameter
72 (Scorer 1949), nonhydrostatic effects (NHE) play a key role in controlling the wave dynamics.
73 Using the stationary phase method, Smith (1979) theoretically studied the far-field OGWs excited

74 by a narrow two-dimensional (2D) ridge, which are nonhydrostatic. A “dispersive tail” was found
75 to trail downstream of the mountain, which was also revealed in a number of numerical simulations
76 (e.g., Klemp and Durran 1983; Xue and Thorpe 1991; Zängl 2003). This suggests that the wave
77 energy can, not only propagate upwards as in the case of hydrostatic OGWs, but also disperse
78 downstream. Owing to nonhydrostatic dispersion, the wave activity above the mountain is weaker
79 than in its hydrostatic counterpart, leading to a suppression of wave breaking (Zängl 2003).
80 Nonetheless, NHE on wave breaking can be modified by the interaction between OGWs and
81 critical levels, as studied in Guarino and Teixeira (2017) for three-dimensional (3D) OGWs excited
82 in directional shear flows past isolated mountains. These modeling results showed that wave
83 breaking tends to be inhibited when the background shear is weak while it is enhanced for stronger
84 wind shear. Besides wave breaking, NHE can also influence the gravity wave momentum flux
85 (GWMF) at the surface. The high-frequency parts of nonhydrostatic OGWs (i.e., short-wavelength
86 components) tend to be trapped in the lower troposphere (e.g., Wurtele et al. 1996; Doyle and
87 Durran 2002). Consequently, the GWMF associated with upward-propagating waves is smaller
88 than that existing in the hydrostatic case (e.g., Xue et al. 2000).

89 The GWMF at the surface is a key parameter in the parameterization schemes of OGWs.
90 It denotes the maximum GWMF that can be absorbed into the mean flow. Changes in the surface
91 GWMF can affect wave breaking at high altitudes (Xu et al. 2020) and thus redistribute the wave
92 momentum deposition, impacting the large-scale circulation in the middle atmosphere (Xu et al.
93 2019). However, NHE are not considered in any OGW parameterization scheme. This is mainly
94 due to the fact that there is no analytical solution for nonhydrostatic OGWs except for very special
95 cases. To compensate for this, some OGW parametrization schemes (e.g., Lott and Miller 1997)
96 filter all orography of horizontal scale smaller than a few km out of the orography that serves as

97 input to the OGW parametrization, assuming that it only causes turbulent orographic form drag
98 (TOFD) which is the object of a separate parametrization (e.g., Beljaars et al. 2004). However, this
99 filtering procedure is somewhat arbitrary, ignoring the influence of the flow characteristics on how
100 non-hydrostatic the OGWs are, and how reduced their GWMF is by NHE. In the present study,
101 this limitation will be overcome.

102 Smith (1980) proposed solving the wave equation of nonhydrostatic OGWs numerically
103 using the Fast Fourier Transform (FFT) technique, which is apparently not suitable for the purpose
104 of OGW parameterization given its computational cost. Alternatively, ray theory has been widely
105 adopted to obtain the asymptotic solutions of nonhydrostatic OGWs. For instance, Smith (1979)
106 derived the far-field approximation of 2D nonhydrostatic OGWs, while Marks and Eckermann
107 (1995) developed a ray-tracing model for 3D nonhydrostatic gravity waves in a rotating, stratified
108 and fully compressible atmosphere. Standard ray theory often utilizes the stationary-phase method
109 and the asymptotic solution is expressed in spatial coordinates (Shutts 1998). This spatial-ray
110 solution is inaccurate directly over the mountain because of the presence of ray caustics there. To
111 overcome this problem, Broutman et al. (2002) expressed the ray solution in the wavenumber
112 rather than spatial domain, i.e., Maslov's method. This eliminates the caustics over the mountain
113 because rays in the spectral domain are well separated. Broutman et al. (2003) further extended
114 the so-called Fourier-ray solution to accommodate nonhydrostatic OGWs, which showed good
115 agreement with numerical simulations. Nonetheless, the Fourier-ray solution also has caustics at
116 the buoyancy-frequency turning point for nonhydrostatic waves. Later, Pulido and Rodas (2011)
117 developed a higher-order ray approximation method, i.e., the Gaussian beam approximation
118 (GBA), for OGWs generated in vertically sheared flows. In the standard ray theory, each ray only
119 consists of a single monochromatic wavenumber (i.e., the characteristic wavenumber). On the

120 contrary, the GBA uses a bundle of rays centered at the characteristic wavenumber (i.e., Gaussian
121 beams) for each ray, and considers diffractive effects. Therefore, the GBA solution is well defined
122 even at caustics. However, all these studies focused on the wave fields rather than on the GWMF
123 and hence OGW parameterization. Based on the GBA, Xu et al. (2017a, 2018) revised a traditional
124 OGW parameterization scheme by explicitly incorporating the horizontal propagation (e.g.,
125 Eckermann et al. 2015; Ehard et al. 2017) and directional absorption (e.g., Shutts 1995; Xu et al.
126 2012; Teixeira and Miranda 2009; Teixeira and Yu 2014) of OGWs. The revised scheme was
127 implemented into the global Weather Research and Forecasting (WRF) model, and helped improve
128 the simulation of the stratospheric polar-night jet in the Northern Hemisphere (Xu et al. 2019).

129 Compared with the traditional parameterization schemes of OGWs, ray-tracing based
130 schemes have to keep track of a number of rays, which requires a significant amount of
131 computation (e.g., Song and Chun 2008; Amemiya and Sato 2016). This approach is thus not
132 suitable for operational use. Teixeira et al. (2008, hereafter T08) studied the surface GWMF
133 associated with vertically-propagating OGWs produced by nonhydrostatic and rotating flow over
134 a 2D ridge. Instead of calculating the GWMF numerically, an asymptotic expression was derived
135 by using Taylor expansion for weakly-nonhydrostatic and weakly-rotating conditions. Fortuitously,
136 the asymptotic expansion was found to be fairly accurate even for nonhydrostatic inertio-gravity
137 waves, i.e., when the nonhydrostatic or rotation effects were not weak. The analytical form of this
138 asymptotic expression of GWMF makes it promising for practical use in OGW parameterizations
139 in numerical models. However, T08 only considered 2D OGWs forced by a ridge, while subgrid-
140 scale OGWs are intrinsically 3D (Lott and Miller 1997; Kim and Doyle 2005). In this work, an
141 asymptotic expression will be derived for 3D GWMF to accommodate the parameterization of 3D

142 nonhydrostatic OGWs. This provides a physically-based, flow-dependent, alternative to simply
143 filtering out the GWMF associated with waves shorter than a prescribed scale.

144 The rest of the paper is organized as follows. Section 2 presents the expression for surface
145 GWMF of 3D nonhydrostatic OGWs from linear mountain wave theory. An asymptotic solution
146 is derived in section 3 for the linear nonhydrostatic GWMF associated with vertically-propagating
147 OGWs. The behavior of this GWMF solution is studied for both isotropic and elliptical mountains
148 in section 4. Finally, the paper is summarized and discussed in section 5.

149

150 **2 Linear theory of nonhydrostatic OGWs**

151 In the case of steady, adiabatic, inviscid, and Boussinesq flow, the governing equation for
152 the perturbed vertical velocity of gravity waves in spectral space is

$$153 \left. \frac{\partial^2 \hat{w}}{\partial z^2} + \left[\frac{N^2 K^2}{\hat{D}(z)^2} - \frac{1}{\hat{D}(z)} \frac{\partial^2 \hat{D}(z)}{\partial z^2} - K^2 \right] \hat{w} = 0, \right. \quad (1)$$

154 where N is the Brunt-Väisälä frequency, $K = \sqrt{k^2 + l^2}$ is the magnitude of horizontal
155 wavenumber vector $\mathbf{K} = (k, l)$, and $\hat{D}(z) = \mathbf{V}(z) \cdot \mathbf{K} = U(z)k + V(z)l$, with $\mathbf{V}(z)$ being a
156 horizontally uniform mean flow. The above equation is similar to Eq. (9) in Xu et al. (2012) except
157 for the last term K^2 within the brackets, which denotes the NHE. The Earth's rotation is neglected
158 because we only consider nonhydrostatic OGWs forced by relatively narrow orography.

159 In the parameterization schemes of OGWs, the mean flow is assumed to be constant when
160 calculating the surface GWMF (e.g., Lott and Miller 1997), although vertical wind shear (either
161 unidirectional or directional) definitely influences the GWMF (e.g., Grubišić et al. 1997; Teixeira
162 et al. 2004; Turner et al. 2019; Xu et al. 2020). Herein, we also make this assumption, to be

163 consistent with existing parameterization schemes. For constant wind, i.e., $\mathbf{V}(z) = \mathbf{V}_0 =$
 164 (U_0, V_0) , Eq. (1) simplifies to

$$165 \quad \frac{\partial^2 \hat{w}}{\partial z^2} + m^2 \hat{w} = 0, \quad (2)$$

166 where $m^2 = \frac{N^2 K^2}{\hat{D}_0^2} - K^2$ is the squared vertical wavenumber, and $\hat{D}_0 = \mathbf{V}_0 \cdot \mathbf{K} = U_0 k + V_0 l =$
 167 $|\mathbf{V}_0| K \cos(\varphi - \psi_0)$, with φ and ψ_0 being the directions of \mathbf{K} and \mathbf{V}_0 respectively. For vertically-
 168 propagating OGWs the magnitude of the horizontal wavenumber should be smaller than
 169 $\left| \frac{N}{V_0 \cos(\varphi - \psi_0)} \right|$. Otherwise, the vertical wavenumber will be imaginary, indicating evanescent waves
 170 that decay exponentially with height.

171 Under the free-slip condition at the bottom boundary, i.e., $w(z = 0) = \mathbf{V}_0 \cdot \nabla h(x, y)$, the
 172 vertical velocity of upward-propagating OGWs can be determined as

$$173 \quad \hat{w}(z) = i \hat{D}_0 \hat{h}(k, l) e^{imz}, \quad (3)$$

174 where $\hat{h}(k, l)$ is the 2D Fourier transform of the terrain elevation $h(x, y)$. In idealized studies of
 175 OGWs and their parameterizations (e.g., Phillips 1984; Lott and Miller 1997; Teixeira and
 176 Miranda 2006), elliptical bell-shaped mountains are often adopted, a convenient example of which
 177 is:

$$178 \quad h(x, y) = h_0 [1 + (x/a)^2 + (y/b)^2]^{-3/2}, \quad (4)$$

179 where h_0 is the mountain amplitude, and a and b are the mountain half widths in the x and y
 180 directions, respectively. The horizontal aspect ratio (i.e., anisotropy) of the elliptical terrain is
 181 quantified by $\gamma = \frac{a}{b}$. The 2D Fourier transform of the terrain elevation is given by

$$182 \quad \hat{h}(k, l) = \frac{h_0 ab}{2\pi} e^{-\sqrt{a^2 k^2 + b^2 l^2}} \quad (5)$$

183 and the GWMF at the surface is equal to

184
$$\boldsymbol{\tau} = -\bar{\rho} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{v}' w' dx dy. \quad (6)$$

185 Here $\bar{\rho}$ is the background air density, and $\mathbf{v}' = (u', v')$ and w' are the perturbed horizontal and
 186 vertical velocities in physical space, respectively. On substitution of the 2D Fourier transforms of
 187 \mathbf{v}' and w' into the above equation and using the polarization relation between \mathbf{v}' and w' , i.e., $\hat{\mathbf{v}} =$
 188 $i \frac{\mathbf{K}}{K^2} \frac{\partial \hat{w}}{\partial z}$ (see the appendix of Xu et al. 2017b), one can readily obtain

189
$$\boldsymbol{\tau} = 4\pi^2 \bar{\rho} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathbf{K}}{K^2} \Im \left(\frac{\partial \hat{w}}{\partial z} \hat{w}^* \right) dk dl, \quad (7)$$

190 where $\Im(\cdot)$ denotes the imaginary part of a complex number and the asterisk indicates complex
 191 conjugate.

192 For the sake of computational convenience, elliptical polar coordinates are introduced, that
 193 is,

194
$$\tilde{k} = ak = \tilde{K} \cos \phi, \quad \tilde{l} = bl = \tilde{K} \sin \phi. \quad (8)$$

195 In this situation, the terrain spectrum has a simple form that only depends on \tilde{K} , i.e.,

196
$$\hat{h}(\tilde{K}) = \frac{h_0 ab}{2\pi} e^{-\tilde{K}}. \quad (9)$$

197 Consequently, the GWMF can be expressed as

198
$$\boldsymbol{\tau} = \frac{8\pi^2 \bar{\rho}}{b} \int_0^\pi \int_0^\infty (\cos \phi, \gamma \sin \phi) (\cos^2 \phi + \gamma^2 \sin^2 \phi)^{-1} \Im \left(\frac{\partial \hat{w}}{\partial z} \hat{w}^* \right) d\tilde{K} d\phi. \quad (10)$$

199 Substituting Eqs. (3) and (9) into the above equation yields

200
$$\boldsymbol{\tau} = \Pi \int_0^\pi \int_0^{[Fr \cos(\phi - \chi)]^{-1}} (\cos \phi, \gamma \sin \phi) \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} \sqrt{1 - [Fr \cos(\phi - \chi) \tilde{K}]^2} \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} d\phi, \quad (11)$$

201

202 where $\Pi = 2\bar{\rho} N h_0^2 b |\tilde{\mathbf{V}}_0|$ and $\chi = \text{atan} \left(\frac{\gamma V_0}{U_0} \right)$. Note that χ is the direction of $\tilde{\mathbf{V}}_0 = (U_0, \gamma V_0)$,
 203 which is similar to the actual wind \mathbf{V}_0 but with the y velocity component scaled by the terrain
 204 anisotropy. Only in the case of isotropic terrain (i.e., $\gamma = 1$) or when the horizontal wind is aligned

205 with the main axes of the orography (i.e., $U_0 = 0$ or $V_0 = 0$) is χ equal to the actual wind direction.
 206 For simplicity, it is still called the horizontal wind direction hereafter, unless otherwise stated.

207 The non-dimensional parameter Fr is defined as $Fr = \frac{|\tilde{V}_0|}{Na}$, which represents a measure of
 208 NHE. It is similar to the traditional Froude number ($Fr = \frac{|V|}{Nh_0}$) that quantifies the nonlinearity of
 209 OGWs (e.g., Miranda and James 1992), but with the mountain amplitude replaced by the mountain
 210 width. It is thus called horizontal Froude number hereafter. Physically, the horizontal Froude
 211 number can be viewed as the ratio between the period of buoyancy oscillation ($1/N$) and the
 212 advection time of airflow past the mountain ($a/|\tilde{V}_0|$). In the limit $Fr \rightarrow 0$, i.e., slow airflow and/or
 213 a broad mountain, the OGWs are predominantly hydrostatic. As Fr increases, NHE are more and
 214 more important. In the limit $Fr \rightarrow \infty$, the airflow can quickly traverse the mountain, with no
 215 internal OGWs excited.

216 In Eq. (11) the upper limit of the integral over \tilde{K} is $[Fr \cos(\phi - \chi)]^{-1}$, which indicates the
 217 contribution to the GWMF coming from internal OGWs, because evanescent waves produce zero
 218 GWMF. This upper limit depends on the directions of both the mean flow and the horizontal
 219 wavenumber. To facilitate the deduction of the asymptotic GWMF expression (see section 3), this
 220 upper limit is set to Fr^{-1} , i.e.,

$$221 \quad \tau_{\text{trunc}} = \Pi \int_0^\pi \int_0^{Fr^{-1}} (\cos\phi, \gamma \sin\phi) \frac{\cos(\phi - \chi)}{\sqrt{\cos^2\phi + \gamma^2 \sin^2\phi}} \sqrt{1 - [Fr \cos(\phi - \chi)\tilde{K}]^2} \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} d\phi. \quad (12)$$

223 This corresponds to an artificial truncation of waves with \tilde{K} between Fr^{-1} and $[Fr \cos(\phi - \chi)]^{-1}$.
 224 The latter value can go up to infinity when $\cos(\phi - \chi) \rightarrow 0$. Nonetheless, as will be shown below,
 225 these high-frequency waves only give a weak contribution to the total GWMF.

226

227 3 Asymptotic solution

228 Generally, a closed analytical form for Eq. (12) does not exist, and the GWMF must be
 229 evaluated by numerical integration. Yet an asymptotic solution can be derived for weakly
 230 nonhydrostatic OGWs at small Fr (see T08). In the limit $Fr \rightarrow 0$, the nonhydrostatic term in Eq.
 231 (12) can be approximated by

$$232 \quad \sqrt{1 - [Fr \cos(\phi - \chi)\tilde{K}]^2} \approx 1 - \frac{1}{2}Fr^2 \cos^2(\phi - \chi)\tilde{K}^2, \quad (13)$$

233 based upon a Taylor series expansion around $Fr = 0$ up to first order. On substitution of Eq. (13)
 234 into (12), the asymptotic GWMF (τ_{asy}) is given by the sum of τ_0 , τ_{asy1} , and τ_{asy2} , as follows

$$235 \quad \tau_0 = \Pi \int_0^\pi (\cos\phi, \gamma\sin\phi) \frac{\cos(\phi-\chi)}{\sqrt{\cos^2\phi+\gamma^2\sin^2\phi}} \left(\int_0^\infty \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} \right) d\phi, \quad (14a)$$

$$236 \quad \tau_{asy1} = -\Pi \int_0^\pi (\cos\phi, \gamma\sin\phi) \frac{\cos(\phi-\chi)}{\sqrt{\cos^2\phi+\gamma^2\sin^2\phi}} \left(\int_{Fr^{-1}}^\infty \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} \right) d\phi, \quad (14b)$$

$$237 \quad \tau_{asy2} = -\frac{1}{2}Fr^2 \Pi \int_0^\pi (\cos\phi, \gamma\sin\phi) \frac{\cos^3(\phi-\chi)}{\sqrt{\cos^2\phi+\gamma^2\sin^2\phi}} \left(\int_0^{Fr^{-1}} \tilde{K}^4 e^{-2\tilde{K}} d\tilde{K} \right) d\phi, \quad (14c)$$

238 with $\tau_0 = (\tau_{0x}, \tau_{0y})$ denoting the GWMF of hydrostatic OGWs. In deriving these equations, we
 239 have used $\int_0^{Fr^{-1}} \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} = \int_0^\infty \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} - \int_{Fr^{-1}}^\infty \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K}$. Using integration by parts, it
 240 is easy to show that

$$241 \quad \int_0^\infty \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} = \frac{1}{4}, \quad (15a)$$

$$242 \quad \int_{Fr^{-1}}^\infty \tilde{K}^2 e^{-2\tilde{K}} d\tilde{K} = \frac{1}{4}(2Fr^{-2} + 2Fr^{-1} + 1)e^{-2Fr^{-1}} = \frac{1}{4}I_2(Fr), \quad (15b)$$

$$243 \quad \int_0^{Fr^{-1}} \tilde{K}^4 e^{-2\tilde{K}} d\tilde{K} = \frac{1}{4}[3 - (2Fr^{-4} + 4Fr^{-3} + 6Fr^{-2} + 6Fr^{-1} + 3)e^{-2Fr^{-1}}] = \frac{1}{4}I_4(Fr). \quad (15c)$$

244 The I_2 term receives contributions from wavenumbers ranging from $\tilde{K} = Fr^{-1}$ to $\tilde{K} = \infty$. The
 245 largest contribution of the integrand comes from $\tilde{K} = 1$ (see the solid line in Fig. 1), which
 246 corresponds to the typical horizontal scale of the orography. On the contrary, the I_4 term is made

247 up of wavenumbers in the range between $\tilde{K} = 0$ and $\tilde{K} = Fr^{-1}$, with the largest contribution from
 248 the integrand being shifted to a higher wavenumber $\tilde{K} = 2$ (i.e., half the orography scale; see the
 249 dashed line in Fig. 1). The response decays rapidly away from $\tilde{K} = 1$ for I_2 and $\tilde{K} = 2$ for I_4 ,
 250 especially towards the high-wavenumber tail of the spectrum (i.e., high-frequency waves). It is
 251 noteworthy that this decay depends crucially on the exponential that results directly from the
 252 Fourier transform of the terrain elevation, but any smooth topography will have a spectrum that
 253 decays towards high wavenumbers (albeit in different ways). Substitution of Eq. (15) into (14)
 254 yields

$$255 \quad \boldsymbol{\tau}_0 = \frac{\Pi}{4} \int_0^\pi (\cos\phi, \gamma \sin\phi) \frac{\cos(\phi-\chi)}{\sqrt{\cos^2\phi + \gamma^2 \sin^2\phi}} d\phi, \quad (16a)$$

$$256 \quad \boldsymbol{\tau}_{asy1} = -I_2(Fr) \boldsymbol{\tau}_0, \quad (16b)$$

$$257 \quad \boldsymbol{\tau}_{asy2} = -\frac{1}{2} Fr^2 I_4(Fr) [R_x(\gamma, \chi) \tau_{0x}, R_y(\gamma, \chi) \tau_{0y}], \quad (16c)$$

258 where

$$259 \quad R_x(\gamma, \chi) = \frac{\int_0^\pi \frac{\cos\phi \cos^3(\phi-\chi)}{\sqrt{\cos^2\phi + \gamma^2 \sin^2\phi}} d\phi}{\int_0^\pi \frac{\cos\phi \cos(\phi-\chi)}{\sqrt{\cos^2\phi + \gamma^2 \sin^2\phi}} d\phi}, \quad (17a)$$

$$260 \quad R_y(\gamma, \chi) = \frac{\int_0^\pi \frac{\sin\phi \cos^3(\phi-\chi)}{\sqrt{\cos^2\phi + \gamma^2 \sin^2\phi}} d\phi}{\int_0^\pi \frac{\sin\phi \cos(\phi-\chi)}{\sqrt{\cos^2\phi + \gamma^2 \sin^2\phi}} d\phi}. \quad (17b)$$

261 $\boldsymbol{\tau}_{asy1}$ is anti-parallel to $\boldsymbol{\tau}_0$, with its magnitude controlled by the I_2 term. $\boldsymbol{\tau}_{asy2}$ is more complicated,
 262 depending not only on Fr but also on γ and χ . Given the difference between R_x and R_y , $\boldsymbol{\tau}_{asy2}$ may
 263 be misaligned with $\boldsymbol{\tau}_0$. This suggests that NHE can change the direction of the GWMF as well as
 264 its magnitude.

265 In order to better understand the NHE, they are quantified by the ratio between the
 266 asymptotic and hydrostatic GWMFs, i.e.,

$$267 \quad \tilde{\tau}_x(Fr, \gamma, \chi) = \frac{\tau_{asyx}}{\tau_{x0}} = 1 - I_2(Fr) - \frac{1}{2}Fr^2 I_4(Fr) R_x(\gamma, \chi), \quad (18a)$$

$$268 \quad \tilde{\tau}_y(Fr, \gamma, \chi) = \frac{\tau_{asyy}}{\tau_{y0}} = 1 - I_2(Fr) - \frac{1}{2}Fr^2 I_4(Fr) R_y(\gamma, \chi). \quad (18b)$$

269 The second term on the right-hand-side (RHS) of Eq. (18) is related to $\mathbf{\tau}_{asy1}$ (hereafter, NHE1 for
 270 short), which only depends on the horizontal Froude number. It denotes the wave components that
 271 are mistaken as vertically-propagating internal waves in the hydrostatic approximation, but are
 272 actually evanescent waves. The third term arises from $\mathbf{\tau}_{asy2}$ (hereafter, NHE2 for short), which is
 273 attributed to the difference between the dispersion relationships of hydrostatic and nonhydrostatic
 274 OGWs, i.e., the K^2 term within the brackets of Eq. (1). As noted above, NHE2 can affect both the
 275 magnitude and direction of the GWMF.

276 The above asymptotic expressions were derived for weakly nonhydrostatic OGWs. In the
 277 limit $Fr \rightarrow 0$, they simplify to

$$278 \quad \tilde{\tau}_x(Fr \rightarrow 0, \gamma, \chi) = 1 - \frac{3}{2}R_x(\gamma, \chi)Fr^2, \quad (19a)$$

$$279 \quad \tilde{\tau}_y(Fr \rightarrow 0, \gamma, \chi) = 1 - \frac{3}{2}R_y(\gamma, \chi)Fr^2. \quad (19b)$$

280 As will be shown in section 4, the relative difference between the asymptotic and exact GWMFs
 281 increases as the horizontal Froude number increases. Therefore, the asymptotic GWMF at $Fr \rightarrow$
 282 ∞ provides an estimate of the upper bound of the bias. Expanding the $e^{-2Fr^{-1}}$ term in Eq. (18) as
 283 $Fr \rightarrow \infty$ using Taylor series, one can readily find that

$$284 \quad \tilde{\tau}_x(Fr \rightarrow \infty, \gamma, \chi) = \left[\frac{4}{3} - \frac{2}{5}R_x(\gamma, \chi) \right] Fr^{-3}, \quad (20a)$$

$$285 \quad \tilde{\tau}_y(Fr \rightarrow \infty, \gamma, \chi) = \left[\frac{4}{3} - \frac{2}{5}R_y(\gamma, \chi) \right] Fr^{-3}. \quad (20b)$$

286 At this highly-nonhydrostatic limit, the GWMF becomes extremely small (proportional to Fr^{-3}),
 287 given the trivial contribution from very small-scale OGWs (see Fig. 1). This result is not only
 288 qualitatively correct, given that, without adopting the approximation expressed by Eq. (13), the
 289 drag would also decrease to zero at high Fr , but even approximately quantitatively correct, as will
 290 be shown next.

291

292 **4 Results**

293 In this section, the NHE will be firstly studied for the simple case of a circular mountain,
 294 i.e., $\gamma = 1$. Then we will investigate the more general case of elliptical mountains with $\gamma \neq 1$. In
 295 the latter case, the mean flow can be either parallel or oblique to the main axes of the mountain,
 296 which will be examined separately. These variants will henceforth be called “parallel flow” and
 297 “oblique flow”, for short.

298 4.1 Isotropic terrain

299 For isotropic terrain, without loss of generality, the horizontal wind direction can be set to
 300 $\chi = 0$ for simplicity, i.e., $\mathbf{V}_0 = (U_0, 0)$. In this case, $\tau_{x0} = \frac{\pi}{4} \bar{\rho} N h_0^2 a U_0$, $\tau_{y0} = 0$, $R_x(1, 0) = \frac{3}{4}$,
 301 $R_y(1, 0) = 0$, and Eq. (18) simplifies to

$$\begin{aligned}
 302 \quad \tilde{\tau}_c(Fr) &= 1 - I_2(Fr) - \frac{3}{8} Fr^2 I_4(Fr) \\
 303 \quad &= 1 - \frac{9}{8} Fr^2 + e^{-2Fr^{-1}} \left(-\frac{5}{4} Fr^{-2} - \frac{1}{2} Fr^{-1} + \frac{5}{4} + \frac{9}{4} Fr + \frac{9}{8} Fr^2 \right), \quad (21)
 \end{aligned}$$

304 where the subscript “c” indicates circular terrain. Clearly, $\tilde{\tau}_c$ only depends on the horizontal Froude
 305 number.

306 The variation of $\tilde{\tau}_c$ with the horizontal Froude number is depicted in Fig. 2. For comparison,
 307 the scaled asymptotic GWMF in the case of 2D ridge is also shown, which is expressed as follows
 308 [cf. Eq. (16) in T08]

309
$$\tilde{\tau}_{2D}(Fr) = \frac{\tau_{asy-2D}}{\tau_{0-2D}} = 1 - \frac{3}{4}Fr^2 + e^{-2Fr^{-1}} \left(-Fr^{-1} + \frac{1}{2} + \frac{3}{2}Fr + \frac{3}{4}Fr^2 \right). \quad (22)$$

310 It is clear that NHE weaken the GWMF. For both 2D and 3D OGWs, the asymptotic GWMFs are
 311 in good agreement with their exact counterparts which are obtained via numerical integration of
 312 Eq. (11) in this work and Eq. (10) in T08, respectively. The GWMF is only slightly overestimated
 313 by Eq. (22) for 2D flow and underestimated by Eq. (12) with respect to Eq. (11) for 3D flow for
 314 moderate Fr . This justifies the choice of Fr^{-1} as the upper limit of the integral in Eq. (11), given
 315 the simplifications this entails. Although adoption of the asymptotic approximation for the GWMF
 316 slightly improves the agreement with Eq. (11), the GWMF is still underestimated by a larger
 317 fraction than it is overestimated in the 2D case. Note that $\tilde{\tau}_c$ is always smaller than its 2D
 318 counterpart. In the limit $Fr \rightarrow 0$, $\tilde{\tau}_c(Fr)$ tends asymptotically to $1 - \frac{9}{8}Fr^2$ while $\tilde{\tau}_{2D}$ varies as
 319 $1 - \frac{3}{4}Fr^2$. In the opposite limit $Fr \rightarrow \infty$, $\tilde{\tau}_c$ and $\tilde{\tau}_{2D}$ tend asymptotically to

320
$$\tilde{\tau}_c(Fr \rightarrow \infty) = \frac{31}{30}Fr^{-3}, \quad (23)$$

321
$$\tilde{\tau}_{2D}(Fr \rightarrow \infty) = \frac{3}{2}Fr^{-2}, \quad (24)$$

322 respectively. $\tilde{\tau}_c$ is proportional to Fr^{-3} which decays faster than $\tilde{\tau}_{2D}$. As shown by Fig. 2, the way
 323 in which $\tilde{\tau}_c$ approaches zero as Fr increases is surprisingly accurate (as found in T08 for $\tilde{\tau}_{2D}$)
 324 given that the asymptotic approximation was developed for small Fr .

325 As stated in section 3, the NHE can be decomposed into two terms: NHE1 and NHE2.
 326 Figure 3 displays these two terms as a function of the horizontal Froude number. The magnitude
 327 of NHE1 exhibits an increasing trend with Fr . At lower horizontal Froude numbers ($Fr < 0.2$), the
 328 NHE1 term is very weak. This is because the lower limit of the integral in Eq. (15b) is given by
 329 Fr^{-1} , hence the NHE1 term mainly comes from high-frequency waves which produce negligible
 330 GWMF (Fig. 1). As Fr increases beyond 0.2 (corresponding to a cutoff horizontal wavenumber of

331 $\tilde{K} = 5$), the magnitude of NHE1 term increases rapidly, reaching up to about 0.7 at $Fr = 1$. As Fr
332 approaches infinity, this term tends asymptotically to -1. The NHE2 term is jointly determined by
333 the squared horizontal Froude number (Fr^2) and I_4 given by Eq. (15c). As the horizontal Froude
334 number increases, each of these two factors increases and decreases, respectively. The latter effect
335 is due to the fact that the upper limit of the integral in Eq. (15c) decreases as Fr increases. As a
336 result, the magnitude of NHE2 firstly increases with Fr , peaking around $Fr = 0.48$ at a maximum
337 of about 0.1. It then starts decreasing as the horizontal Froude number increases. It is clear that
338 NHE2 plays a more important role in the flow regimes with low Fr whereas NHE1 dominates
339 above about $Fr = 0.4$.

340 4.2 Anisotropic terrain: parallel flow

341 For OGWs generated by elliptical mountains, we firstly study the special case of horizontal
342 wind parallel to the main axes of the orography, which are assumed to be aligned in the x and y
343 directions, i.e., $\chi = 0$ (mean flow along the x axis) or $\chi = \pm \frac{\pi}{2}$ (mean flow along the y axis). In
344 this situation, $\tilde{\mathbf{\tau}}$ only depends on the horizontal Froude number and on the terrain anisotropy.

345 Taking $\chi = 0$ for example, i.e., $\mathbf{V}_0 = (U_0, 0)$, one obtains that $\tau_{y0} = 0$, $R_y(\gamma, 0) = 0$, and

$$346 \quad R_{x0}(\gamma) = R_x(\gamma, 0) = \frac{\int_0^\pi \frac{\cos^4 \phi}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} d\phi}{\int_0^\pi \frac{\cos^2 \phi}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} d\phi}. \quad (25)$$

347 Hereafter, the subscript “0” denotes the case with $\chi = 0$. The black line in Fig. 4 shows the
348 variation of $R_{x0}(\gamma)$ with γ . Clearly, $R_{x0}(\gamma)$ increases as γ increases (i.e., from a ridge normal to
349 the flow to a ridge along the flow direction), showing substantial changes (by about 30%) from
350 $\gamma = \frac{1}{10}$ to $\gamma = 10$. The fastest variation occurs near $\gamma = 1$.

351 To better reveal the influence of terrain anisotropy, the relative variation of $\tilde{\tau}_{x0}(\gamma, Fr)$ with
 352 respect to $\tilde{\tau}_c$ is examined, which is defined as

$$353 \quad \Delta\tilde{\tau}_{x0}(\gamma, Fr) = \frac{\tilde{\tau}_{x0}(\gamma, Fr) - \tilde{\tau}_c(Fr)}{\tilde{\tau}_c(Fr)} = \frac{\tilde{\tau}_{x0}(\gamma, Fr)}{\tilde{\tau}_c(Fr)} - 1. \quad (26)$$

354 At $Fr = 0$, $\Delta\tilde{\tau}_{x0}$ is always equal to zero (Fig. 5). As the horizontal Froude number increases, the
 355 $\Delta\tilde{\tau}_{x0}$ curves quickly diverge. In the case of mean flow perpendicular to the long axis of the
 356 mountain ($\gamma < 1$), $\Delta\tilde{\tau}_{x0}$ is greater than zero, i.e., $\tilde{\tau}_{x0}(\gamma, Fr) > \tilde{\tau}_c(Fr)$. This means that the
 357 GWMF is less reduced than in the isotropic case, i.e., weakening of NHE. This is consistent with
 358 the 2D-3D comparison presented in Fig. 2. In contrast, when the mean flow is aligned with the
 359 long axis of the mountain ($\gamma > 1$), NHE are enhanced, as suggested by the negative $\Delta\tilde{\tau}_{x0}$.

360 The $\Delta\tilde{\tau}_{x0}$ curves become more and more flat as the horizontal Froude number increases,
 361 tending asymptotically to their limits at $Fr \rightarrow \infty$, i.e.,

$$362 \quad \Delta\tilde{\tau}_{x0}(\gamma, Fr \rightarrow \infty) = \frac{9}{31} \left[1 - \frac{4}{3} R_{x0}(\gamma) \right], \quad (27)$$

363 which is obtained on substitution of Eqs. (20a) and (23) into Eq. (26). It is clear that the influence
 364 of terrain anisotropy is controlled by $R_{x0}(\gamma)$. When the mean flow is aligned with the long axis of
 365 the mountain, (the magnitude of) $\Delta\tilde{\tau}_{x0}$ is more notably enhanced than it is suppressed in the case
 366 of mean flow perpendicular to the long axis of the mountain. For instance, at $Fr = 1$, $\Delta\tilde{\tau}_{x0}$ exceeds
 367 3% at $\gamma = 8$ while it is less than 3% at $\gamma = \frac{1}{8}$. This difference is attributed to the asymmetric
 368 distribution of $R_{x0}(\gamma)$ about $\gamma = 1$ (see the black line in Fig. 4).

369 While $R_{x0}(\gamma)$ changes substantially with γ , that is not so much the case of $\Delta\tilde{\tau}_{x0}$. For two
 370 arbitrary γ , say, (γ_1, γ_2) , the difference between their $\Delta\tilde{\tau}_{x0}$ gradually saturates as $Fr \rightarrow \infty$, i.e.,

$$371 \quad \Delta\tilde{\tau}_{x0}(\gamma_1, Fr \rightarrow \infty) - \Delta\tilde{\tau}_{x0}(\gamma_2, Fr \rightarrow \infty) = \frac{12}{31} [R_{x0}(\gamma_2) - R_{x0}(\gamma_1)]. \quad (28)$$

372 This means that the influence of terrain anisotropy on $R_{x0}(\gamma)$ can be only partially projected onto
373 $\Delta\tilde{\tau}_{x0}$, since the latter is at most $\frac{12}{31} \approx 40\%$ of the former. From Eq. (25), $R_{x0}(\gamma)$ equals $\frac{2}{3}$ and 1 at
374 $\gamma = 0$ and $\gamma \rightarrow \infty$, respectively. Bounded by the lower and upper limits of $R_{x0}(\gamma)$, the variation
375 of $\Delta\tilde{\tau}_{x0}$ with γ is thus always smaller than $\frac{12}{31} \times \left(1 - \frac{2}{3}\right) = \frac{4}{31} \approx 12.9\%$. When compared to NHE
376 in the isotropic orography case, i.e., $R_{x0}(1) = \frac{3}{4}$, the maximum positive and negative differences
377 are $\frac{12}{31} \times \left(1 - \frac{3}{4}\right) = \frac{3}{31} \approx 9.7\%$ and $\frac{12}{31} \times \left(\frac{2}{3} - \frac{3}{4}\right) = -\frac{1}{31} \approx -3.2\%$, respectively.

378 From the above analysis, we can see that NHE in the parallel-flow case are only weakly
379 affected by terrain anisotropy. Instead, it is the horizontal Froude number that greatly impacts $\tilde{\tau}_{x0}$,
380 and this occurs both in the cases of circular mountains and 2D ridges (see section 4.1). Physically,
381 when the mean flow is parallel to the main axis of the elliptical terrain, e.g., when $\chi = 0$, as studied,
382 the horizontal Froude number is simplified to $Fr = \frac{|\tilde{V}_0|}{Na} = \frac{U_0}{Na}$. Thus, the terrain width in the cross-
383 flow direction has little contribution to the flow advection time.

384 4.3 Anisotropic terrain: oblique flow

385 In this section, the general case of mean flow oblique to the main axes of the elliptical bell-
386 shaped mountain is examined to understand more thoroughly the impacts of terrain anisotropy and
387 horizontal wind direction on the asymptotic GWMF expression.

388 In addition to $\chi = 0$, Figure 4 also shows the variation of $R_x(\gamma, \chi)$ as a function of γ for
389 three different horizontal wind directions, i.e., $\chi = \frac{\pi}{8}, \frac{\pi}{4}$ and $\frac{3\pi}{8}$. These wind directions are chosen
390 in the range of $\left[0, \frac{\pi}{2}\right)$, but the same results can be obtained for χ in the range of $\left[0, -\frac{\pi}{2}\right)$. This is
391 because $R_x(\gamma, \chi)$ is symmetric about $\chi = 0$, i.e., $R_x(\gamma, \chi) = R_x(\gamma, -\chi)$ in accordance with Eq.
392 (17a). (Note that $R_x(\gamma, \chi)$ is ill-defined at $\chi = \pm\frac{\pi}{2}$ where τ_{x0} vanishes.) The variation of $R_y(\gamma, \chi)$

393 is not presented herein, but can be inferred from that of $R_x(\gamma, \chi)$ because $R_x(\gamma, \chi) =$
394 $R_y\left(\frac{1}{\gamma}, \frac{\pi}{2} - \chi\right)$. In the situation with $\chi = \frac{\pi}{8}$, $R_x(\gamma, \chi)$ increases as γ increases, which is similar to
395 the case with $\chi = 0$. When χ equals $\frac{\pi}{4}$ or $\frac{3\pi}{8}$, $R_x(\gamma, \chi)$ instead decreases as γ increases. This
396 suggests a change in the trend of $R_x(\gamma, \chi)$ with γ for a horizontal wind direction between $\chi = \frac{\pi}{8}$
397 and $\chi = \frac{\pi}{4}$, at which $R_x(\gamma, \chi)$ should be independent of γ . As can be seen below, this occurs at
398 $\chi = \frac{\pi}{6}$.

399 The distribution of $R_x(\gamma, \chi)$ in γ - χ parameter space is shown in Fig. 6, with γ and χ in the
400 ranges of $\left[\frac{1}{10}, 10\right]$ and $\left[0, \frac{\pi}{2}\right)$, respectively. $R_x(\gamma, \chi)$ is always equal to $\frac{3}{4}$ at $\chi = \frac{\pi}{6}$, which can be
401 obtained analytically from Eq. (17a). Remember that $R_x(\gamma, \chi) \equiv \frac{3}{4}$ at $\gamma = 1$ as well (see section
402 4.1). Therefore, the γ - χ space can be divided into four quadrants by the lines $\chi = \frac{\pi}{6}$ and $\gamma = 1$. In
403 the third and fourth quadrants ($0 \leq \chi < \frac{\pi}{6}$), $R_x(\gamma, \chi)$ has an increasing trend with γ . The more the
404 horizontal wind is aligned with the long axis of the elliptical mountain, the more markedly terrain
405 anisotropy affects $R_x(\gamma, \chi)$. The greatest variation of $R_x(\gamma, \chi)$ with γ ($R_{x0}(\gamma \rightarrow \infty) - R_{x0}(\gamma \rightarrow$
406 $0)$) occurs at $\chi = 0$, which takes the value $\frac{1}{3}$, as derived in section 4.2. In the first and second
407 quadrants (i.e., $\frac{\pi}{6} < \chi < \frac{\pi}{2}$), $R_x(\gamma, \chi)$ decreases instead as γ increases, and the influence of terrain
408 anisotropy becomes larger with χ . In the limit of $\chi = \frac{\pi}{2}$, $R_x\left(\gamma, \frac{\pi}{2}\right)$ is ill-defined, yet it is equivalent
409 to $R_y\left(\frac{1}{\gamma}, 0\right)$ which is well defined. From Eq. (17b), $R_x\left(\gamma \rightarrow 0, \frac{\pi}{2}\right) = R_y(\gamma \rightarrow \infty, 0) = 1$, and
410 $R_x\left(\gamma \rightarrow \infty, \frac{\pi}{2}\right) = R_y(\gamma \rightarrow 0, 0) = \frac{1}{3}$. As a result, the greatest variation of $R_x(\gamma, \chi)$ with γ is $\frac{2}{3}$, i.e.
411 twice that for $\chi = 0$. Similarly, the greatest variations of $R_x(\gamma, \chi)$ with χ (i.e., variations along the

412 vertical rather than horizontal direction in the graph) on the left- and right semi-planes of the γ - χ
 413 parameter space are $\frac{1}{3}$ and $\frac{2}{3}$, respectively.

414 As in the parallel-flow case, the relative variation of $\tilde{\tau}_x(\gamma, Fr)$ with respect to $\tilde{\tau}_c$ is also
 415 examined here, which is defined as

$$416 \quad \Delta\tilde{\tau}_x(\gamma, \chi, Fr) = \frac{\tilde{\tau}_x(\gamma, \chi, Fr) - \tilde{\tau}_c(Fr)}{\tilde{\tau}_c(Fr)} = \frac{\tilde{\tau}_x(\gamma, \chi, Fr)}{\tilde{\tau}_c(Fr)} - 1. \quad (29)$$

417 As $Fr \rightarrow \infty$, $\tilde{\tau}_x$ tends asymptotically to

$$418 \quad \tilde{\tau}_x(\gamma, \chi, Fr \rightarrow \infty) = \frac{9}{31} \left[1 - \frac{4}{3} R_x(\gamma, \chi) \right]. \quad (30)$$

419 For two pairs of (γ, χ) , e.g., (γ_1, χ_1) and (γ_2, χ_2) , the difference between their $\tilde{\tau}_x$ is

$$420 \quad \tilde{\tau}_x(\gamma_1, \chi_1, Fr \rightarrow \infty) - \tilde{\tau}_x(\gamma_2, \chi_2, Fr \rightarrow \infty) = \frac{12}{31} [R_x(\gamma_2, \chi_2) - R_x(\gamma_1, \chi_1)]. \quad (31)$$

421 Again, this means that the influences of terrain anisotropy and horizontal wind direction on
 422 $R_x(\gamma, \chi)$ have a relatively small impact on $\tilde{\tau}_x$. From Fig. 6, the global maximal variation of
 423 $R_x(\gamma, \chi)$ with γ and χ is $\frac{2}{3}$. Thus, under the influence of both terrain anisotropy and horizontal
 424 wind direction, $\tilde{\tau}_x$ can change by $\frac{12}{31} \times \frac{2}{3} \approx 25.8\%$ at most as Fr tends to infinity. Compared to the
 425 NHE in the isotropic terrain case, the maximum positive and negative differences are
 426 $\frac{12}{31} \times \left(1 - \frac{3}{4}\right) = \frac{3}{31} \approx 9.7\%$ and $\frac{12}{31} \times \left(\frac{1}{3} - \frac{3}{4}\right) = -\frac{5}{31} \approx -16.1\%$, respectively. At small horizontal
 427 Froude number, the impacts of terrain anisotropy and horizontal wind direction are rather weak,
 428 as will be shown below.

429 Figure 7 gives the distributions of $\Delta\tilde{\tau}_x$ on the γ - χ plane at four different horizontal Froude
 430 numbers: $Fr = 0.1, 0.3, 0.5$, and 1.0 , respectively. Positive $\Delta\tilde{\tau}_x$ is found in the first and third
 431 quadrants, indicating an amplification of the NHE compared to the case of isotropic orography.
 432 Conversely, NHE are weakened in the second and fourth quadrants, given the negative values of

433 $\Delta\tilde{\tau}_x$ existing there. At $Fr = 0.1$ (Fig. 7a) $\Delta\tilde{\tau}_x$ is extremely small, implying that the terrain
 434 anisotropy and horizontal wind direction have negligible influence on the NHE. At $Fr = 0.3$ (Fig.
 435 7b), the impacts of terrain anisotropy and horizontal wind direction increase by more than 10 times
 436 compared to those at $Fr = 0.1$. When the horizontal Froude number further increases to $Fr = 0.5$
 437 and 1.0 (Figs. 7c, 7d), there occurs a consistent increase in the magnitude of $\Delta\tilde{\tau}_x$, which can reach
 438 up to 0.1 in the first quadrant (i.e., $\gamma > 1$ and $\frac{\pi}{6} < \chi < \frac{\pi}{2}$).

439 Figure 8 displays the variation of $\tilde{\tau}_x$ as a function of Fr . Two elliptical mountains with $\gamma =$
 440 $\frac{1}{8}$ (dashed lines) and $\gamma = 8$ (solid lines) are selected, along with two horizontal wind directions
 441 $\chi = \frac{\pi}{8}$ (blue lines) and $\chi = \frac{3\pi}{8}$ (red lines). From the above analysis, these configurations of terrain
 442 anisotropy and horizontal wind direction tend to have a significant influence on the NHE. However,
 443 as can be seen from Fig. 8, $\tilde{\tau}_x$ is still mainly determined by Fr . At $Fr = 0.1$, $\tilde{\tau}_x = 0.99$, i.e., the
 444 OGWs are almost purely hydrostatic. As Fr increases, $\tilde{\tau}_x$ decreases rapidly to about 0.65 at $Fr =$
 445 0.5, and further reduces to about 0.27 at $Fr = 1.0$. Compared with the horizontal Froude number,
 446 terrain anisotropy and horizontal wind direction only play a minor role. This is due to the fact that
 447 these two factors only affect the NHE2 term [see Eq. (18)]. At small horizontal Froude number
 448 ($Fr < 0.2$), the NHE2 term is of greater importance than NHE1 (Fig. 2), but $\Delta\tilde{\tau}_x$ is too weak to
 449 exert a profound influence on $\tilde{\tau}_x$ (Fig. 7a). At moderate to large horizontal Froude number ($Fr >$
 450 0.4), while $\Delta\tilde{\tau}_x$ is significantly enhanced (Figs. 7c, 7d), the NHE2 term is exceeded by NHE1, thus
 451 contributing less to $\tilde{\tau}_x$.

452 4.4 Surface pressure perturbation

453 Theoretically, the GWMF is equal to the pressure drag at the surface (e.g., Teixeira et al.
 454 2004). In this section, the surface pressure perturbations are investigated to help understand the
 455 impact of NHE on the GWMF. Herein, we only focus on the simple case of mean flow over circular

456 bell-shaped mountains, because the horizontal wind direction and orography anisotropy play a
457 minor role on the NHE (as we have just seen).

458 Figure 9 depicts the distribution of the surface pressure perturbation obtained via numerical
459 integration of Eqs. (A4). Note that the pressure perturbations are scaled with $\bar{\rho}N|\tilde{\mathbf{V}}|h_0$. At $Fr =$
460 0.1, the pressure field (Fig. 9a) shows a left-right anti-symmetric pattern about the orography
461 center, with positive and negative regions on the windward and leeward slope respectively (Smith
462 1980; Teixeira et al. 2004). In this weakly nonhydrostatic case, the pressure perturbation mainly
463 arises from vertically-propagating OGWs, with little contribution from evanescent waves (Figs.
464 9b, 9c). At $Fr = 0.5$, however, the surface pressure perturbation ceases to be perfectly anti-
465 symmetric about the mountain center (Fig. 9d). The maximum on the windward slope weakens
466 slightly as compared to that at $Fr = 0.1$, while the minimum on the lee slope also weakens notably
467 and moves downstream. In addition, a secondary pressure minimum occurs near the orography
468 center. This more complex pressure pattern is due to an enhanced pressure contribution from
469 evanescent waves (Fig. 9f), which is symmetric about the orography center (and thus produces
470 zero surface pressure drag). Concurrently, the pressure perturbation associated with vertically-
471 propagating OGWs weakens (Fig. 9e), giving rise to the reduction of GWMF.

472 Using the Taylor series expansion of the vertical wavenumber at small Fr (expressed by
473 Eq. (13)), one can also derive an asymptotic expression for the pressure perturbation associated
474 with vertically-propagating OGWs (see details in Appendix A), which is decomposed into three
475 parts (namely, p_0 , p_1 and p_2) corresponding to $\boldsymbol{\tau}_0$, $\boldsymbol{\tau}_{asy1}$ and $\boldsymbol{\tau}_{asy2}$, respectively.

476 Figure 10 shows the distribution of the asymptotic surface pressure perturbation at $Fr =$
477 0.1, which is also scaled by $\bar{\rho}N|\tilde{\mathbf{V}}|h_0$. The total asymptotic pressure perturbation (Fig. 10a) agrees
478 well with that in Fig. 9a. It is dominated by the hydrostatic part (Fig. 10b), because NHE are very

479 weak at $Fr = 0.1$ (see Fig. 2). The maximum (minimum) pressure perturbation occurs about one
 480 half-width away from the orography center, suggesting that the horizontal scale of the dominant
 481 wave field is comparable to that of the mountain. This is consistent with the power spectrum of τ_0 ,
 482 which peaks at $\tilde{K} = 1$, i.e., $K = a^{-1}$ (Fig. 1). The p_1 pressure perturbation is extremely small (Fig.
 483 10c), given the small magnitude of τ_{asy1} at this low horizontal Froude number (Fig. 3). A wave-
 484 train pattern is found both upstream and downstream of the mountain, which can be ascribed to
 485 the $\cos\left(\frac{\mu}{Fr}\right)$ and $\sin\left(\frac{\mu}{Fr}\right)$ terms in Eq. (A9b). This pattern is indiscernible in Fig. 10a because of
 486 its small magnitude. The horizontal wavelength of p_1 is very short, since it originates mainly from
 487 the high-frequency part of the wave spectrum [Eq. (A7b)]. Similar to p_0 , the p_2 pressure
 488 perturbation is anti-symmetric about the orography center (Fig. 10d), but with negative (positive)
 489 perturbations on the upslope (downslope) side. Consequently, p_2 produces a pressure gradient
 490 force opposed to that of p_0 , contributing negatively to the total surface pressure drag. Moreover,
 491 the p_2 pressure perturbation is mainly confined to the region within one half-width of the mountain
 492 to the orography center. This is also in agreement with the power spectrum of τ_{asy2} which peaks
 493 at $\tilde{K} = 2$ (Fig. 1).

494 Figure 11 is similar to Fig. 10, but for $Fr = 0.5$. Compared to that at $Fr = 0.1$, the total
 495 pressure perturbation is substantially reduced (Fig. 11a). The pressure perturbation extrema only
 496 correspond to about 70% of those at $Fr = 0.1$. The scaled p_0 (Fig. 11b) is independent of Fr , so it
 497 is exactly the same as in Fig. 10b. The p_1 pressure perturbation (Fig. 11c) increases markedly in
 498 magnitude, reaching up to 60% of p_0 . The p_2 pressure perturbation is also enhanced (Fig. 11d).
 499 However, unlike in the case with $Fr = 0.1$, p_2 is smaller than p_1 . This agrees with the major role
 500 played by the NHE1 term at moderate-to-large horizontal Froude numbers (see Fig. 3). Moreover,
 501 while the p_1 and p_2 pressure perturbations still display a wave-train pattern upstream and

502 downstream of the mountain, their horizontal wavelengths have increased significantly. Taking p_1
503 as an example, the dominant wavelength is approximately twice the orography half-width. This is
504 because, at $Fr = 0.5$, p_1 is composed of wavenumbers ranging from $\tilde{K} = 2$ to ∞ [see Eq. (A7b)].
505 In this spectral range, the greatest response of τ_{asy1} corresponding to p_1 occurs at $\tilde{K} = 2$ (Fig. 1).
506 Owing to the enhanced p_1 pressure perturbation, the extrema of the total pressure perturbation
507 slightly move away from the orography center (Fig. 11a), implying an increase in the dominant
508 wavelength. This is reasonable, since short waves are removed by the NHE from the range of
509 waves that contribute to the GWMF.

510

511 **5 Summary and discussion**

512 It has been widely recognized that the parameterization of subgrid-scale orographic gravity
513 waves (OGWs) is essential for accurate numerical weather forecast and climate prediction. Many
514 efforts have been made to improve the representation of orographic gravity wave momentum flux
515 (GWMF) and its deposition into the mean flow in numerical models. With the development of
516 high-resolution global numerical weather prediction (NWP) and general circulation models
517 (GCMs), the horizontal scale of unresolved OGWs is becoming increasingly small. As a result,
518 the GWMF can be significantly impacted by nonhydrostatic effects (NHE). However, these effects
519 are not accounted for in even the state-of-the-art parameterization schemes, since there is in general
520 no analytical solution for nonhydrostatic OGWs. In some parametrizations (e.g., Lott and Miller
521 1997), the GWMF reduction that is known to occur for highly non-hydrostatic waves is mimicked
522 rather artificially by filtering the orography that is fed into the OGW parametrization. The present
523 study proposes the more physical approach of explicitly evaluating the NHE approximately.

524 Using linear gravity wave theory, we have derived an asymptotic solution for the surface
525 GWMF of 3D OGWs, which is an extension of the 2D asymptotic expression studied in T08. The
526 intensity of the NHE can be quantified by the non-dimensional parameter called here the horizontal
527 Froude number, i.e., $Fr = \frac{|\tilde{V}_0|}{Na}$. This parameter is akin to the inverse non-dimensional mountain
528 half width $\frac{Na}{U}$ used in previous studies (e.g., Durran and Klemp 1983; Xue and Thorpe 1991; Zängl
529 2003) but with U replaced by $\tilde{V}_0 = (U_0, \gamma V_0)$. This extended definition is necessary due to the
530 horizontal anisotropy of the isolated orography that generates the 3D OGWs.

531 Based upon an asymptotic approach, the NHE are divided into two components (NHE1
532 and NHE2). The first component accounts for the high-frequency parts of the wave spectrum (i.e.,
533 short waves) that are mistaken as hydrostatic, upward-propagating waves in the hydrostatic
534 approximation. The GWMF associated with NHE1 is parallel but opposite to the hydrostatic
535 GWMF. The second component is due to the difference between the dispersion relationships of
536 hydrostatic and nonhydrostatic OGWs. While NHE1 only depends on the horizontal Froude
537 number, NHE2 also depends on the terrain anisotropy and horizontal wind direction. In the
538 presence of NHE, both the magnitude and direction of GWMF can be changed.

539 The asymptotic GWMF expression derived here was investigated for OGWs forced by both
540 circular and elliptical mountains for flows with various orientations. In the isotropic orography
541 case, NHE only depend on the horizontal Froude number, which is the same dependence as in the
542 2D-ridge case studied by T08. Compared to its 2D counterpart, the 3D GWMF is more strongly
543 reduced by NHE. Considering the two parts of the NHE, NHE1 is weaker than NHE2 at lower
544 horizontal Froude number, but its magnitude grows rapidly as the horizontal Froude number
545 increases. On the contrary, NHE2 firstly increases but then starts decreasing with the horizontal

546 Froude number, with this change of trend occurring at about $Fr = 0.48$. Consequently, NHE1 starts
547 to be dominant in the reduction of the GWMF above about $Fr = 0.4$.

548 For OGWs generated by anisotropic terrain, when the mean flow is perpendicular to the
549 long axis of the orography ($\gamma < 1$), the GWMF is less reduced than in the isotropic case,
550 suggesting a weakening of the NHE. This is consistent with the results of OGWs forced by 2D
551 ridges. Conversely, NHE are enhanced when the mean flow is parallel to the long axis or the
552 orography ($\gamma > 1$). In the parallel-flow case, the NHE vary by no more than 12.9% with the terrain
553 anisotropy, and this occurs as the horizontal Froude number tends asymptotically to infinity. Since
554 this corresponds to a situation in which τ approaches zero, the relevance of this effect is even more
555 limited. When the mean flow is oblique to the main axes of the mountain, NHE exhibit a greater
556 variation under the joint influence of terrain anisotropy and horizontal wind direction, with a
557 maximum value twice that of the parallel-flow case. Nevertheless, in either case, it is still the
558 horizontal Froude number that dominates the variation of the NHE.

559 Given the relatively weak influence of terrain anisotropy and horizontal wind direction on
560 the NHE, the asymptotic solution of the GWMF for isotropic terrain [i.e., Eq. (21)], which is
561 simply a function of the horizontal Froude number, may be used to quantify the NHE with a good
562 accuracy. Benefiting from the analytical form of this expression, the parameterization schemes for
563 hydrostatic OGWs can be easily extended to nonhydrostatic conditions, which will inevitably
564 occur in high-resolution NWP and GCMs. It is noteworthy that the horizontal Froude number
565 depends on the horizontal scale of subgrid-scale orography, which is constrained by the model's
566 horizontal resolution. Since the NHE are scale-aware (or scale-dependent), they make the
567 parametrization itself scale-aware. Recently, variable-resolution numerical models have generated
568 a growing interest (e.g., Skamarock et al. 2012; Davis et al. 2016; Zhou et al. 2019; Zhang et al.

569 2019), as they can significantly reduce the computational costs, while allowing for high-resolution
 570 modelling in areas of specific interest. A nonhydrostatic parameterization scheme will be
 571 particularly useful for models with variable-resolution meshes, as it can adjust the parameterized
 572 GWMF in the fine-resolution regions where NHE are expected to be important, while having little
 573 influence in the coarse-resolution areas.

574 In our upcoming research, a traditional hydrostatic OGW parameterization scheme will be
 575 revised taking into account NHE, based on the asymptotic expressions derived in the present study.
 576 Then the revised scheme will be implemented in a high-resolution numerical model (with a grid
 577 spacing on the order of 10 km) to investigate the impacts of NHE on the vertical momentum
 578 transport of subgrid-scale OGWs and their consequences for the large-scale circulation.

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583
 584 **Appendix A: Derivation of the asymptotic pressure perturbation at the surface**

585 According to Eq. (7) in Xu et al. (2017b), for 3D OGWs generated by constant flow over
 586 an isolated mountain, the polarization relation between the pressure and vertical velocity
 587 perturbations in spectral space has the simple form:

$$588 \quad \hat{p}(k, l, z) = -i \frac{\bar{\rho}}{K^2} \hat{D} \frac{\partial \hat{w}(z)}{\partial z}. \quad (\text{A1})$$

589 Substitution of Eq. (3) into the above equation yields

$$590 \quad \hat{p}(k, l, z) = i \bar{\rho} \frac{\hat{D}^2}{K^2} m e^{imz} \hat{h}(k, l). \quad (\text{A2})$$

591 Using inverse 2D Fourier transforms, the pressure perturbation in physical space is given by

592
$$p(x, y, z) = Re \left[i\bar{\rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\bar{D}^2}{K^2} m \hat{h}(k, l) e^{i(kx+ly+mz)} dk dl \right], \quad (A3)$$

593 where $Re(\cdot)$ denotes the real part of a complex number. For the elliptical bell-shaped mountain
 594 under consideration, and using polar coordinates for the horizontal wavenumber [see Eq. (8)], the
 595 pressure perturbation of nonhydrostatic OGWs at $z = 0$ is

596
$$p(x, y, 0) = p(S, \Psi, 0) = Re \left[\frac{i}{\pi} \bar{\rho} N |\tilde{\mathbf{V}}| h_0 \int_0^{\pi} \int_0^{\infty} \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} \times \right.$$

597
$$\left. \sqrt{1 - [\tilde{K} Fr \cos(\phi - \chi)]^2} \tilde{K} e^{-\tilde{K}} e^{i\tilde{K} S \cos(\phi - \Psi)} d\tilde{K} d\phi \right], \quad (A4a)$$

598 which can be divided into two parts, i.e.,

599
$$p_{GW}(S, \Psi, 0) = Re \left[\frac{i}{\pi} \bar{\rho} N |\tilde{\mathbf{V}}| h_0 \int_0^{\pi} \int_0^{[Fr \cos(\phi - \chi)]^{-1}} \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} \times \right.$$

600

601
$$\left. \sqrt{1 - [\tilde{K} Fr \cos(\phi - \chi)]^2} \tilde{K} e^{-\tilde{K}} e^{i\tilde{K} S \cos(\phi - \Psi)} d\tilde{K} d\phi \right]. \quad (A4b)$$

602
$$p_{evanescent}(S, \Psi, 0) = Re \left[\frac{i}{\pi} \bar{\rho} N |\tilde{\mathbf{V}}| h_0 \int_0^{\pi} \int_{[Fr \cos(\phi - \chi)]^{-1}}^{\infty} \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} \times \right.$$

603

604
$$\left. \sqrt{1 - [\tilde{K} Fr \cos(\phi - \chi)]^2} \tilde{K} e^{-\tilde{K}} e^{i\tilde{K} S \cos(\phi - \Psi)} d\tilde{K} d\phi \right]. \quad (A4c)$$

605 for vertically-propagating OGWs and evanescent waves, respectively. In the deduction of the
 606 above equations, the following elliptical polar coordinate in physical space was introduced for
 607 convenience:

608
$$X = \frac{x}{a} = S \cos \Psi, \quad Y = \frac{y}{b} = S \sin \Psi, \quad (A5)$$

609 where $S = \frac{1}{a} \sqrt{x^2 + (\gamma y)^2}$ and $\Psi = \text{atan} \left(\frac{\gamma y}{x} \right)$.

610 By expanding the vertical wavenumber for small Fr [see Eq. (13)], the asymptotic surface
 611 pressure perturbation associated with vertically propagating OGWs can be approximated by the
 612 sum of p_0 , p_1 and p_2 , namely,

$$613 \quad p_0(S, \Psi, 0) = Re \left[\frac{i}{\pi} \bar{\rho} N |\tilde{\mathbf{V}}| h_0 \int_0^\pi \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} G_0(\phi, S, \Psi) d\phi \right], \quad (\text{A6a})$$

$$614 \quad p_1(S, \Psi, 0) = Re \left[-\frac{i}{\pi} \bar{\rho} N |\tilde{\mathbf{V}}| h_0 \int_0^\pi \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} G_1(\phi, S, \Psi) d\phi \right], \quad (\text{A6b})$$

$$615 \quad p_2(S, \Psi, 0) = Re \left[-\frac{i}{2\pi} Fr^2 \bar{\rho} N |\tilde{\mathbf{V}}| h_0 \int_0^\pi \frac{\cos^3(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} G_2(\phi, S, \Psi) d\phi \right], \quad (\text{A6c})$$

616 with G_0 , G_1 and G_2 given, respectively, by

$$617 \quad G_0(\phi, S, \Psi) = \int_0^\infty \tilde{K} e^{\tilde{K}[iS\cos(\phi - \Psi) - 1]} d\tilde{K} = Q^{-2}, \quad (\text{A7a})$$

$$618 \quad G_1(\phi, S, \Psi) = \int_{Fr^{-1}}^\infty \tilde{K} e^{\tilde{K}[iS\cos(\phi - \Psi) - 1]} d\tilde{K} = Q^{-2} e^{-QFr^{-1}} (1 + QFr^{-1}), \quad (\text{A7b})$$

$$619 \quad G_2(\phi, S, \Psi) = \int_0^{Fr^{-1}} \tilde{K}^3 e^{\tilde{K}[iS\cos(\phi - \Psi) - 1]} d\tilde{K} \\ 620 \quad = Q^{-4} [6 - e^{-QFr^{-1}} (Q^3 Fr^{-3} + 3Q^2 Fr^{-2} + 6QFr^{-1} + 6)], \quad (\text{A7c})$$

621 and

$$622 \quad Q(\phi, S, \Psi) = 1 - iS\cos(\phi - \Psi) = 1 - i\mu(\phi, S, \Psi). \quad (\text{A8})$$

623 Clearly, p_0 is the pressure perturbation of purely hydrostatic OGWs while p_1 and p_2 are the pressure
 624 perturbations corresponding to $\mathbf{\tau}_{asy1}$ and $\mathbf{\tau}_{asy2}$.

625 Finally, after some lengthy but straightforward algebraic manipulations, one can obtain the
 626 three components of the surface pressure perturbation associated with vertically-propagating
 627 OGWs:

$$628 \quad p_0(S, \Psi, 0) = -\frac{\bar{\rho} N |\tilde{\mathbf{V}}| h_0}{\pi} \int_0^\pi \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} \frac{2\mu}{(1 + \mu^2)^2} d\phi, \quad (\text{A9a})$$

$$629 \quad p_1(S, \Psi, 0) = \frac{\bar{\rho} N |\tilde{\mathbf{V}}| h_0}{\pi} \int_0^\pi \frac{\cos(\phi - \chi)}{\sqrt{\cos^2 \phi + \gamma^2 \sin^2 \phi}} \frac{1}{(1 + \mu^2)^2} \frac{J_1(\mu) \cos\left(\frac{\mu}{Fr}\right) + J_2(\mu) \sin\left(\frac{\mu}{Fr}\right)}{e^{Fr^{-1}}} d\phi, \quad (\text{A9b})$$

$$630 \quad p_2(S, \Psi, 0) = \frac{\bar{\rho}N|\tilde{V}|h_0}{2\pi} \int_0^\pi \frac{\cos^3(\phi-\chi)}{\sqrt{\cos^2\phi+\gamma^2\sin^2\phi}} \frac{Fr^2}{(1+\mu^2)^4} \left[J_0(\mu) - \frac{J_3(\mu)\cos\left(\frac{\mu}{Fr}\right) + J_4(\mu)\sin\left(\frac{\mu}{Fr}\right)}{e^{Fr-1}} \right] d\phi, \quad (\text{A9c})$$

631 where

$$632 \quad J_0(\mu) = 24(1 - \mu^2)\mu, \quad (\text{A10a})$$

$$633 \quad J_1(\mu) = \mu \left(2 + \frac{1+\mu^2}{Fr} \right), \quad (\text{A10b})$$

$$634 \quad J_2(\mu) = 1 - \mu^2 + \frac{1+\mu^2}{Fr}, \quad (\text{A10c})$$

$$635 \quad J_3(\mu) = \mu \left[24(1 - \mu^2) - \frac{6(\mu^2-3)(1+\mu^2)}{Fr} + \frac{6(1+\mu^2)^2}{Fr^2} + \frac{(1+\mu^2)^3}{Fr^3} \right], \quad (\text{A10d})$$

$$636 \quad J_4(\mu) = 6(\mu^4 - 6\mu^2 + 1) + \frac{6(1+3\mu^2)(1-\mu^2)}{Fr} + \frac{3(1-\mu^2)(1+\mu^2)^2}{Fr^2} + \frac{(1+\mu^2)^3}{Fr^3}. \quad (\text{A10e})$$

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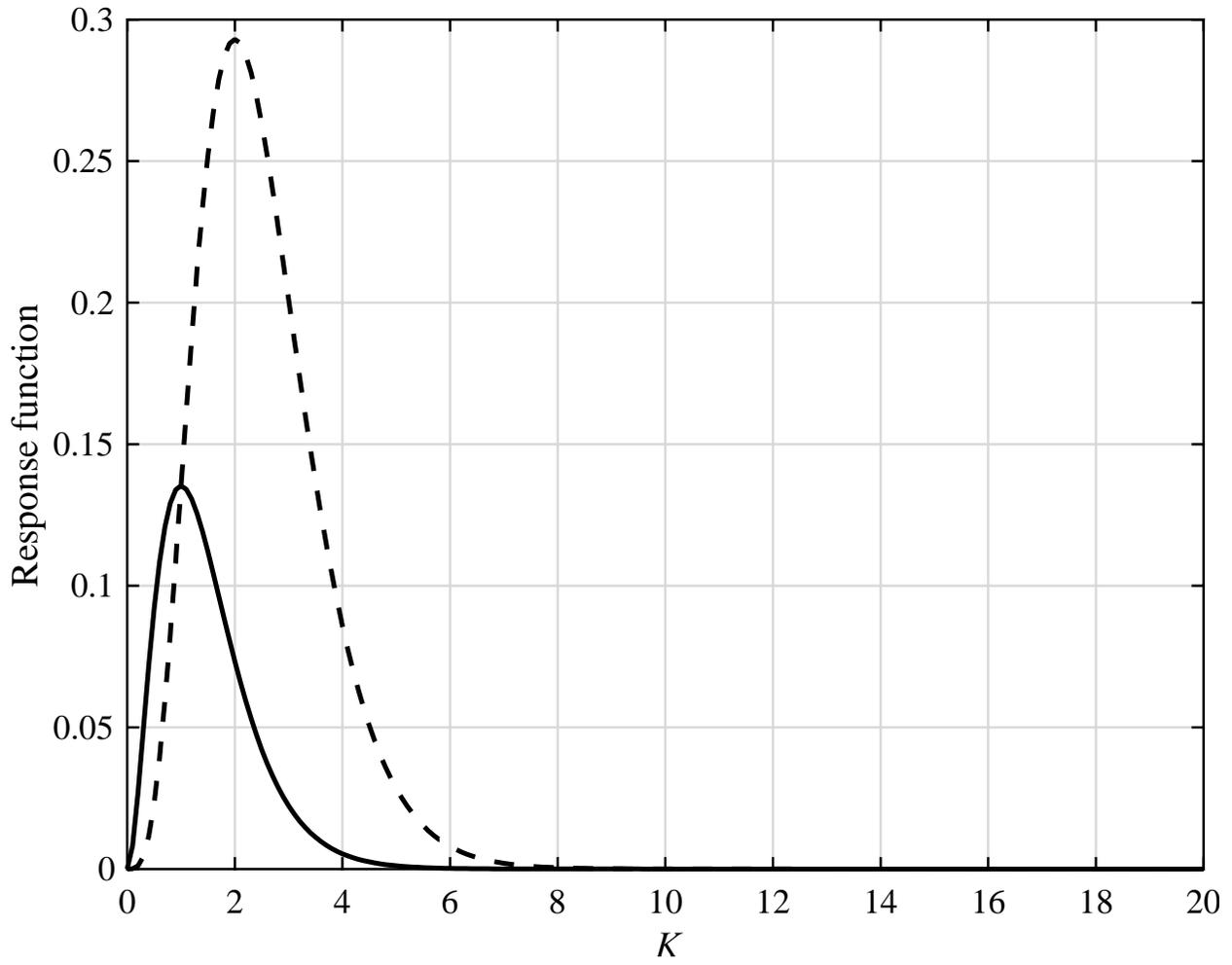
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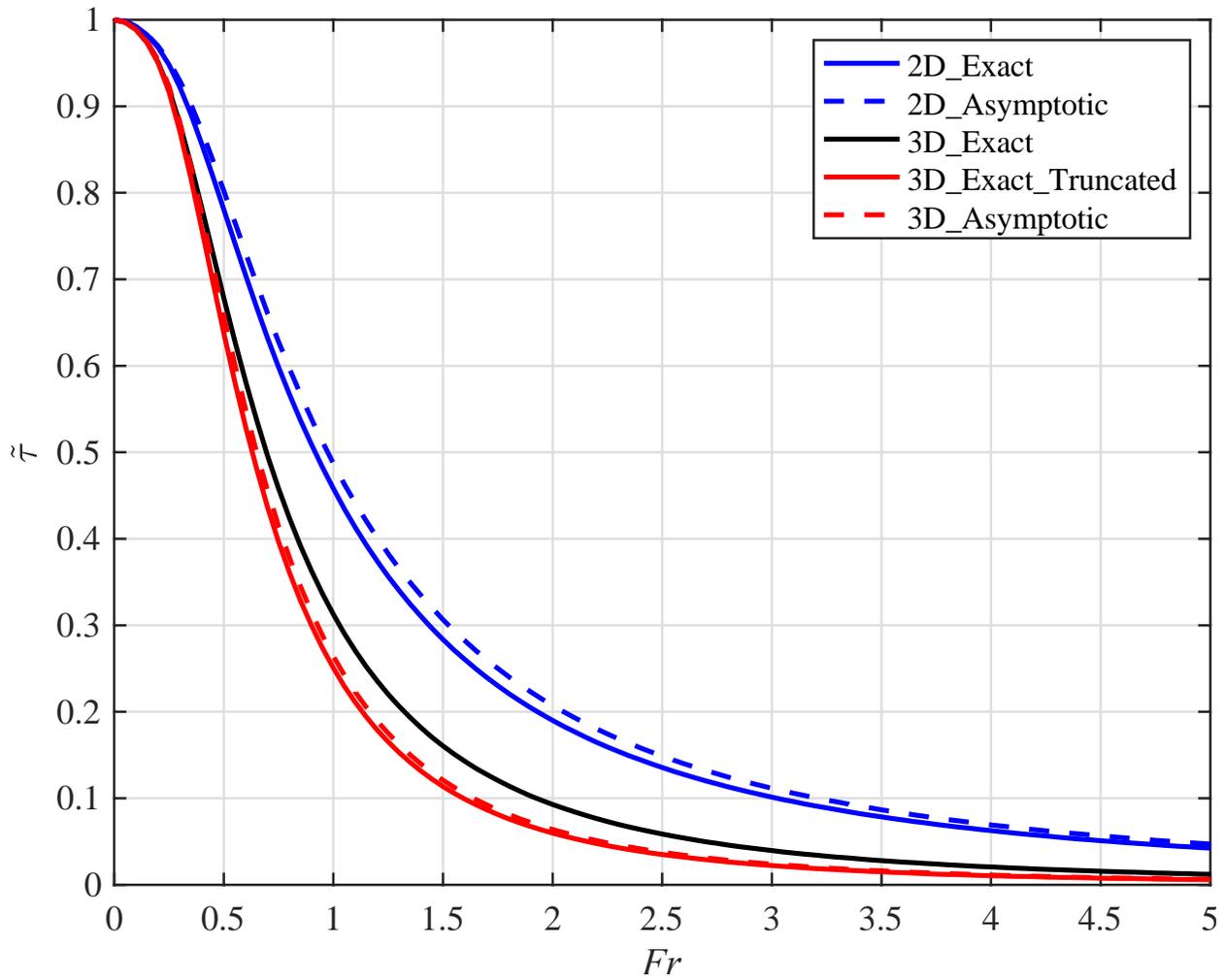
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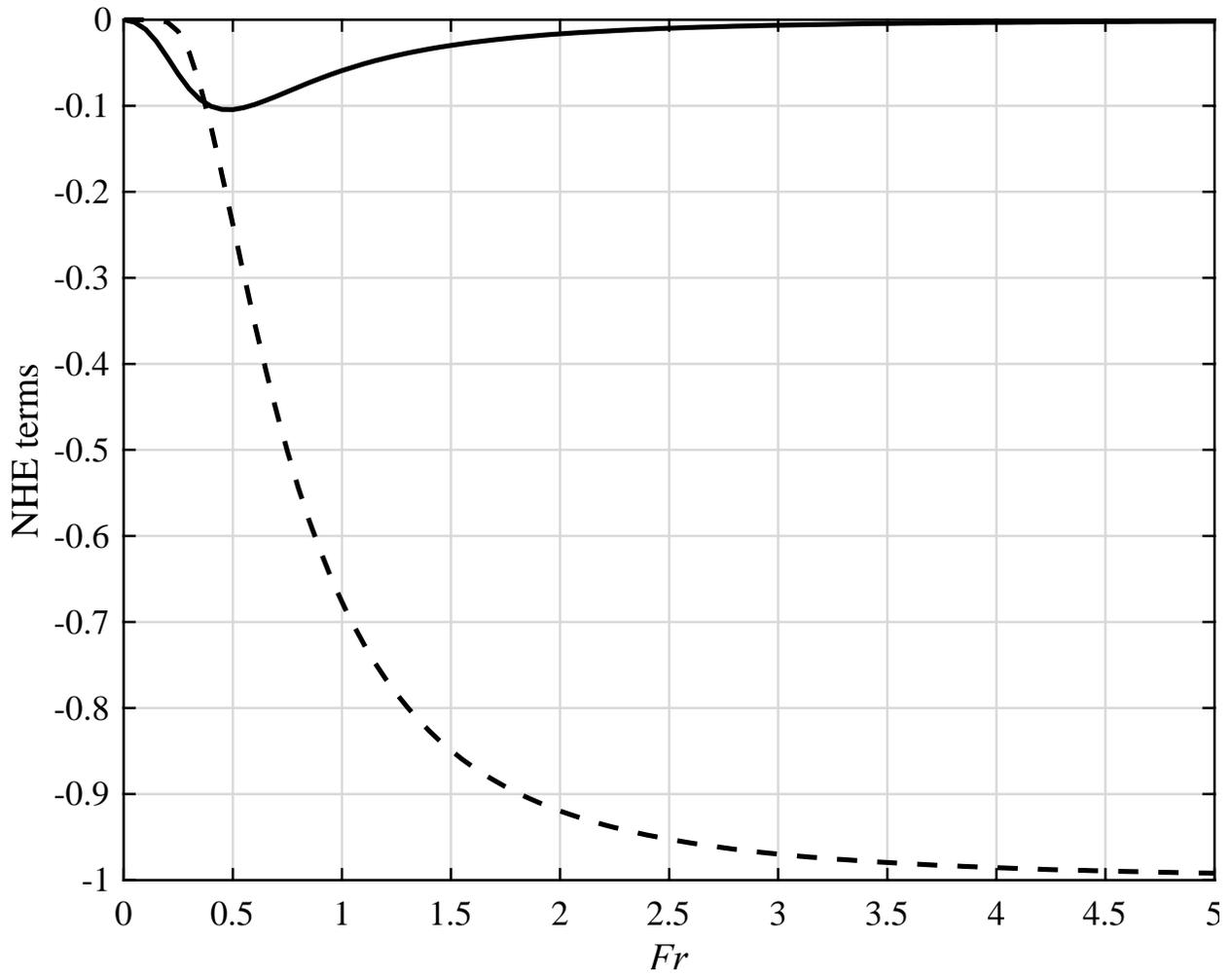
787 Fig. 1 Response functions $\tilde{K}^2 e^{-2\tilde{K}}$ (solid) and $\tilde{K}^4 e^{-2\tilde{K}}$ (dashed).

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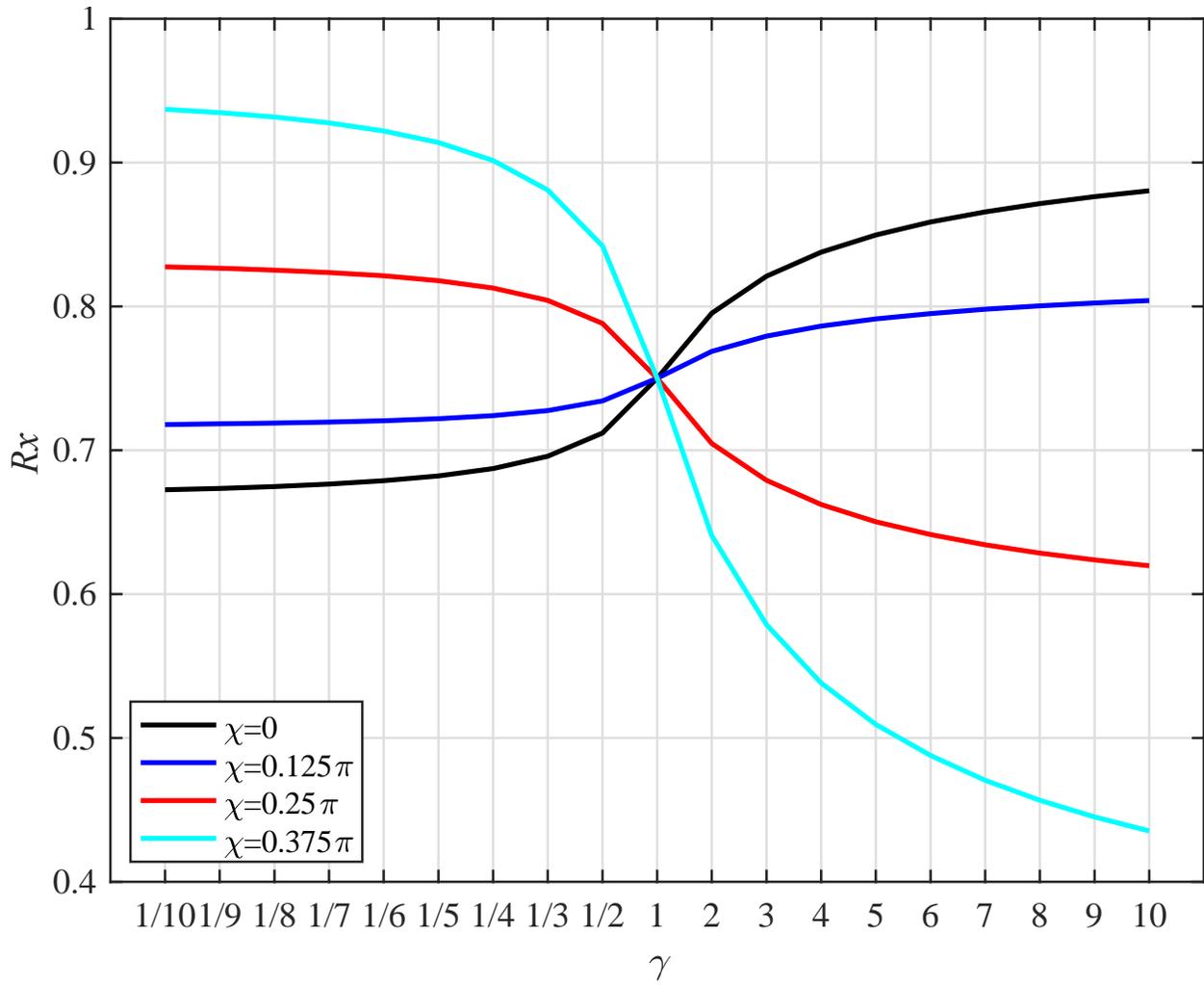
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790 Fig. 2 Variation of the normalized GWMF ($\tilde{\tau}$) with the horizontal Froude number (Fr). Blue lines
 791 are for the nonhydrostatic OGWs forced by 2D bell-shaped ridges, while the black and red lines
 792 are for those forced by 3D circular bell-shaped mountains. The normalization is made with respect
 793 to their hydrostatic counterparts.



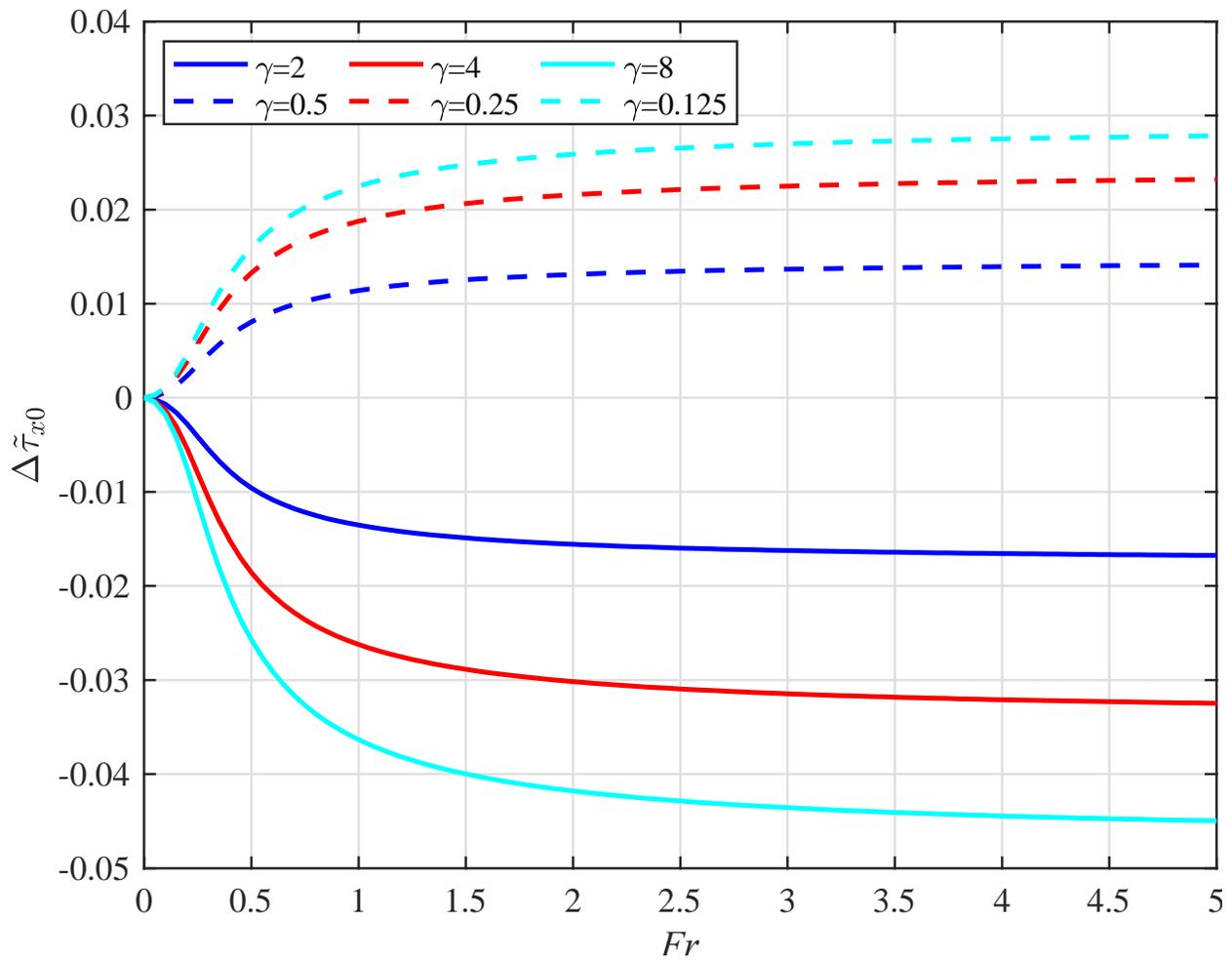
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795 Fig. 3 Variations of the NHE1 (dashed) and NHE2 (solid) terms with the horizontal Froude number
 796 (Fr) in the case of isotropic terrain.



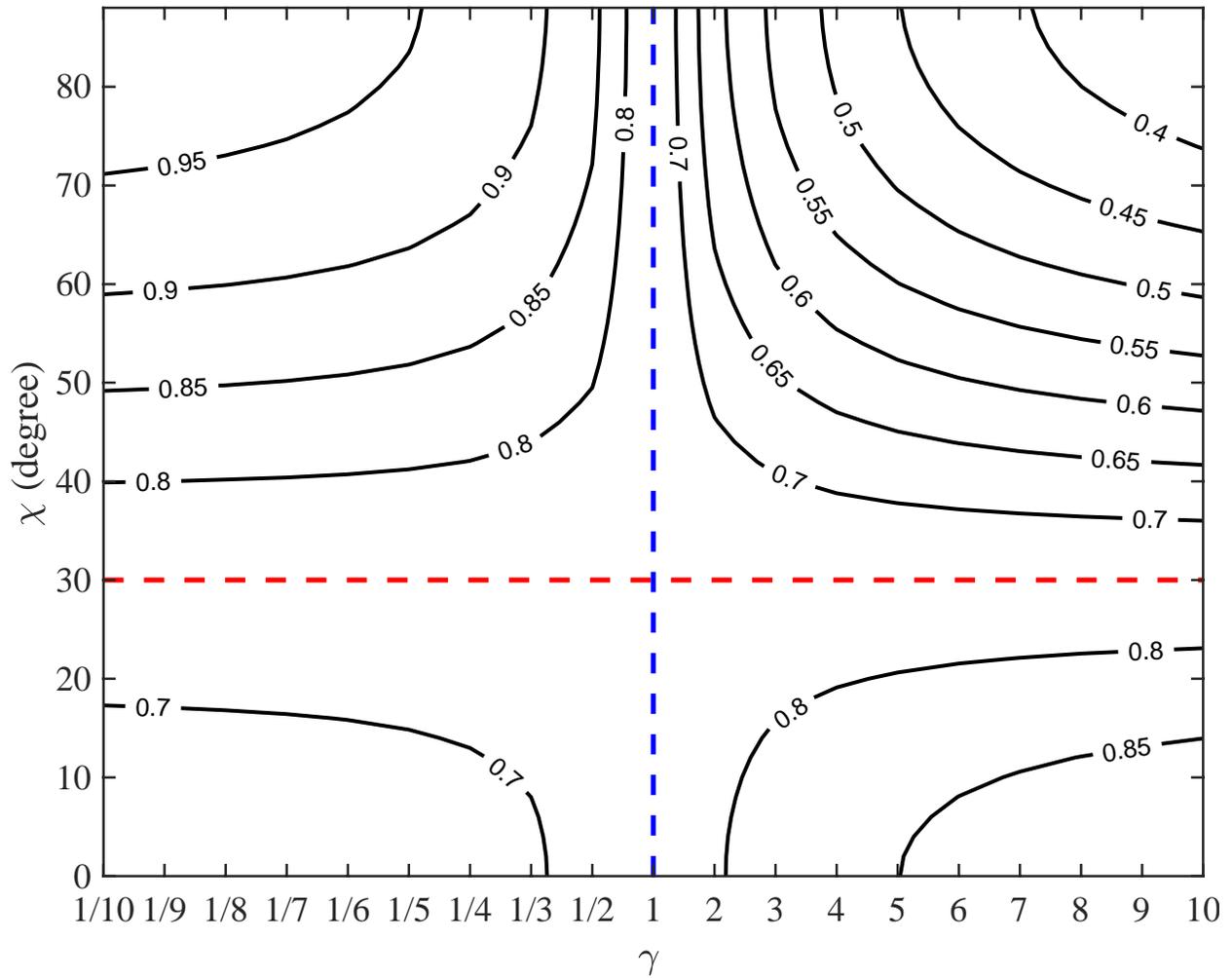
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798 Fig. 4 Variation of $R_x(\gamma, \chi)$ as a function of terrain anisotropy (γ) for different horizontal wind
 799 directions (χ).

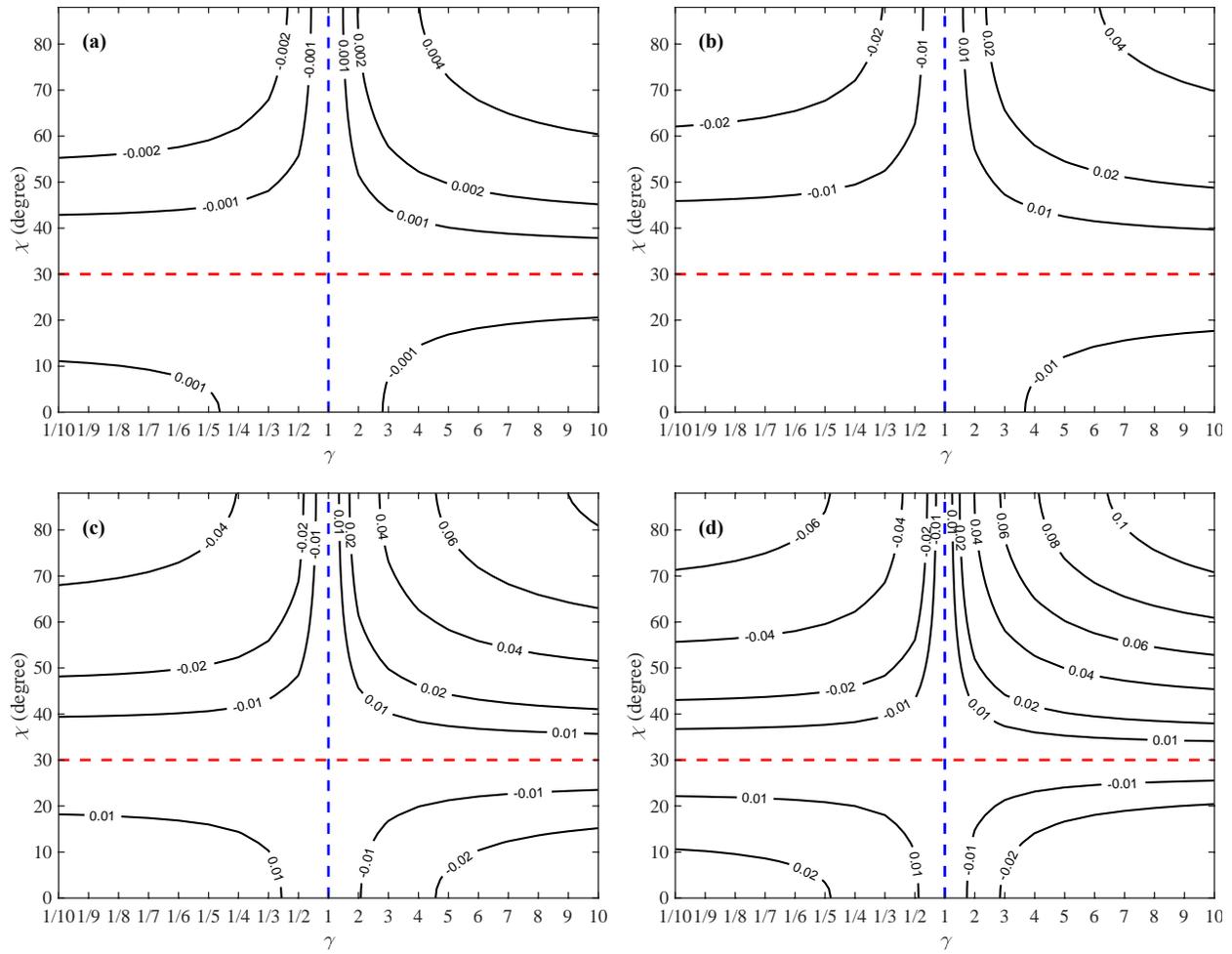


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801 Fig. 5 Variation of $\Delta\tilde{\tau}_{x0}$ in the parallel-flow case as a function of horizontal Froude number (Fr)
 802 for different terrain anisotropies (γ).

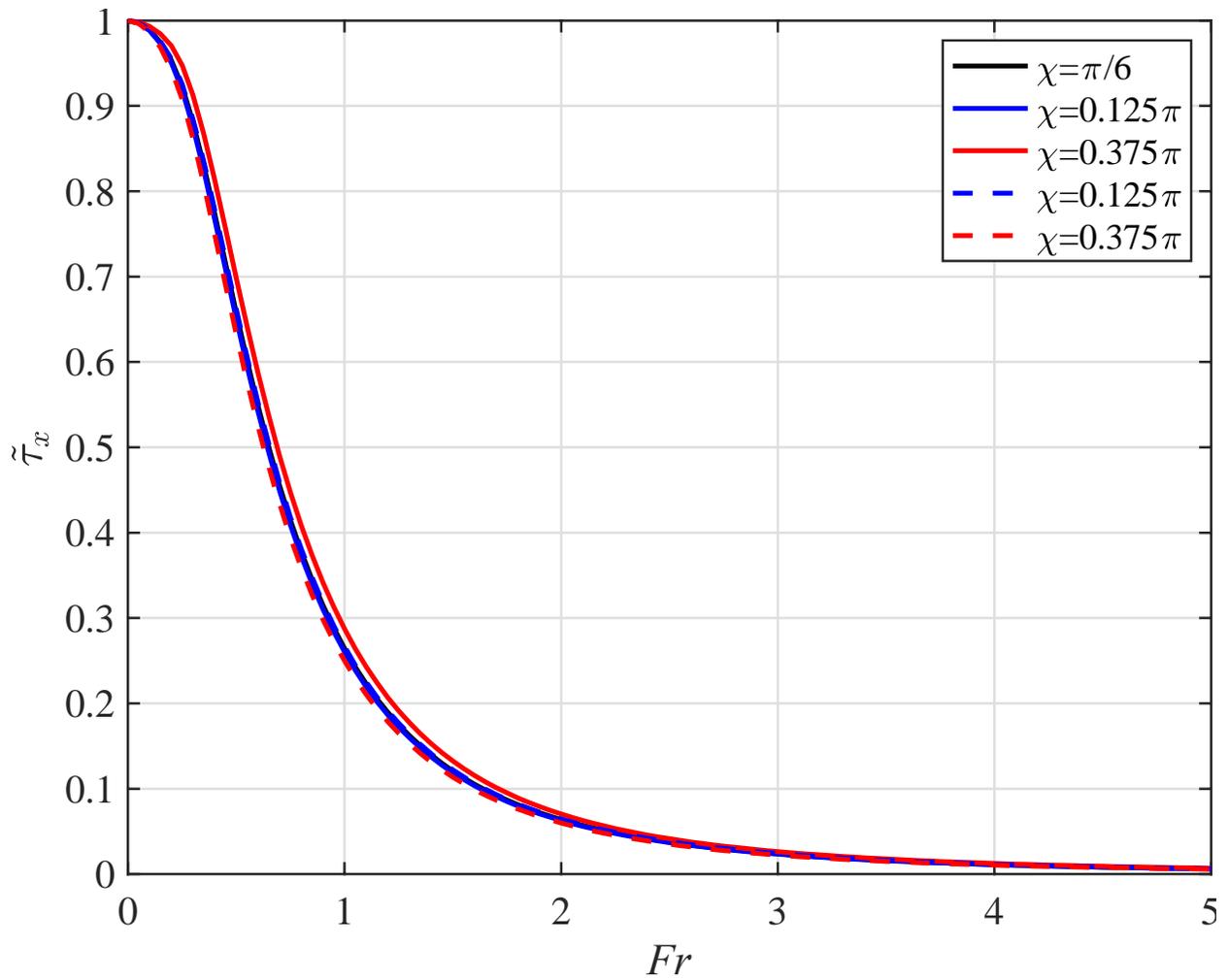


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 804 Fig. 6 Distribution of $R_x(\gamma, \chi)$ in γ - χ parameter space. The red line represents $\chi = \frac{\pi}{6}$ while the
 805 blue line indicates $\gamma = 1$.



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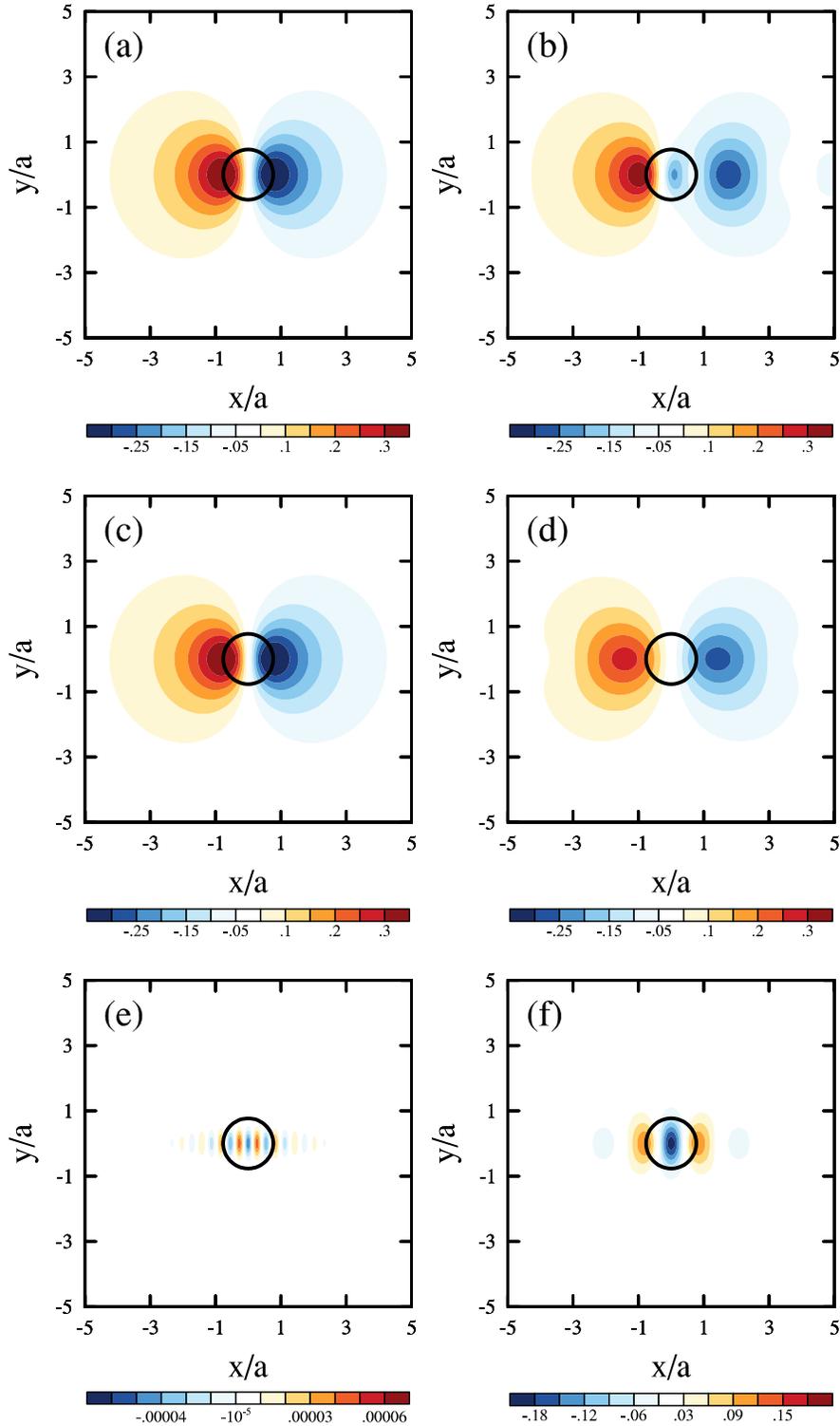
807 Fig. 7 Distribution of $\Delta\tilde{\tau}_x$ in γ - χ parameter space at different horizontal Froude numbers: (a) Fr
 808 $= 0.1$, (b) $Fr = 0.3$, (c) $Fr = 0.5$, and (d) $Fr = 1.0$. The red line represents $\chi = \frac{\pi}{6}$ while the blue line
 809 indicates $\gamma = 1$.



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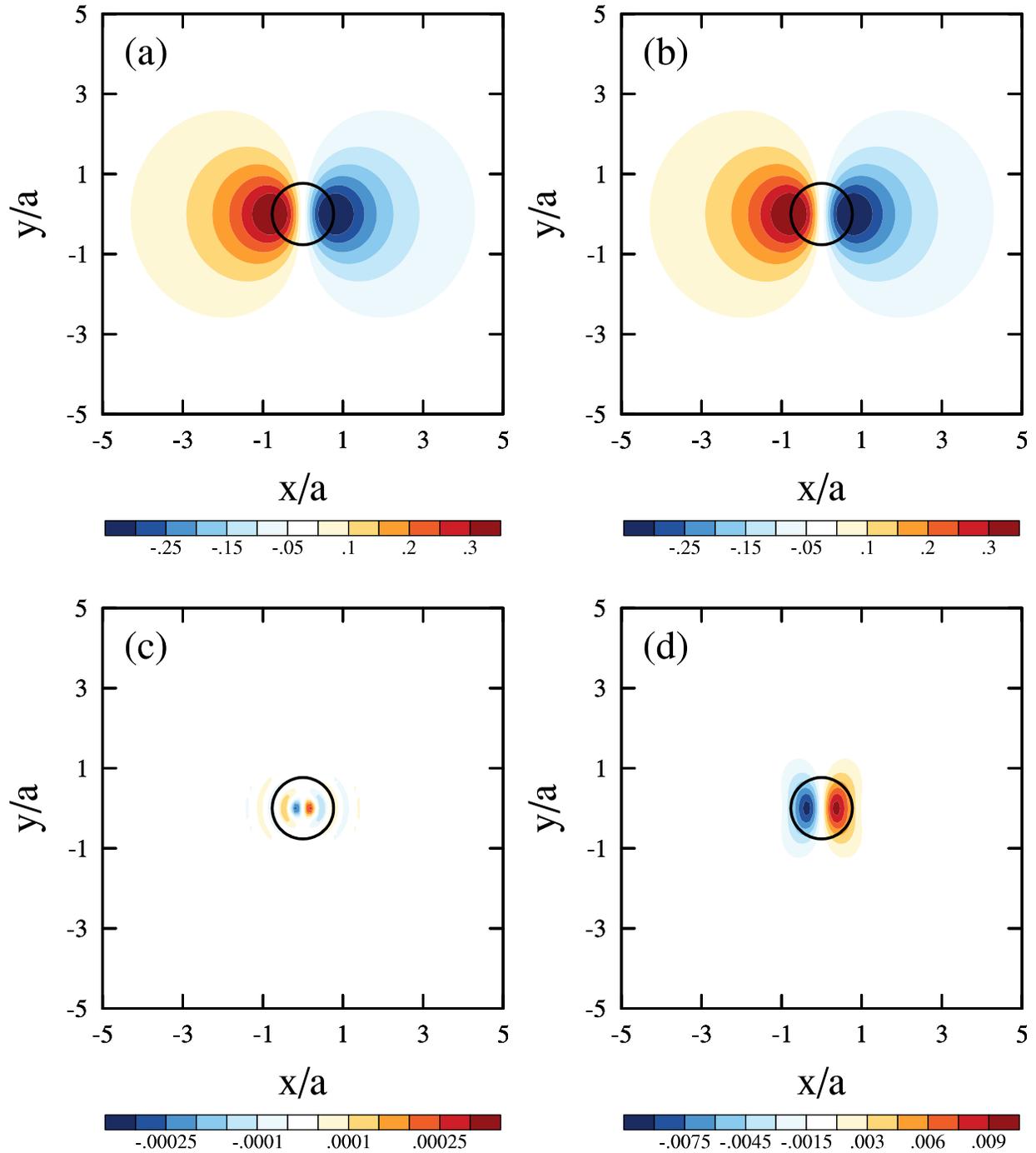
811 Fig. 8 Variation of the x -component of the normalized GWMF ($\tilde{\tau}_x$) in the oblique-flow case as a
 812 function of the horizontal Froude number (Fr). Solid and dashed lines are for $\gamma = 8$ and $\gamma = \frac{1}{8}$,
 813 respectively.

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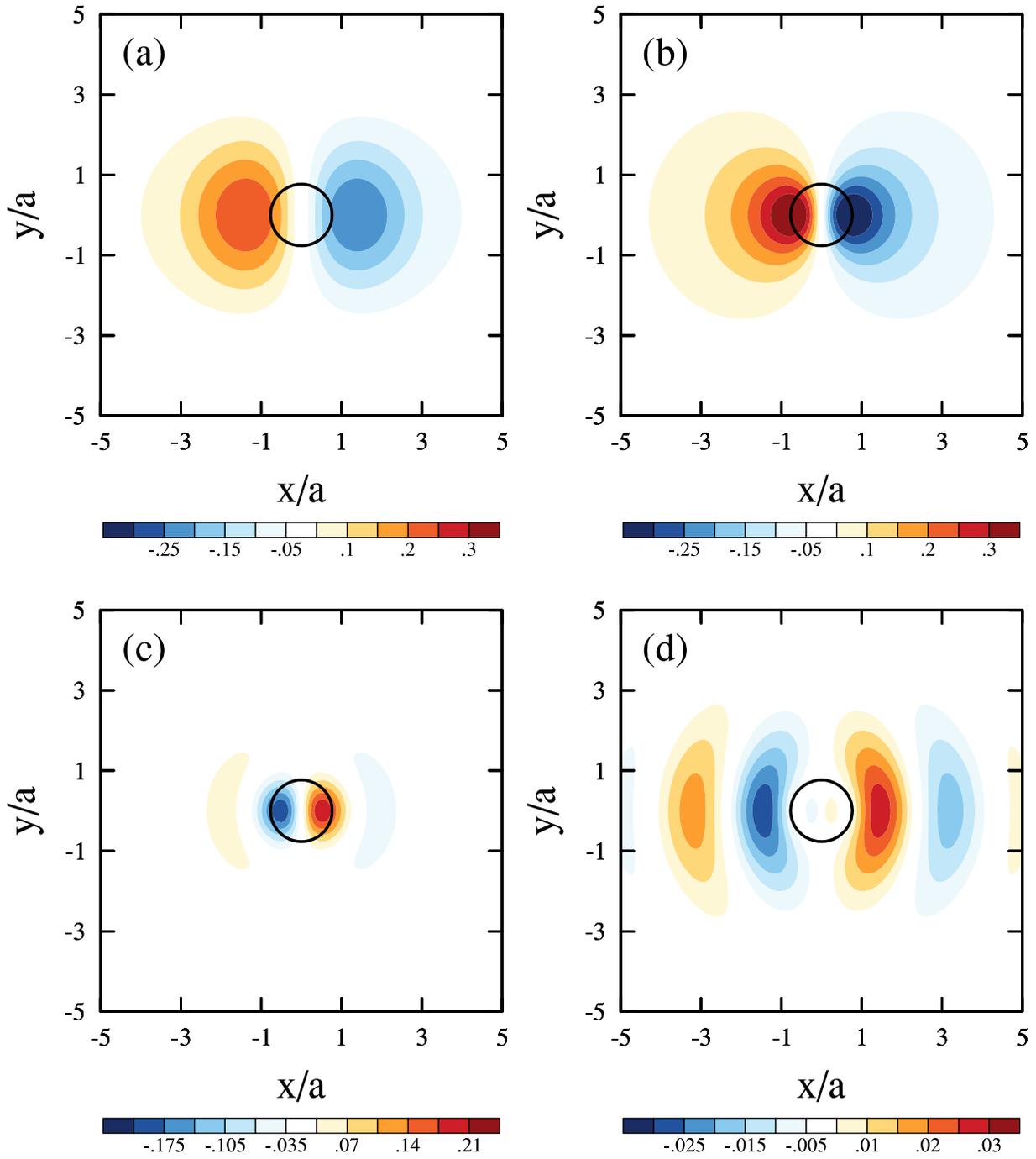
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Fig. 9 Exact surface pressure perturbation (top) of nonhydrostatic OGWs forced by a circular bell-shaped mountain, which is the sum of p_{GW} (middle) and $p_{evanescent}$ (bottom). See appendix for details. (a) (c) and (e) are for $Fr = 0.1$, while (b) (d) and (f) are for $Fr = 0.5$. The pressure perturbations are scaled with $\bar{\rho}N|\tilde{V}|h_0$. The axes are scaled by the mountain half width a . The black circle indicates the contour of $0.5 h_0$, with h_0 being the maximum elevation of the mountain.



821

822 Fig. 10 (a) Asymptotic surface pressure perturbation of nonhydrostatic vertically propagating
 823 OGWs forced by a circular bell-shaped mountain at $Fr = 0.1$, which is the sum of (b) p_0 , (c) p_1 and
 824 (d) p_2 (see appendix for details). The pressure perturbations are scaled with $\bar{\rho}N|\tilde{\mathbf{V}}|h_0$. The axes are
 825 scaled by the mountain half width a . The black circle indicates the contour $0.5 h_0$, with h_0 being
 826 the maximum elevation of the mountain.



827

828 Fig. 11 Same as Fig. 10 but for $Fr = 0.5$.